BOOK 1
The Calculus of First Order Diffusion

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Introduction to BOOK 1

First order diffusions are the topic of this first book, BOOK 1, of this dissertation. BOOK 1 is split into two main sections, FOUNDATIONS 1 and FOUNDATIONS II, and two smaller sections following each of these parts, Ito Calculus and *A Compendium of Random Walks*. BOOK 1 contains the foundations for the remainder of the thesis; concepts introduced in BOOK 1 are utilized all the way through BOOK 2 and BOOK 3. Since the diffusion calculus of this book is first order, and consequently simpler, it is introduced here and such concepts are then assumed knowledge, and knowledge that is then familiar, for BOOK 2 and a natural progression into BOOK 3. The concepts introduced in BOOK 1 are essential to explaining the bias and structural inaccuracy in the risk-neutral option pricing formulae and the development of the formulae for the accurate pricing of the option risk, that of the unbiased premium.

Diffusion: a spreading of something more widely over time (Oxford Dictionary)

In this thesis, the spreading relates to probability. The general definition and application of diffusion refers to the dispersion of many small particles, which is the correct definition of the application of this thesis when probability is viewed in the context of relative frequency. The graphics of the transition distribution, for each diffusion, shows visually a dispersing probability corridor making the context of this interpretation clear.

The first main part, FOUNDATIONS I, is concerned with continuous diffusions in continuous time with a continuous and deterministic rate of accumulation of the infinitesimal contributions of the components of an orthogonal normally distributed basis function. In the second main part, FOUNDATIONS II, the rate of accumulation of basis contributions and aggregation of probability occurs randomly through time according to a continuous time discrete counting process, the Poisson process. The summation of homogenous random variables according to a Poisson process is referred to as a compound Poisson process, and this work generalizes this to a Poisson process generating the terms in the diffusion sum, rather than simply a time invariant homogenous Poisson distribution, and makes the claim such processes are also first order diffusion (with a unit-root) like the diffusions of FOUNDATIONS I. Discussion of generalizations of the compound Poisson diffusions to non-unit-roots and higher order diffusion is deferred until BOOK 3 of the dissertation. Implementing the theory in an applied setting, in the context of mathematical finance, security dynamics, and option valuation, receives a greater emphasis in FOUNDATIONS II and in the second and third books of this dissertation.

First order continuous diffusion, the topic of FOUNDATIONS I, involves the convolution and modulated convolution of the components of the basis variable z (a Gaussian-Hilbert space) onto the diffusion variable X. As such, convolution is the whole point of the diffusion calculus, the calculus of the probability generated in S by z and whence measured by X. This is the contents of BOOK 1 described in the words of BOOK 3. The properties of the convolved probability can alternatively be expressed in terms of time correlation in the measured variable X. This notion of convolution is generalized in BOOK 3 to higher orders and deconvolution.

Definition of Call and Put Options

A financial derivative is a deferral of a transaction in the spot financial markets, i.e., a transaction that is deferred rather than made on the spot. Derivative markets are therefore derived from, and an extension to, the spot financial markets. The date to which the spot transaction is deferred is called either the expiry date or the maturity date of the derivative and the derivative is said to mature at that time.

A call option on a market asset is defined by the time at which the option matures and the exercise price. The call option gives the owner the right, but not the obligation, to buy the asset for the exercise price – in the case of European calls, on the expiry date, and in the case of American calls, on any date up to and including the expiry date. A call option is a deferred long transaction and is therefore a long future with the right to walk away from the transaction if it is unfavourable. Initially, only European options will be considered and any discussion will relate to this form of option unless explicitly stated otherwise.

Whenever the value of the underlying asset is above the exercise value, the call option is said to be in-the-money. Whenever the value of the underlying asset is below the exercise value, the call option is said to be out-of-the-money. The extent of the 'moneyness' is referred to as the call option's intrinsic value. At the expiry date, the intrinsic value becomes the payoff of the option. If the value of the underlying asset is below the exercise value on the expiry date, the call option owner will choose not to exercise the call on the asset and the call option expires worthless and is said to expire out-of-the-money. If the value of the underlying asset is above the exercise value on the expiry date, the call option owner will choose to exercise the call by calling the asset to purchase for the exercise value – the value of the in-the-money exercise is then the intrinsic value at that time. This is the excess of the value of the asset over what is paid for it by the option holder. The delivery of the exercise

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may be physical or financial – the asset may be physically delivered for the exercise price or the call option writer may close out the position financially. The value of the exchange is sometimes referred to as the excess function at expiry. In mathematical terms[1], the mathematical function is stated in terms of the max function or with the symbol $\vee$.

A put option on a market asset is defined by the time at which the option matures and the exercise price. The put option gives the owner the right, but not the obligation, to sell the asset for the exercise price on the expiry date. A put option is a deferred short transaction and is therefore a short future with the right to walk away from the transaction if it is unfavourable. Whenever the value of the underlying asset is below the exercise value, the put option is said to be in-the-money. Whenever the value of the underlying asset is above the exercise value, the put option is said to be out-of-the-money. The extent of the 'moneyness' is referred to as the put option's intrinsic value. If the value of the underlying asset is above the exercise value on the expiry date, the put option owner will choose not to exercise the put on the asset and the put option expires worthless and is said to expire out-of-the-money. If the value of the underlying asset is below the exercise value on the expiry date, the put option owner will choose to exercise the put by selling (putting) the asset to the put option writer for the exercise value. The value of the in-the-money exercise is then worth what the asset is sold for less the actual value of the asset. The delivery may be physical or financial. The value of the exchange is sometimes referred to as the shortfall or residual function at expiry.

[1]Klugman, S. A., Panjer, H. H. and Willmot, G. E., (1998) *Loss Models*. John Wiley & Sons.

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FOUNDATIONS I: CONTINUOUS TIME PROCESSES WITH CONTINUOUS GENERATORS

Introduction to FOUNDATIONS I

The processes of FOUNDATIONS I are introduced and looked at thoroughly in terms of

> **(1)** the partial differential equations of the diffusion transition density function, known as the forward and backward equations, and
> **(2)** the differential equation of the random variable of the diffusion transition distribution which defines the time sequence of the diffusion as a first order differential equation of the diffusion variable X as a function of time. This is later referred to as the diffusion equation.

The focus in the first book, BOOK 1, is the equations of the transition density function, their development, and their solution, for various types of first order diffusion. This thesis has sought to cover, rigorously, each type that has a transition distribution that has a known statistical form. This thesis considers univariate diffusion only. Solution to the forward equation, where applicable, is by the technique of transformation to the standard heat equation, $f_t = \frac{1}{2} f_{xx}$. This is the forward equation of standard Brownian motion $\int dB_t$. Alternatively, the forward equation is solved by transforming to the equivalent equation in the moment generating function, solving for the moment generating function subject to the initial conditions in the backward variables, and then either inverting the moment generating function or inferring the transition distribution by the uniqueness between a distribution and its moment generating function. For both methods, once the transition distribution is known, so is the dependent variable of the forward equation, the transition density of the diffusion. It is then shown that each of the first order diffusions are obtainable by ito-transform pairs (and convolution ito-transform pairs), providing a third method of solving the forward equation. This is a method of identification and inferring the solution by uniqueness of a process and its forward equation and its relation to standard Brownian motion or the more general additive process, the Ornstein-Uhlenbeck process. Since transforming to the standard heat equation is an equivalent route to solution as the identification of an ito-transform pair, Ito's Lemma is an alternative to transforming the forward equation.

A unit root is $(1 - B) / h$ in a discrete setting and d / dt in a continuous setting, as applied to the diffusion equation. This concept is fundamental to the stochastic calculus and receives a major emphasis in BOOK 3. It is labeled as a unit root, regardless of whether the basis function and operators are continuous or discrete, for consistency with the literature. There is no difference between $(1 - B) / h$ or d / dt since unit roots are fractal at every time scale down to and including dt. The processes to be discussed, those that have transition distributions of a known statistical form, are transformations and combinations of transformations of functions of the basic first order diffusion, standard Brownian motion. Standard Brownian motion is the direct integration of a standardized Gaussian Hilbert space, a continuous set of uncorrelated normal random variables that generate the diffusion. As such, Brownian motion is a unit root first order diffusion, the direct integration resulting from the unit root. The generalized process of standard Brownian motion is the Ornstein-Uhlenbeck process, which is an exponentially weighted integration of the standardized Gaussian Hilbert space, i.e., a modulated Brownian motion. When the exponential index is zeroed, the process reverts to standard Brownian motion. The exponential index is the infinitesimal characteristic root of the diffusion and, when this is zero (null modulation), it reverts to a unit discrete characteristic root. Again, this thesis usually refers to the discrete root (i.e., unit root) for consistency with existing literature. The defining feature of diffusion, that of integrations of a Gaussian diffusion basis, is explored in depth in the last part of this exposition, BOOK 3, whereby the theory of diffusion of arbitrary order is developed. Scaling and adding a non-zero mean to the underlying set of uncorrelated normal variables results in arithmetic Brownian motion in the case of direct integration and the Ornstein-Uhlenbeck process in the case of the exponentially weighted integration. These first order diffusions are additive processes. When these processes are transformed to exponentials, each become multiplicative. The resulting processes are geometric Brownian motion and the geometric Ornstein-Uhlenbeck process. Such diffusions are called log-diffusions since they are ordinary (additive) diffusions on logarithms. This terminology is consistent with the nomenclature of normal and log-normal random variables, although it is more correct to think of them as exponential diffusions. These diffusions, Brownian motions, Ornstein-Uhlenbeck processes, and the geometric equivalents are the diffusions that have forward and backward equations that are transformable to the standard heat equation. There is another process which is also given consideration which has a forward equation that is transformable to the standard heat equation, a particular form of the square-root process, which is the square of an Ornstein-Uhlenbeck process. This is referred to by this thesis as a two-parameter square-root diffusion. The square-root terminology is again

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consistent with the nomenclature of log-diffusions as well as that of the existing literature regarding the square-root process and normal and log-normal random variables. These diffusions are more appropriately referred to as quadratic diffusions. Convolutions of of homogenous two-parameter square-root diffusions result in the three-parameter square-root process. The square-root transformation of the three-parameter square-root diffusion results in the Bessel process. The Bessel process is the radius of a multi-dimensional Ornstein-Uhlenbeck process. While it may be possible to transform the equation to a simpler form, the forward equations of these latter processes, those obtained from homogeneous convolutions and transformations of these convolutions, are not transformable to the standard heat equation. Such processes are special forms of ito-transform pairs of what I call convolution ito-transform pairs. This terminology, convolution ito-transform pairs, also includes transformations of convolved diffusions.

The processes that have their forward equations solved and transition distributions derived, are standard Brownian motion, arithmetic Brownian motion, geometric Brownian motion, the Ornstein-Uhlenbeck process, the geometric Ornstein-Uhlenbeck process, the square-root process, and the Bessel process.

The classification and terminology of arithmetic and geometric Brownian motion and arithmetic and geometric Ornstein-Uhlenbeck (additive and multiplicative first order unit and non-unit-root[2] diffusions) concerns the fact that the successive diffusion increments of the additive process are independent and identically distributed (the ratios are not) and the transition distribution is closed under the operation of summation and arithmetic averaging while, on the other hand, the successive ratios of the multiplicative process are independent and identically distributed and the transition density is closed under the operation of multiplication and geometric averaging. This is a direct consequence of the exponential-logarithmic transformation and inverse transform in exactly the same way as the exponential and logarithmic transformations transform between arithmetic and geometric averages. In distributional terms, the cumulants[3] of an arithmetic diffusion sum linearly while the cumulants of a geometric diffusion sum proportionately. As such, geometric Brownian motion is sometimes referred to as proportional Brownian motion.

The general first order normal diffusion is defined by the stochastic differential equation, which is a random variable, and is equivalent to assigning a random variable to an event, in this case, a differential process of random variables relating to a dynamic time continuum of statistical outcomes. From this equation the forward and backward equations are derived which are differential equations of ordinary non-random functions, namely, the probability functions of the random variable of the stochastic differential equation.

Any first order diffusion is expressible as any one of

1) stochastic differential equation (differential form of the diffusion equation)
2) integral solution to the stochastic differential equation (integral form of the diffusion equation)
3) infinitesimal parameters of the stochastic differential equation (given that it is first order)
4) forward equation of the transition density function (or moment generating function)
5) backward equation of the transition density function (or moment generating function)
6) ito transform pair (or convolution ito transform pair, given the convolution index)
7) transition distribution or density (since first order diffusions are Markov processes)

Any one of these aspects implies the others since each is a characteristic of a given diffusion variable. In FOUNDATIONS I, continuous occurrence, the focus is directed at mathematically forming and solving the forward partial differential equation of the transition density function. The focus is then turned to obtaining the transition distribution, to the same processes for which the forward equation is solved, by ito calculus and ito-transform pairs. FOUNDATIONS II, discrete occurrence, then looks at and develops theory whereby the summations of cumulants, for additive processes, sum according to a compound Poisson process. It is shown that, in the limit, this form of diffusion is equivalent to the continuous occurrence diffusions of this first part of BOOK 1, FOUNDATIONS I. A rationale and a discussion of the utility of compound Poisson diffusions is presented for market microstructure transaction data. A slightly different approach is taken to this form of diffusion although the results are completely analogous, in particular, the forward and backward equations play a central role.

---

[2] the unit root terminology refers to the discrete root; in terms of the root of the infinitesimal derivative, this is a zero characteristic root; I have kept this terminology so that it is consistent with the current journal and textbook literature.

[3] some of the authors of probability theory refer to cumulants as semi-invariants, such as Feller and the Russian authors including Shiryayev

i. Feller, W., (1966) An Introduction to Probability Theory and Its Applications, Volume 2. John Wiley & Sons.
ii. Shiryayev, A. N., (1984) *Probability*. Springer Series in Soviet Mathematics. Springer-Verlag.

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Central to this entire thesis, including the forward and backward equations which relate to the convolution of density functions and the implied aggregation of random variables, is the fact that cumulants of all orders are additive for independent random variables and that adding independent random variables is equivalent to adding their cumulants;

Lemma

For a given sum of *n* independent random variables

$$S_n = X_1 + X_2 + \ldots + X_n = \sum_{j=1}^{n} X_j$$

which is equivalently expressed in terms of convolution of density functions,

$$f_{S_n}(x) = f_{X_1} * f_{X_2} * \ldots * f_{X_n}(x) = f_{\Sigma_{j=1}^{n} X_j}(x) = f_X^{*n}$$

the following property holds for a cumulant of arbitrary order *r*

$$\kappa_r(S_n) = \kappa_r\left(\sum_{j=1}^{n} X_j\right) = \sum_{j=1}^{n} \kappa_r(X_j)$$

That is, for independent random variables, cumulants are additive so that linear additions of independent random variables is equivalent to either **(1)** the convolution of the density functions or **(2)** the linear summation of the cumulants. Convolution implies the summation of random variables which, in turn, implies the summation of the cumulants.

Proof

For any sequence of independent but not necessarily identically distributed random variables, $X_1, X_2, ..., X_n$, the moment generating functions $M_{X_1}(\theta), M_{X_2}(\theta),..., M_{X_n}(\theta)$ can be expressed in terms of the cumulant generating functions $\exp(K_{X_1}(\theta))$, $\exp(K_{X_2}(\theta))$, …, $\exp(K_{X_n}(\theta))$. The sum S_n of the sequence $X_1, X_2, \ldots, X_n$

$$S_n = X_1 + X_2 + \ldots + X_n$$

has a moment generating function that is the product of the moment generating functions for the individual terms in the sum

$$\begin{aligned} M_{S_n}(\theta) &= M_{X_1+X_2+\ldots+X_n}(\theta) \\ &= E\left(e^{\theta X_1+\theta X_2+\ldots+\theta X_n}\right) \\ &= E\left(e^{\theta X_1}\right)E\left(e^{\theta X_2}\right)\ldots E\left(e^{\theta X_n}\right) \\ &= M_{X_1}(\theta)M_{X_2}(\theta)\ldots M_{X_n}(\theta) \end{aligned}$$

In terms of the cumulant generating functions, $M_{S_n}(\theta) = M_{X_1}(\theta)M_{X_2}(\theta)\ldots M_{X_n}(\theta)$ is expressed as $M_{S_n}(\theta) = \exp\left(K_{X_1}(\theta) + K_{X_2}(\theta) + \ldots + K_{X_n}(\theta)\right)$. The cumulant generating function of the sum is then the logarithm, $K_{S_n}(\theta) = K_{X_1}(\theta) + K_{X_2}(\theta) + \ldots + K_{X_n}(\theta)$, which is a linear sum of the individual cumulant generating functions. Because differentiation is a linear operator, the r^{th} derivative of the cumulant generating function of the sum is equal to the r^{th} derivative of the *n* cumulant generating functions of the terms in the sum, $K_{S_n}^{(r)}(\theta) = K_{X_1}^{(r)}(\theta) + K_{X_2}^{(r)}(\theta) + \ldots + K_{X_n}^{(r)}(\theta)$. For a zero cumulant generating function argument, $\kappa_r(X_j) = K_{X_j}^{(r)}(0)$ so that the r^{th} cumulant of the sum of random variables is the sum of the r^{th} order cumulants of the *n* terms of the summation $\kappa_r(S_n) = K_{S_n}^{(r)}(0)$

$$\kappa_r(S_n) = \kappa_r(X_1 + X_2 + \ldots + X_n) = \kappa_r(X_1) + \kappa_r(X_2) + \ldots + \kappa_r(X_n)$$

The proof is complete.

Hence, convolution and deconvolution pertain to, and used synonymously with, the addition and subtraction of random variables; and this further pertains to the diffusion of probability by the convolution of the components of the diffusion basis, i.e., a first order diffusion is defined in terms of continually convolved components of a Gaussian-Hilbert space onto the first order diffusion variable. In BOOK 3, this idea is extended to higher order

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convolutions[4], and introducing deconvolution (for forward moving forward dynamics), and nesting these convolutions and deconvolutions with heterogeneous moderators.

As a statement of the differential form of the diffusion equation from BOOK 3, the definitions then of each of the first order diffusion variables considered in FOUNDATIONS I can be written down in terms of how the convolution of the basis components take place;

Brownian motion is the homogenous convolution of basis components z onto the diffusion variable X ;
Proportional Brownian motion is the homogenous convolution of basis components z onto the logarithm of the diffusion variable $\log X$;
Modulated Brownian motion is the modulated convolution of basis components z onto the diffusion variable X ;
Proportional moderated Brownian motion is the modulated convolution of basis components z onto the logarithm of the diffusion variable $\log X$;
The two parameter square root process is the modulated convolution of basis components z onto the square root of the diffusion variable $\sqrt{X}$;
The three parameter square root process is the convolution of two or more diffusion variables, $X_1 + X_2 + \ldots + X_n$ obtained by homogenous convolutions or modulated convolutions of basis components z onto the square roots of each of the diffusion variables $\sqrt{X_1}$, $\sqrt{X_2}$, …, $\sqrt{X_n}$ (the convolution of two or more two parameter square root variables);
Bessel process (Radius of multidimensional moderated Brownian motion) is the square root of the convolution of two or more diffusion variables $\sqrt{X_1 + X_2 + \ldots + X_n}$ obtained by homogenous or modulated convolutions of basis components z onto the square root of each diffusion variables $\sqrt{X_1}$, $\sqrt{X_2}$, …, $\sqrt{X_n}$, i.e., the square root of the three parameter square root process;
Logistic Brownian motion is the homogenous or modulated convolution of basis components z onto the logarithm of the odds ratio of the diffusion variable $\log [X / (1 - X)]$ (inverse logistic transform).

The Supplement to BOOK 3 extends these to harmonic and hyperbolic and modulated harmonic and hyperbolic integrations of a diffusion basis function.

Stochastic Differential Equation of a First Order Normal Diffusion

The processes considered in FOUNDATIONS I are continuous first order diffusions, in continuous time with normally distributed differential increments. First order diffusions are called Markov processes by the literature. This terminology refers to the fact that first order diffusions are first order summations or transformations of first order summations of an uncorrelated Gaussian Hilbert space; a differenced summation of independent increments leaves independent increments which then implies the (first order) Markov property. In the case of continuous diffusions, the differenced summation is the differentiation of the integration of the continuous uncorrelated diffusion basis. The first order differential equation relating the diffusion random variable to the diffusion basis, as a function of time, defines the class of first order diffusion.

The conditional diffusion variable is $dX(t) \mid X(t) = x$ or $[X(t + dt) - X(t)] \mid X(t) = x$, at any time t , and shorthand for this differential increment omits the conditional variable at the lower endpoint of the transition interval, and expresses the increment simply as $dX(t)$. The first order differential equation of first order diffusion is then

$$dX_t = \mu(x,t)dt + \sigma(x,t)dB_t$$

or as a derivative

$$\frac{dX_t}{dt} = \mu(x,t) + \sigma(x,t)\frac{dB_t}{dt}$$

conditional on a Dirac distributed backward variable

$$\lim_{dt \to 0} X_{t+dt} \sim \delta(x)$$
$$\sim \lim_{\varepsilon \to 0} U(x - \varepsilon, x + \varepsilon)$$

where B_t is standard Brownian motion

[4] convolution of convolution, convolution of convolution of convolution, etc.

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$B(t) \sim N(0,t)$

and dB_t is differential standard Brownian motion $dB_t \sim N(0, dt)$. The differential Brownian motion in the stochastic increment is synonymous with the diffusion basis. This basis generates the stochastic variation in the time progression of the diffusion variable. The forward equation, for an arbitrary first order diffusion, is derived from the moment generating function of the diffusion variable.

The moment generating function of the stochastic differential conditional on $X(t) = x$ is

$$M_{dX(t)}(\theta) = \exp\left(\mu(x,t)\theta + \tfrac{1}{2}\sigma^2(x,t)\theta^2\right)$$

and the logarithm is the cumulant generating function

$$\log M_{dX(t)}(\theta) = \mu(x,t)\theta + \tfrac{1}{2}\sigma^2(x,t)\theta^2$$

which expresses the fact that any diffusion is alternatively described by the continuous summation of the cumulants, for which the only two that are non-zero in the case of Gaussian diffusion are $\mu(x, t)$ and $\sigma^2(x, t)$.

The Infinitesimal Parameters of a First Order Normal Diffusion

In relation to first order normal diffusion, $\mu(x, t)$ and $\sigma^2(x, t)$ are the infinitesimal parameters that specify the conditional mean and variance of the stochastic differential; that of the diffusion variable over an infinitesimal transition interval, i.e., the diffusion variable from time t to $t + dt$, given the value of the variable at time t exact is $X(t) = x$. The functions $\mu(x, t)$ and $\sigma^2(x, t)$ fully specify a first order normal diffusion for the diffusion equation

$$\frac{dX_t}{dt} = \mu(x,t) + \sigma(x,t)\frac{dB_t}{dt}$$

The Forward and Backward Equations

The processes considered in this section involve those that occur continuously through time and measure a continuous variable. Due to the fact that only first order diffusions are considered, these possess the Markov[5] property. The Markov property defines processes whereby the transition distributions for any two non-overlapping time intervals for a given diffusion variable are uncorrelated. This is the property arising from the fact that a first order diffusion is a first order cumulation of independent random variables. The diffusion variable then depends only upon where it was observed last, say, with value y at time s ($t > s$). A hypothetical value x from the distribution of values at time t is the forward diffusion variable and t is the forward time variable. The value y at which it was last, at time s, is the backward diffusion variable and s is the backward time variable. The forward and backward variables, therefore, relate to the two extremities of an interval of time, referred to as the transition interval, the earlier of which is observed or known and the latter of which being some time into the unknown future over which the diffusion undergoes a statistically measurable net change.

The random variable measuring the distribution of diffusion over the variable x at a given time t is denoted $X(t)$. The density function of this random variable is $f_{X(t)}(x)$. In the context of Markov processes, or first order diffusion, these have meaning only with reference to the backward variables so that the transition interval is finite. This is acknowledged by conditioning the random variable to which the forward variables relate on those of the backward variables. Hence, it is the random variable $X(t) \mid X(s) = y$ that the transition function of the forward and backward equations relate to. The transition density function of the process is a probability density function for a distribution of values x at time t, $X(t)$, given that the value of the process was y at time s, $X(s) = y$. This is written as $f_{X(t)\,|X(s)}(x \mid y)$. When the conditioning variable is omitted, by default the conditioning variable is a known value of the diffusion variable at time zero, $X(0) = x_0$. The functions $f_{X(t)}(x)$ and $f_{X(t)}(x \mid x_0)$ are therefore synonymous shorthand for the function $f_{X(t)\,|X(0)}(x \mid x_0)$. The formal definitions of the relevant items of nomenclature are

- The **transition interval** or **transition period** is the time interval between the backward time variable s and the forward time variable t.

[5] The extension of the Markov property from BOOK 3 extends A. A. Markov's idea of a process with independent increments.

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- The **transition distribution** is the distribution of the statistically measureable **net change over the transition interval**, between the backward diffusion variable y and the conditional forward distribution $X(t) \mid X(s) = y$.
- $f_{X(t)\,|X(s)}(x \mid y)$ is the **transition density function** for the distribution of the diffusion at a particular time and measures the probability that the process will be at value x at time t given that the process was at the value y at time s .

The forward and backward variables are the variables of the dynamics of the diffusion. The forward variables are located at some point in the future beyond the time at which the backward variables apply, so that they describe a statistical distribution. In a purely statistical sense, the forward diffusion variable is the only variable of the density function while the other three appear as parameters. In a non-statistical sense, each are variables. The forward and backward equations describe the dynamics of the density function, as with respect to each of these variables.

The equation describing the distributional dynamics of the forward motion (the future of the diffusion) through time at the upper extremity of the transition interval is the forward equation. This is an equation in the forward variables x and t . This equation is also referred to as the Fokker-Planck equation in mathematical physics. The equation describing the dynamics of the backward motion for the differential dynamics of the initial conditions at the lower extremity of the transition interval is the backward equation. This is an equation in the backward variables y and s . Since the equations are describing statistical movements of the diffusion distribution of values over time, the dependent variable that measures the value of the transition at any time is the transition density function $f_{X(t)\,|X(s)}(x \mid y)$. Since the forward (prospective) dynamics of the diffusion are dependent upon the two forward variables, the forward equation is a partial differential equation with dependent variable $f_{X(t)\,|X(s)}(x \mid y)$ and independent variables x and t . Since the backward (retrospective) dynamics of the diffusion are dependent upon the two backward variables, the backward equation is a partial differential equation with dependent variable $f_{X(t)\,|X(s)}(x \mid y)$ and independent variables y and s . The forward and backward equations are equations of the same dependent variable (the transition density function) and are formal adjoints of one another. The transition density function satisfies the forward equation as well as the backward equation since the two equations are adjoints and therefore the transition density is the dependent variable of both equations. The forward equation, for an arbitrary first order normal diffusion, is

$$\frac{\partial}{\partial t} f_{X(t)|X(s)}(x \mid y) = \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,t)\, f_{X(t)|X(s)}(x \mid y)\right) - \frac{\partial}{\partial x}\left(\mu(x,t)\, f_{X(t)|X(s)}(x \mid y)\right)$$

and the backward equation is

$$-\frac{\partial}{\partial s} f_{X(t)|X(s)}(x \mid y) = \tfrac{1}{2}\sigma^2(y,s)\frac{\partial^2}{\partial y^2} f_{X(t)|X(s)}(x \mid y) + \mu(y,s)\frac{\partial}{\partial y} f_{X(t)|X(s)}(x \mid y)$$

$\partial f / \partial t$ and $\partial f / \partial s$ are partial time derivatives of the transition density, i.e., derivatives of the density with respect to either the forward (t) or backward (s or t_0) time variable. As such, it describes the time dynamics of the transition density. The side of the forward or backward equation that contains this partial derivative, therefore, is appropriately referred to as the time side. Derivatives in either the diffusion variable x or backward diffusion variable y, such as $\partial^2 f / \partial x^2$, describe the shape of the diffusion density. As such, the side of the forward or backward equation that contains the partial derivatives in either x or y is appropriately referred to as the diffusion side.

The point source of the transition surface is a Dirac delta function at the backward variables. The transition density at the backward variables of any forward is always a singularity from which probability monotonically diffuses with forward extensions of the transition interval. The dependent variables of the forward and backward equations of the transition density functions, as well as the option premiums, revert to equations of delta functions as $\sigma \to 0$ (and S reverts to R) and the transition surface becomes a Dirac delta traced out along the transition mean. When the forward equation in the Dirac trace is transformed to the cumulant generating function for $\sigma \to 0$, it is the equation of an ordinary regular function of the conventional calculus on R (Reals). This thesis has been written specifically for $\sigma > 0$.

The diffusion of probability ($\sigma > 0$) is a continuum of convolving random variables.

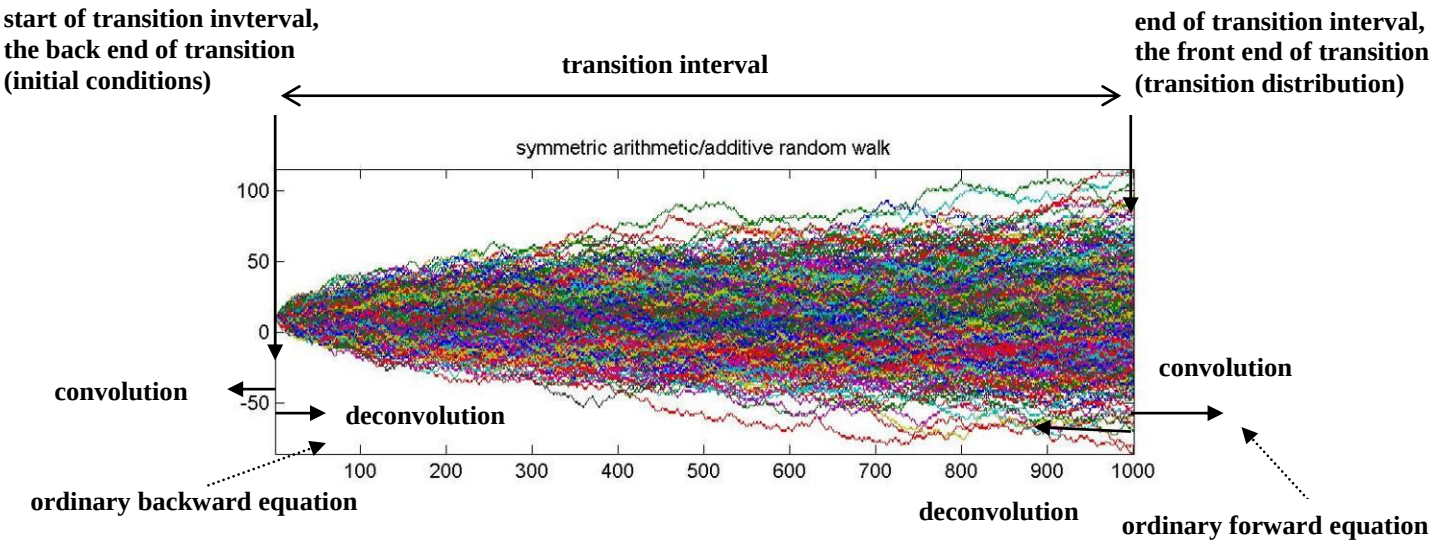
- As time moves forward, the process convolves forward with cumulative differential sums of probability. This is alternatively described by a continuous summation (continuous infinitesimal addition) of cumulants of the continuous sequence of random variables of the infinitesimal differentials that form the Gaussian diffusion basis for which is integrated into a cumulative sum.

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- As time moves forward and one observes the history of the process as it unfolds through time with evolving initial conditions (for fixed forward variables), the process deconvolves by a continuum of successive differential deconvolutions. This is alternatively described by a continuous reduction in the cumulants.

These are the differential dynamics that the forward and backward equations describe. The forward equation describes f^{*t}, the continuous t-fold convolution of the infinitesimal normal components of the diffusion basis onto the diffusion variable (there is no standard notation for deconvolution so the same description of the backward equation is not, as it stands, possible).

When time is also considered as moving backward as well as forward, there are actually four equations, one pair for the forward variables and one pair for the backward variables. One pair relates to the dynamics of the front end of the transition interval at the forward variables, for forward and backward moving time, and the other pair relates to the dynamics of the backend of the transition interval at the backward variables, for forward and backward moving time. The equations of the former pair are the forward equations and relate to the transition distribution. The equations of the latter pair are the backward equations and relate to the initial conditions of the transition interval and distribution. Hence, with bracketed terms relating to backward moving time, the forward equation is an equation of convolution (deconvolution), describing the convolving (deconvolving) evolution of the probability distribution of the diffusion as it extends forward (contracts backward) through time. The backward equation, on the other hand, is an equation of deconvolution (convolution), describing the deconvolving (convolving) dynamics of the distribution of the diffusion as the initial conditions are moved forward (extended backward) through time. The backward equation for forward moving time, a deconvolution equation, relates to PDEs of option premiums and is of primary importance to option replicating portfolios. This is the topic of BOOK 2 .



For diffusion basis function $X_1, X_2, ..., X_{n-1}, X_n$ and diffusion aggregation $X_1 + X_2 + ... + X$

(1) Forward convolution (forward variables with time running forward)

describes the differential dynamics of how the forward diffusion variable moves forward through time whereby successive basis components are convolved onto the endpoint of the transition interval as the diffusion extends forward;

$X_1 + X_2 + ... + X_{n-1} + X_n \Leftrightarrow X_1 + X_2 + ... + X_{n-1} + X_n + X_{n+1}$ (convolve X_{n+1} onto front end)

(2) Forward deconvolution (forward variables with time running backward)

describes the differential dynamics of how the forward diffusion variable moves backward through time whereby successive basis components are deconvolved from the endpoint of the transition interval as the diffusion contracts backward;

$X_1 + X_2 + ... + X_{n-1} + X_n \Leftrightarrow X_1 + X_2 + ... + X_{n-1}$ (deconvolve X_n from front end)

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(3) Backward convolution (backward variables with time running backward)

describes the differential dynamics of how the backward diffusion variable moves backward through time whereby successive basis components are convolved onto the beginning of the transition interval as the beginning of the diffusion extends backward;

$X_1 + X_2 + ... + X_{n-1} + X_n \Leftrightarrow X_0 + X_1 + X_2 + ... + X_{n-1} + X_n$ (convolve X_0 onto back end)

(4) Backward deconvolution (backward variables with time running forward)

describes the differential dynamics of how the backward diffusion variable moves forward through time whereby the successive basis components are deconvolved from the beginning of the transition interval as the beginning of the diffusion contracts forward.

$X_1 + X_2 + ... + X_{n-1} + X_n \Leftrightarrow X_2 + ... + X_{n-1} + X_n$ (deconvolve X_1 from back end)

Conversion of either pair from, say, forward moving time to backward moving time is accomplished by a simple time transformation; either a simple sign change or a sign change plus a constant, for example t to $T - t$. This simply changes the sign of the derivative. Conversion from the forward equation to the backward equation is accomplished by deducing the adjoint with the conjugate density and integration by parts.

This work is primarily interested in the equations **(1)** and **(4)** where

- **i)** the forward diffusion variable is matched with the **forward moving** forward time variable extending the diffusion further into the unobserved future for fixed initial values (the backward variables), and where
- **ii)** the backward diffusion variable is matched with the **forward moving** backward time variable, for fixed forward variables, contracting the diffusion along a filtration of successively observed or known values towards the singularity in probability that occurs when the backward variables meet the forward variables. This is the equation for which option PDEs relate.

These are the forward convolution and backward deconvolution equations, respectively, and both relate to forward moving time.

The next section is the first mathematical section and involves deriving the forward and backward equations for an arbitrary first order diffusion. The following sections then involve solving these equations for particular diffusions with a focus on solving the forward equation. The backward equation takes prominence in BOOK 2 of this thesis which discusses and develops the mathematics of option replicating portfolios and differential equations for option premia. This is where the initial conditions are of interest and are followed as the backward variables, as initial conditions, evolve towards a fixed forward time variable of an option expiry.

Deriving and Solving the Forward and Backward Equations

The derivation of the forward and backward equations of a general diffusion can be obtained by a variety of methods. The derivations differ depending on whether it is the forward or backward equation due to the fixed constructs of mathematics and mathematical statistics. The approaches taken are

(1) differentiating the transition MGF
(2) expanding the transition PDF in the Chapman-Kolmogorov equation

The derivation of the forward equation proceeds by the technique of moment generating functions which emphasizes a diffusion as a progressive accumulation of infinitesimal cumulants. The derivation of the backward equation, on the other hand, proceeds by reference the Chapman-Kolmogorov equation for the law of total probability, simplified by the Markov property of first order diffusion, whereby the transition density function is expanded around the backward variables.

(1) Derivation[6] of the Forward equation for an Arbitrary First Order Diffusion (by Differentiating the Transition MGF)

The transition MGF for a transition interval from s to t is defined as

$$M_{X(t)|X(s)}(\theta) = E\left(e^{X(t)\theta} \mid X(s) = y\right)$$

[6]This derivation follows the same reasoning as that by
Cox, D. R. and Miller, H. D., (1967) *The Theory of Stochastic Processes*. John Wiley & Sons.

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Differentiating the MGF partially with respect to the forward time variable t

$$\begin{aligned}\frac{\partial}{\partial t}M_{X(t)|X(s)}(\theta) &= \frac{M_{X(t+dt)|X(s)}(\theta)-M_{X(t)|X(s)}(\theta)}{dt}\\ &= \frac{E\left(e^{X(t+dt)\theta}\mid X(s)\right)-E\left(e^{X(t)\theta}\mid X(s)\right)}{dt}\\ &= \frac{E\left(e^{X(t+dt)\theta-X(t)\theta+X(t)\theta}\mid X(s)\right)-E\left(e^{X(t)\theta}\mid X(s)\right)}{dt}\\ &= E\left(\frac{e^{dX(t)\theta+X(t)\theta}-e^{X(t)\theta}}{dt}\mid X(s)\right)\\ &= E\left(\frac{E\left(e^{dX(t)\theta+X(t)\theta}\mid X(t)\right)-E\left(e^{X(t)\theta}\mid X(t)\right)}{dt}\mid X(s)\right)\\ &= E\left(e^{X(t)\theta}\frac{E\left(e^{dX(t)\theta}-1\mid X(t)\right)}{dt}\mid X(s)\right)\\ &= E\left(e^{X(t)\theta}\frac{E\left(dX(t)\theta+\frac{1}{2}dX(t)\theta^2+o\left(dX(t)^2\right)\mid X(t)\right)}{dt}\mid X(s)\right)\end{aligned}$$

from a Taylor series expansion in the exponential function in $dX(t)$ and nested conditioning for a differential transition interval from t to $t + dt$ within the outer expectation. Noting that the diffusion equation of the first order variable, conditional on $X(t) = x$, is

$$dX(t) = \mu(x,t)dt + \sigma(x,t)dB(t)$$

which has the differential transition distribution (for the transition over an infinitesimal transition interval from t to $t + dt$

$$dX(t) \sim N\left(\mu(x,t)dt,\, \sigma^2(x,t)dt\right)$$

so that the relevant moments conditional on $X(t) = x$ are

(1) $E\left(dX(t)\mid X(t)=x\right)=\mu(x,t)dt$

(2) $E\left(dX(t)^2\mid X(t)=x\right)=\sigma^2(x,t)dt+o(dt)$

(3) $E\left(o[dX(t)^2\mid X(t)=x]\right)=o(dt)$

These are the infinitesimal parameters. The partial time derivative of the MGF becomes

$$\begin{aligned}\frac{\partial}{\partial t}M_{X(t)|X(s)}(\theta) &= E\left(e^{X(t)\theta}\frac{\theta\mu(x,t)dt+\frac{1}{2}\theta^2\sigma^2(x,t)dt+o(dt)}{dt}\mid X(s)\right)\\ &= E\left(\theta e^{X(t)\theta}\mu(x,t)+\tfrac{1}{2}\theta^2e^{X(t)\theta}\sigma^2(x,t)\mid X(s)\right)+\frac{o(dt)}{dt}\end{aligned}$$

This can be written in terms of the transition density function $f_{X(t)|X(s)}(x\mid y)$ in integral form as

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-\infty}^{\infty}e^{\theta x}f_{X(t)|X(s)}(x\mid y)dx &= \int_{-\infty}^{\infty}\theta e^{\theta x}\mu(x,t)\,f_{X(t)|X(s)}(x\mid y)dx+\tfrac{1}{2}\int_{-\infty}^{\infty}\theta^2e^{\theta x}\sigma^2(x,t)\,f_{X(t)|X(s)}(x\mid y)dx+\frac{o(dt)}{dt}\\ &= \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}e^{\theta x}\right)\mu(x,t)\,f_{X(t)|X(s)}(x\mid y)dx+\tfrac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\partial^2}{\partial x^2}e^{\theta x}\right)\sigma^2(x,t)\,f_{X(t)|X(s)}(x\mid y)dx+\frac{o(dt)}{dt}\end{aligned}$$

that is,

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$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{X(t)|X(s)}(x \mid y)dx = -\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x}\Big(\mu(x,t)\, f_{X(t)|X(s)}(x \mid y)\Big)dx + \tfrac{1}{2}\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2}\Big(\sigma^2(x,t)\, f_{X(t)|X(s)}(x \mid y)\Big)dx + \frac{o(dt)}{dt}$$

Letting $o(dt) \to 0$ and inverting the transition MGF leaves the forward equation for the transition density

$$\frac{\partial}{\partial t} f_{X(t)|X(s)}(x \mid y) = \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\Big(\sigma^2(x,t)\, f_{X(t)|X(s)}(x \mid y)\Big) - \frac{\partial}{\partial x}\Big(\mu(x,t)\, f_{X(t)|X(s)}(x \mid y)\Big)$$

which is essentially the partial derivative of the diffusion probability distribution, and describes the dynamics of the convolution of the infinitesimal basis components onto the front end of the diffusion as the forward variables *x* and *t* progress forward through time.

(2) Derivation of the Backward Equation for an Arbitrary First Order Diffusion (by Expanding the Chapman-Kolmogorov Equation)

From the Chapman-Kolmogorov equation expressing the rule of total probability simplified by the Markov property of first order diffusion, the transition density function can be written as

$$f_{X(t)|X(s-ds)}(x \mid y) = \int_{-\infty}^{\infty} f_{X(t)|X(s)}(x \mid u)\, f_{X(s)|X(s-ds)}(u \mid y)du$$

Expanding the first transition function in the integral in a power series around the backward diffusion variable

$$\begin{aligned} & f_{X(t)|X(s-ds)}(x \mid y) \\ &= \int_{-\infty}^{\infty} \left\{ f_{X(t)|X(s)}(x \mid y) + (u-y)\frac{\partial}{\partial y} f_{X(t)|X(s)}(x \mid y) + \tfrac{1}{2}(u-y)^2 \frac{\partial^2}{\partial y^2} f_{X(t)|X(s)}(x \mid y) + \ldots \right\} f_{X(s)|X(s-ds)}(u \mid y)du \\ &= \left(\int_{-\infty}^{\infty} f_{X(s)|X(s-ds)}(u \mid y)du \right) f_{X(t)|X(s)}(x \mid y) + \left(\int_{-\infty}^{\infty} (u-y)\, f_{X(s)|X(s-ds)}(u \mid y)du \right) \frac{\partial}{\partial y} f_{X(t)|X(s)}(x \mid y) \\ &\quad + \tfrac{1}{2}\left(\int_{-\infty}^{\infty} (u-y)^2\, f_{X(s)|X(s-ds)}(u \mid y)du \right) \frac{\partial^2}{\partial y^2} f_{X(t)|X(s)}(x \mid y) + \ldots \end{aligned}$$

Conditional on $X(s-ds) = y$

$$X(s) - X(s-ds) \sim N\Big(\mu(y,s)ds, \sigma^2(y,s)ds\Big)$$

The coefficients are

$$\int_{-\infty}^{\infty} f_{X(s)|X(s-ds)}(u \mid y)du = 1$$

$$\int_{-\infty}^{\infty} (u-y)\, f_{X(s)|X(s-ds)}(u \mid y)du = \mu(y,s)ds$$

$$\int_{-\infty}^{\infty} (u-y)^2\, f_{X(s)|X(s-ds)}(u \mid y)du = \sigma^2(y,s)ds + o(ds)$$

and $\int_{-\infty}^{\infty} (u-y)^n\, f_{X(s)|X(s-ds)}(u \mid y)du = o(ds)$ for $n > 2$. Hence, the series expansion can be written as

$$f_{X(t)|X(s-ds)}(x \mid y) = f_{X(t)|X(s)}(x \mid y) + \mu(y,s)\frac{\partial}{\partial y} f_{X(t)|X(s)}(x \mid y)ds + \tfrac{1}{2}\sigma^2(y,s)\frac{\partial^2}{\partial y^2} f_{X(t)|X(s)}(x \mid y)ds + o(ds)$$

Subtracting $f_{X(t)|X(s)}(x \mid y)$ from both sides and dividing through by *ds* gives

$$-\frac{f_{X(t)|X(s)}(x \mid y) - f_{X(t)|X(s-ds)}(x \mid y)}{ds} = \tfrac{1}{2}\sigma^2(y,s)\frac{\partial^2}{\partial y^2} f_{X(t)|X(s)}(x \mid y) + \mu(y,s)\frac{\partial}{\partial y} f_{X(t)|X(s)}(x \mid y) + \frac{o(ds)}{ds}$$

Taking the infinitesimal limit $ds \to 0$ so that $o(ds)/ds \to 0$ and the left-hand-side is the definition of a backward derivative in the backward time variable

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$$-\frac{\partial}{\partial s} f_{X(t)|X(s)}(x \mid y) = \tfrac{1}{2}\sigma^2(y,s)\frac{\partial^2}{\partial y^2} f_{X(t)|X(s)}(x \mid y) + \mu(y,s)\frac{\partial}{\partial y} f_{X(t)|X(s)}(x \mid y)$$

which is the backward equation which describes the dynamics of the deconvolution of the infinitesimal basis components from the back end of the diffusion as the backward variables (initial conditions) progress forward through time.

Converting Between the Forward and Backward Equations[7]

The Method

The forward and backward equations are adjoint, adjoints in the diffusion variable and also in the time variable. In terms of quantum mechanics, the two equations are said to be Hermitian conjugates. To obtain the adjoint to the forward equation (the backward equation), the forward equation is multiplied by the conjugate transition density function $f_{X(s)|X(t)}(x_0 \mid x)$ and integrated over the forward diffusion variable, from the backward to the forward diffusion variable $\int_{x_0}^{x}$, and then again over the forward time variable, from the backward to the forward time variable $\int_{t_0}^{t}$. Integrating by parts then interchanges the order of the transition density and its conjugate. Differentiating the integrals with respect to the forward variables leaves the backward equation and two null additive terms of the difference between two sides of the forward and backward conjugate equations. With some further trivial bookkeeping of canceling a common factor (the transition density function which has been transposed with the conjugate density) and interchanging the symbolic notation of the forward and backward, the backward equation then remains in the standard form.

The relevance of the conversion concerns the fact that option partial differential equations are backward equations. If one has the forward equation of the dynamics of the underlying asset, then this equation can be converted to the backward equation which can then further be transformed to the partial differential equation of the option premium and other related statistics.

The Mathematics

From the forward equation of a general first order diffusion

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)] - \frac{\partial}{\partial x}[\mu(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)]$$

Multiplying both sides by the conjugate of the transition density $f_{X(t_0)|X(t)}(x_0 \mid x)$,

$$f_{X(t_0)|X(t)}(x_0 \mid x)\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$= \tfrac{1}{2} f_{X(t_0)|X(t)}(x_0 \mid x)\frac{\partial^2}{\partial x^2}[\sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)] - f_{X(t_0)|X(t)}(x_0 \mid x)\frac{\partial}{\partial x}[\mu(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)]$$

integrating the front end of the diffusion by integrating over the forward variables gives

$$\int_{t_0}^{t}\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0 \mid y)\frac{\partial}{\partial s} f_{X(s)|X(t_0)}(y \mid x_0)\,dy\,ds$$
$$= \int_{t_0}^{t}\int_{x_0}^{x}\left(f_{X(t_0)|X(s)}(x_0 \mid y)\frac{\partial^2}{\partial y^2}[\sigma^2(y,s)\, f_{X(s)|X(t_0)}(y \mid x_0)] + f_{X(t_0)|X(s)}(x_0 \mid y)\frac{\partial}{\partial y}[\mu(y,s)\, f_{X(s)|X(t_0)}(y \mid x_0)]\right) dy\,ds$$

The terms to be integrated are

[7] Contribution of the author of this thesis.

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(1) $\int_{t_0}^{t}\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0\,|\,y)\frac{\partial}{\partial s} f_{X(s)|X(t_0)}(y\,|\,x_0)dyds$

(2) $\int_{t_0}^{t}\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0\,|\,y)\frac{\partial}{\partial y}[\mu(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)]dyds$

(3) $\int_{t_0}^{t}\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0\,|\,y)\frac{\partial^2}{\partial y^2}[\sigma^2(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)]dyds$

Only the integration with respect to the derivative are required and the remaining integration is removed by differentiation;

(1) $\int_{t_0}^{t} f_{X(t_0)|X(s)}(x_0\,|\,y)\frac{\partial}{\partial s} f_{X(s)|X(t_0)}(y\,|\,x_0)ds$

$= f_{X(t)|X(t_0)}(y\,|\,x_0) f_{X(t_0)|X(t)}(x_0\,|\,y) - f_{X(t_0)|X(t_0)}(y\,|\,x_0) f_{X(t_0)|X(t_0)}(x_0\,|\,y)$

$-\int_{t_0}^{t} f_{X(s)|X(t_0)}(y\,|\,x_0)\frac{\partial}{\partial s} f_{X(t_0)|X(s)}(x_0\,|\,y)ds$

by the parts

$u = f_{X(t_0)|X(s)}(x_0\,|\,y)$ $\qquad \frac{du}{ds} = \frac{\partial}{\partial s} f_{X(t_0)|X(s)}(x_0\,|\,y)$

$\frac{dv}{ds} = \frac{\partial}{\partial s} f_{X(s)|X(t_0)}(y\,|\,x_0)$ $\qquad v = f_{X(s)|X(t_0)}(y\,|\,x_0)$

(2) $\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0\,|\,y)\frac{\partial}{\partial y}[\mu(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)]dy$

$= \mu(x,s)\, f_{X(s)|X(t_0)}(x\,|\,x_0) f_{X(t_0)|X(s)}(x_0\,|\,x) - \mu(x_0,s)\, f_{X(s)|X(t_0)}(x_0\,|\,x_0) f_{X(t_0)|X(s)}(x_0\,|\,x_0)$

$-\int_{x_0}^{x} \mu(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)\frac{\partial}{\partial y} f_{X(t_0)|X(s)}(x_0\,|\,y)dy$

by the parts $u = f_{X(t_0)|X(s)}(x_0\,|\,y)$ and $\frac{du}{dy} = \frac{\partial}{\partial y} f_{X(t_0)|X(s)}(x_0\,|\,y)$

for $\frac{dv}{dy} = \frac{\partial}{\partial y}[\mu(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)]$ and $v = \mu(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)$

(3) $\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0\,|\,y)\frac{\partial^2}{\partial y^2}[\sigma^2(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)]dy$

$= f_{X(t_0)|X(s)}(x_0\,|\,x)\frac{\partial}{\partial x}[\sigma^2(x,s)\, f_{X(s)|X(t_0)}(x\,|\,x_0)] - f_{X(t_0)|X(s)}(x_0\,|\,x_0)\frac{\partial}{\partial x_0}[\sigma^2(x_0,s)\, f_{X(s)|X(t_0)}(x_0\,|\,x_0)]$

$-\sigma^2(x,s)\, f_{X(s)|X(t_0)}(x\,|\,x_0)\frac{\partial}{\partial x} f_{X(t_0)|X(s)}(x_0\,|\,x) + \sigma^2(x_0,s)\, f_{X(s)|X(t_0)}(x_0\,|\,x_0)\frac{\partial}{\partial x_0} f_{X(t_0)|X(s)}(x_0\,|\,x_0)$

$+\int_{x_0}^{x} \sigma^2(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)\frac{\partial^2}{\partial y^2} f_{X(t_0)|X(s)}(x_0\,|\,y)dy$

by the parts on the first application

$u = f_{X(t_0)|X(s)}(x_0\,|\,y)$ $\qquad \frac{du}{dy} = \frac{\partial}{\partial y} f_{X(t_0)|X(s)}(x_0\,|\,y)$

$\frac{dv}{dy} = \frac{\partial^2}{\partial y^2}[\sigma^2(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)]$ $\qquad v = \frac{\partial}{\partial y}[\sigma^2(y,s)\, f_{X(s)|X(t_0)}(y\,|\,x_0)]$

and the parts of the second application

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$$u = \frac{\partial}{\partial y} f_{X(t_0)|X(s)}(x_0 \mid y) \qquad \frac{du}{dy} = \frac{\partial^2}{\partial y^2} f_{X(t_0)|X(s)}(x_0 \mid y)$$

$$\frac{dv}{dy} = \frac{\partial}{\partial y}[\sigma^2(y,s)\, f_{X(s)|X(t_0)}(y \mid x_0)] \qquad v = \sigma^2(y,s)\, f_{X(s)|X(t_0)}(y \mid x_0)$$

Differentiating each of the double integrals with respect to each of the forward variables leaves the terms

(1) $$\frac{\partial^2}{\partial t \partial x}\int_{t_0}^{t}\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0 \mid y)\frac{\partial}{\partial s} f_{X(s)|X(t_0)}(y \mid x_0)\,dyds$$

$$= \frac{\partial}{\partial t}\left(f_{X(t)|X(t_0)}(x \mid x_0) f_{X(t_0)|X(t)}(x_0 \mid x)\right) - f_{X(t)|X(t_0)}(x \mid x_0)\frac{\partial}{\partial t} f_{X(t_0)|X(t)}(x_0 \mid x)$$

(2) $$\frac{\partial^2}{\partial t \partial x}\int_{t_0}^{t}\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0 \mid y)\frac{\partial}{\partial y}[\mu(y,s)\, f_{X(s)|X(t_0)}(y \mid x_0)]\,dyds$$

$$= \frac{\partial}{\partial x}\left(\mu(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0) f_{X(t_0)|X(t)}(x_0 \mid x)\right) - \mu(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)\frac{\partial}{\partial x} f_{X(t_0)|X(t)}(x_0 \mid x)$$

(3) $$\frac{\partial^2}{\partial t \partial x}\int_{t_0}^{t}\int_{x_0}^{x} f_{X(t_0)|X(s)}(x_0 \mid y)\frac{\partial^2}{\partial y^2}[\sigma^2(y,s)\, f_{X(s)|X(t_0)}(y \mid x_0)]\,dyds$$

$$= \frac{\partial}{\partial x}\left(f_{X(t_0)|X(t)}(x_0 \mid x)\frac{\partial}{\partial x}[\sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)] - \sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)\frac{\partial}{\partial x} f_{X(t_0)|X(t)}(x_0 \mid x)\right)$$

$$+ \sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)\frac{\partial^2}{\partial x^2} f_{X(t_0)|X(t)}(x_0 \mid x)$$

Considering only the derivatives that remain in the expression following the integration by parts and subsequent differentiation,

$$\frac{\partial}{\partial t}\left(f_{X(t)|X(t_0)}(x \mid x_0) f_{X(t_0)|X(t)}(x_0 \mid x)\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial x}\left(f_{X(t_0)|X(t)}(x_0 \mid x)\frac{\partial}{\partial x}[\sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)] - \sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)\frac{\partial}{\partial x} f_{X(t_0)|X(t)}(x_0 \mid x)\right)$$

$$- \frac{\partial}{\partial x}\left(\mu(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0) f_{X(t_0)|X(t)}(x_0 \mid x)\right)$$

or

$$f_{X(t)|X(t_0)}(x \mid x_0)\frac{\partial}{\partial t} f_{X(t_0)|X(t)}(x_0 \mid x) + f_{X(t_0)|X(t)}(x_0 \mid x)\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0)$$

$$= \frac{1}{2} f_{X(t_0)|X(t)}(x_0 \mid x)\frac{\partial^2}{\partial x^2}[\sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)] - \frac{1}{2}\sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)\frac{\partial^2}{\partial x^2} f_{X(t_0)|X(t)}(x_0 \mid x)$$

$$- \mu(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)\frac{\partial}{\partial x} f_{X(t_0)|X(t)}(x_0 \mid x) - f_{X(t_0)|X(t)}(x_0 \mid x)\frac{\partial}{\partial x}[\mu(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)]$$

or

$$0 = f_{X(t)|X(t_0)}(x \mid x_0)\left(\frac{\partial}{\partial t} f_{X(t_0)|X(t)}(x_0 \mid x) + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2} f_{X(t_0)|X(t)}(x_0 \mid x) + \mu(x,t)\frac{\partial}{\partial x} f_{X(t_0)|X(t)}(x_0 \mid x)\right)$$ RHS–LHS of backward equation

$$+ f_{X(t_0)|X(t)}(x_0 \mid x)\left(\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) - \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)] + \frac{\partial}{\partial x}[\mu(x,t)\, f_{X(t)|X(t_0)}(x \mid x_0)]\right)$$ RHS–LHS of forward equation

which is identically zero, the two terms on the right containing factors equal to the difference between the time and diffusion sides of the backward and forward equations respectively. The roles of the symbols of the forward and backward variables are interchanged for the backward equation. These are only function arguments. Hence,

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both are zero and the equation in the non-integral terms of the various applications of integration by parts is identically zero. The equation reduces to

$$-f_{X(t)|X(t_0)}(x\,|\,x_0)\frac{\partial}{\partial t}f_{X(t_0)|X(t)}(x_0\,|\,x)$$
$$=\frac{1}{2}\sigma^2(x,t)\,f_{X(t)|X(t_0)}(x\,|\,x_0)\frac{\partial^2}{\partial x^2}f_{X(t_0)|X(t)}(x_0\,|\,x)+\mu(x,t)\,f_{X(t)|X(t_0)}(x\,|\,x_0)\frac{\partial}{\partial x}f_{X(t_0)|X(t)}(x_0\,|\,x)$$

The position of the transition density and the conjugate density have hence been transposed. Canceling the factor of the transition density on both sides of the equation leaves

$$-\frac{\partial}{\partial t}f_{X(t_0)|X(t)}(x_0\,|\,x)=\frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2}f_{X(t_0)|X(t)}(x_0\,|\,x)+\mu(x,t)\frac{\partial}{\partial x}f_{X(t_0)|X(t)}(x_0\,|\,x)$$

which is the backward equation in the conjugate density. Further interchanging the symbols of the original forward and backward variables in the conjugate density leaves the backward equation in the standard notation,

$$-\frac{\partial}{\partial t_0}f_{X(t)|X(t_0)}(x\,|\,x_0)=\frac{1}{2}\sigma^2(x_0,t_0)\frac{\partial^2}{\partial x_0^2}f_{X(t)|X(t_0)}(x\,|\,x_0)+\mu(x_0,t_0)\frac{\partial}{\partial x_0}f_{X(t)|X(t_0)}(x\,|\,x_0)$$

This technique can similarly be used to obtain the forward equation from the backward equation. The technique proceeds by multiplying the backward equation by the conjugate transition density, integrating by parts over the backward variables (the backward diffusion variable and the backward time variable) up to the forward variables, and subsequently differentiating with respect to the backward variables. Canceling the common factor in the transition density and interchanging the symbols of the forward and backward variables leaves the forward equation in the standard notation.

This technique can be used to obtain the backward equation from the forward equation, or visa-versa, for any first order diffusion with a normal basis (defined as has having no cumulants higher than the second), as well as diffusion with basis functions with generalized Gaussian distributions that have non-zero cumulants at orders greater than two, such as the generalized heat equation. Skew requires three applications of parts and kurtosis requires four. The mathematics is not displayed in this thesis due to the bulk of the mathematical working. The principles are identical to those in this chapter and also apply to the forward and backward equations of diffusion variables of higher order which is discussed again in BOOK 3.

Solution to the Forward Equation

The forward equation is solved in terms of an initial condition and the backward equation in terms of boundary conditions. The initial condition occurs at the backward variables of the transition interval and the boundary conditions occur at the forward variables of the transition interval. Emphasis is placed on solving the forward equation since it is this equation that relates to the forward propagation of the statistical dynamics as described by the forward transition density. The initial condition common to all forward equations of diffusion variables is a Dirac singularity. A Dirac singularity is defined as $\int \delta_y(x)\,dx = 1$, and in the context of probability and the transition distribution, this is an infinitesimal uniform distribution that has infinitesimal width dt and infinite height $1/dt$ such that the area sums to one. The Dirac singularity of the transition density is located at the point where the value of the forward and backward variables coincide y, $\delta_y(x)$. That is,

$$\lim_{t\to s}\big(X(t)\,|\,X(s)=y\big)\sim\lim_{\varepsilon\to 0}U(y-\varepsilon,y+\varepsilon)$$
$$\sim\delta_y$$

The moment generating function of the Dirac singularity δ_y is

$$\lim_{t\to s}M_{X(t)|X(s)=y}(\theta)\;=\int_{-\infty}^{\infty}e^{\theta x}\,\delta_y(x)\,dx=e^{\theta y}$$

by the definition of a Dirac delta function. This is the required function for the intial condition in solving the forward equation by means of transforming to the transition moment generating function. When the basis variance zeroes, $\sigma\to 0$, the transition variance zeroes and the transition surface becomes a Dirac singularity traced out over the transition mean. This thesis refers to this function, for obvious reasons, as a Dirac trace.

There are three alternative methods of solving a forward equation. These are

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1) transformation to the standard heat equation
2) technique of moment generating functions
3) identification of an ito-transform pair (or convolution pair) by reference to the infinitesimal parameters

Consideration is given to each of these three methods. The most rudimentary case of the forward equation, the standard heat equation $f_t = \frac{1}{2} f_{xx}$ is relatively simple to solve since standard Brownian motion is a process of linear summation of a single cumulant, namely, the second. Transformations of the diffusion variable can often reduce a given equation to the standard heat equation from which the random variable and the transition density of the process can be obtained by the inverse transformation from standard Brownian motion, i.e., an inverse transformation from a normal density with variance t. This is equivalent to an ito transform pair. For the processes considered in this part of BOOK 1, it is possible to transform the forward equation to the standard heat equation whenever the diffusion is a transformation of ordinary diffusion. It is not possible for a convolved set of transformed ordinary diffusions.

The forward equation is solved by the technique of moment generating functions for each of the processes in FOUNDATIONS I, whether or not they are transformable to the standard heat equation. The technique based upon transforming to the moment generating function converts the forward partial differential equation in the transition density function to the forward equation in the transition moment generating function. The resulting equation is solved for the moment generating function, and then by the uniqueness between a distribution and its moment generating function (the Polya criterion[8]), the transition density function is surmised. One could alternatively obtain the density by inverting the moment generating function along the Bromwhich contour or with an inverse Fourier transform of the MGF with the parameter made complex. The technique of transforming to, and subsequently solving, the equation in the moment generating function, rather than the density, transforms the derivatives and powers in the diffusion variable into multiplications and derivatives in the moment generating function parameter. This is the basis for Bartlett's formula. The key results of this conversion, stated in words are as follows;

(1) The moment generating function of derivatives of the transition density with respect to the forward diffusion variable results in the moment generating function times a power of the moment generating function parameter, the power equal to the order of the derivative, with alternating sign;
(2) The moment generating function of multiplications of powers of the forward diffusion variable and the transition density function result in derivatives of the moment generating function with respect to the moment generating function parameter, the order of the derivatives equal to the power of the forward diffusion variable.

These two conversions can be combined and are used repeatedly throughout FOUNDATIONS I as well as the remaining books of this thesis. The technique of solving the forward equations by transforming to the transition moment generating function is discussed again in detail in BOOK 3 and generalized for pure diffusion of arbitrary order as an extension to that of BOOK 1. The reader is reminded that a moment generating function M is a characteristic function ϕ with a complex parameter. This thesis avoids unnecessary or redundant complexity for which a characteristic function has if all the moments of a variable are real in which case the MGF is sufficient and algebraically parsimonious.

Transforming Derivatives in the Transition Density to Functions of the MGF and the MGF parameter

Differential and partial differential equations are formed by differentiation and solved by integration. For this reason, it is felt by this thesis that the basis of the transformations used to transform the forward equation in the density function to that in the MGF

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^n}{\partial x^n}\left(x^m f_{X(t)}(x)\right) dx = (-\theta)^n \frac{\partial^m}{\partial \theta^m} M_{X(t)}(\theta)$$

be kept consistent with the task for which they are utilized, namely, differentiation (by the product rule) for forming the equations and integration (by parts) for solving them. This is consistent with what the forward and backward equations describe mathematically; the rate and manner by which density functions and moments of

[8] Grimmet, G. and Stirzaker, D., (2001) *One Thousand Exercises in Probability.* Oxford University Press.

note that, together with this thesis, this sort of book is the sort of training I am envisaging for a derivatives actuarial practice area.

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the basis convolve onto the diffusion variable. Since formation of the forward equations is considered before it is solved, the approach by differentiation is considered first and that by integration is considered second. As a refresher

(1) $Du(x)v(x) = u(x)Dv(x) + v(x)Du(x)$

(2) $D^{-1}u(x)Dv(x) = u(x)v(x) - D^{-1}v(x)Du(x)$

are the product rule and parts respectively. These are inverse algorithms.

A. Transformations by Differentiation

The transition moment generating function $M_{X(t)}(\theta)$ is a function that is independent of the forward diffusion variable x, since it has been integrated out expressing the distribution in terms of moments rather than probability; hence the MGF parameter θ features rather than the forward diffusion variable x. Therefore the derivative with respect to the forward diffusion variable is zero

$$\frac{\partial}{\partial x} M_{X(t)}(\theta) = 0$$

i. MGF of n^{th} Order Curvature of Transition Density Function by Differentiation

The definition of the transition MGF is

$$M_{X(t)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f_{X(t)}(x)dx$$

Differentiating the MGF with respect to the forward diffusion variable

$$\begin{aligned}
\frac{\partial}{\partial x} M_{X(t)}(\theta) &= \frac{\partial}{\partial x} \int_{-\infty}^{\infty} e^{\theta x} f_{X(t)}(x)dx \\
&= \theta \int_{-\infty}^{\infty} e^{\theta x} f_{X(t)}(x)dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(t)}(x)dx \\
&= \theta M_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(t)}(x)dx \\
&= 0
\end{aligned}$$

hence

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(t)}(x)dx = -\theta M_{X(t)}(\theta)$$

Differentiating the MGF again with respect to the forward diffusion variable

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} M_{X(t)}(\theta) &= \frac{\partial}{\partial x} \left(\theta M_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(t)}(x)dx \right) \\
&= \theta \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(t)}(x)dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(t)}(x)dx \\
&= -\theta^2 M_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(t)}(x)dx \\
&= 0
\end{aligned}$$

hence

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(t)}(x)dx = \theta^2 M_{X(t)}(\theta)$$

Similarly, differentiating a third time with respect to the forward diffusion variable,

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$$\frac{\partial^3}{\partial x^3} M_{X(t)}(\theta) = \frac{\partial}{\partial x}\left(-\theta^2 M_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(t)}(x)dx \right)$$
$$= \theta \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(t)}(x)dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^3}{\partial x^3} f_{X(t)}(x)dx$$
$$= \theta^3 M_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^3}{\partial x^3} f_{X(t)}(x)dx$$
$$= 0$$

gives

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^3}{\partial x^3} f_{X(t)}(x)dx = -\theta^3 M_{X(t)}(\theta)$$

Continuing with *n* differentiations, $\frac{\partial^n}{\partial x^n} M_{X(t)}(\theta) = 0$ gives

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^n}{\partial x^n} f_{X(t)}(x)dx = (-\theta)^n M_{X(t)}(\theta)$$

and is the technique consistent with the formation of the forward and backward equations.

ii. MGF of m^{th} Order Weighted Transition Density Function by Differentiation

Differentiating the MGF with respect to the MGF parameter *m* times

$$\frac{\partial^m}{\partial \theta^m} M_{X(t)}(\theta) = \frac{\partial^m}{\partial \theta^m} \int_{-\infty}^{\infty} e^{\theta x} f_{X(t)}(x)dx$$
$$= \int_{-\infty}^{\infty} e^{\theta x} x^m f_{X(t)}(x)dx$$

iii. MGF of n^{th} Order Curvature of m^{th} Order Weighted Transition Density Function by Differentiation

From the m^{th} order partial derivative of the MGF with respect to the MGF parameter,

$$\frac{\partial^m}{\partial \theta^m} M_{X(t)}(\theta) = M^{(m)}_{X(t)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} x^m f_{X(t)}(x)dx$$

differentiating with respect to the forward diffusion variable

$$\frac{\partial^{m+1}}{\partial x \partial \theta^m} M_{X(t)}(\theta) = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} e^{\theta x} x^m f_{X(t)}(x)dx$$
$$= \theta \int_{-\infty}^{\infty} e^{\theta x} x^m f_{X(t)}(x)dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x^m f_{X(t)}(x)dx$$
$$= \theta M^{(m)}_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x^m f_{X(t)}(x)dx$$
$$= 0$$

hence

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x^m f_{X(t)}(x)dx = -\theta M^{(m)}_{X(t)}(\theta)$$

Differentiating again with respect to the forward diffusion variable

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$$\frac{\partial^{m+2}}{\partial x^2 \partial \theta^m} M_{X(t)}(\theta) = \frac{\partial}{\partial x}\left(\theta M^{(m)}_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x^m f_{X(t)}(x) dx \right)$$
$$= \theta \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x^m f_{X(t)}(x) dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} x^m f_{X(t)}(x) dx$$
$$= -\theta^2 M^{(m)}_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} x^m f_{X(t)}(x) dx$$
$$= 0$$

hence

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} x^m f_{X(t)}(x) dx = \theta^2 M^{(m)}_{X(t)}(\theta)$$

Similarly, differentiating a third time with respect to the forward diffusion variable,

$$\frac{\partial^{m+3}}{\partial x^3 \partial \theta^m} M_{X(t)}(\theta) = \frac{\partial}{\partial x}\left(-\theta^2 M^{(m)}_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} x^m f_{X(t)}(x) dx \right)$$
$$= \theta \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} x^m f_{X(t)}(x) dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^3}{\partial x^3} x^m f_{X(t)}(x) dx$$
$$= \theta^3 M^{(m)}_{X(t)}(\theta) + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^3}{\partial x^3} x^m f_{X(t)}(x) dx$$
$$= 0$$

gives

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^3}{\partial x^3} x^m f_{X(t)}(x) dx = \theta^3 M^{(m)}_{X(t)}(\theta)$$

Continuing with *n* differentiations, $\frac{\partial^{m+n}}{\partial x^n \partial \theta^m} M_{X(t)}(\theta) = 0$ gives

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^n}{\partial x^n} x^m f_{X(t)}(x) dx = (-\theta)^n M^{(m)}_{X(t)}(\theta)$$

Again, this is the technique that is consistent with the formation of the forward and backward equations. These results are now obtained with integration by parts rather than by differentiation with the product rule. It is deemed best by this thesis to differentiate the MGF in relation to forming the equations and to use integration for solving them. Hence, for solving the equations, the transformation should be obtained by parts which is presented now.

B. Transformations by Integration

i. MGF of n^{th} Order Curvature of Transition Density Function by Integration

$$I_{n,0}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^n}{\partial x^n} f_{X(t)}(x) dx \qquad \text{where } I_{0,0}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \left(\frac{\partial}{\partial x} \right)^0 f_{X(t)}(x) dx = M_{X(t)}(\theta)$$

Integration by parts with $u = e^{\theta x}$ and $dv = \frac{\partial^n}{\partial x^n} f_{X(t)}(x) dx$ for $du = \theta e^{\theta x}$ and $v = \frac{\partial^{n-1}}{\partial x^{n-1}} f_{X(t)}(x)$ gives

$$I_{n,0}(\theta) = \left[e^{\theta x} \frac{\partial^n}{\partial x^n} f_{X(t)}(x) \right]_{x=-\infty}^{x=\infty} - \theta \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^{n-1}}{\partial x^{n-1}} f_{X(t)}(x) dx$$

and, noting the condition $f_{X(t)}(\pm\infty) \to 0$, results in the recursion

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$$I_{n,0}(\theta) = (-\theta) I_{n-1,0}(\theta) \qquad \text{subject to } I_{0,0}(\theta) = M_{X(t)}(\theta)$$

Solving

$$I_{n,0}(\theta) = (-\theta) I_{n-1,0}(\theta) = (-\theta)^2 I_{n-2,0}(\theta) = (-\theta)^3 I_{n-3,0}(\theta) = ... = (-\theta)^{n-1} I_{0,1}(\theta) = (-\theta)^n I_{0,0}(\theta)$$

that is,

$$I_{n,0}(\theta) = (-\theta)^n M_{X(t)}(\theta)$$

ii. MGF of *m*th Order Weighted Transition Density Function by Integration

$$I_{0,m}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} x^m f_{X(t)}(x) dx \qquad \text{where } I_{0,0}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} x^0 f_{X(t)}(x) dx = M_{X(t)}(\theta)$$

Noting the identity of the fundamental theorem of calculus $DD^{-1} = 1$,

$$I_{0,m}(\theta) = \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial} e^{\theta x} x^m f_{X(t)}(x) dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta} \left(\int e^{\theta x} x^m d\theta \right) f_{X(t)}(x) dx = \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} e^{\theta x} x^{m-1} f_{X(t)}(x) dx$$

which results in the recursion

$$I_{0,m}(\theta) = \frac{\partial}{\partial \theta} I_{0,m-1}(\theta) \qquad \text{subject to } I_{0,0}(\theta) = M_{X(t)}(\theta)$$

Solving

$$I_{0,m}(\theta) = \frac{\partial}{\partial \theta} I_{0,m-1}(\theta) = \left(\frac{\partial}{\partial \theta} \right)^2 I_{0,m-2}(\theta) = \left(\frac{\partial}{\partial \theta} \right)^3 I_{0,m-3}(\theta) = \ldots = \left(\frac{\partial}{\partial \theta} \right)^{m-1} I_{0,1}(\theta) = \left(\frac{\partial}{\partial \theta} \right)^m I_{0,0}(\theta)$$

that is,

$$I_{0,m}(\theta) = \left(\frac{\partial}{\partial \theta} \right)^m M_{X(t)}(\theta)$$

iii. MGF of *n*th Order Curvature of *m*th Order Weighted Transition Density Function by Integration

$$I_{n,m}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^n}{\partial x^n} \left(x^m f_{X(t)}(x) \right) dx \text{ where } I_{0,0}(\theta) = M_{X(t)}(\theta)$$

By parts,

$$I_{n,m}(\theta) = (-\theta) I_{n-1,m}(\theta)$$

solves in *n* for

$$I_{n,m}(\theta) = (-\theta)^n I_{0,m}(\theta)$$

and directly by the previous result,

$$I_{n,m}(\theta) = (-\theta)^n \frac{\partial}{\partial \theta} I_{0,m-1}(\theta)$$

which solves in *m* for

$$I_{n,m}(\theta) = (-\theta)^n \left(\frac{\partial}{\partial \theta} \right)^m I_{0,0}(\theta)$$

that is,

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$$I_{n,m}(\theta) = (-\theta)^n \left(\frac{\partial}{\partial \theta} \right)^m M_{X(t)}(\theta)$$

■

Following solving the forward equation, consideration is given to that of the stationary (steady state) distribution for diffusion that is recurrent. When a diffusion reaches the distribution of the steady state, the time dependence of the forward and backward equations vanish leaving an ordinary differential equation in diffusion derivatives. The solution is then obtained by two applications of direct integration. As one may well appreciate, solving the forward equation for the steady state transition density function is a much simplified task since it is an ordinary differential equation rather than a partial differential equation. In the cases of Brownian motion and geometric Brownian motion, which have unit characteristic roots, the large t transition distributions are infinite uniforms which standardize to a normal bell with variance proportional to t regardless of the magnitude of time and hence retain their statistical properties. That is, a unit-root does not vanish over any timeframe and has the same statistical properties anywhere from $dt \to 0$ to $1 / dt \to \infty$ (inclusive). This is because unit roots are persistent. Unit root diffusion is also statistically fractal. Brownian motion, which has a unit-root, is the dividing line between recurrence and transience of first order diffusion and is at the cross-road – persistence is the intermittent dividing line between recurrence and transience. The literature has referred to Brownian motion as being null-recurrent. The statistical properties of the transition distribution do not change over any time horizon. The recurrent forms of the Ornstein-Uhlenbeck process, the geometric Ornstein-Uhlenbeck process, the square-root processes, and the Bessel process each possess stationary limiting distributions that are time invariant. The stationary time invariant distributions are normal, log-normal, gamma, and log-gamma corresponding to continuous processes of normal, log-normal, gamma, and log-gamma basis functions which form a variants of continuous Hilbert space. So long as a diffusion is recurrent (not persistent or transient) and consequently mean reverting, there exists a long-run time-invariant distribution.

As a summary of the processes to be covered and to state concisely the mathematical definitions in terms of the defining features, and to present the mathematical results in advance, the following set of tables are presented;

1) the stochastic differential and infinitesimal parameters;
2) the integrating factor of the diffusion sides of the forward and backward equations;
3) the forward equation;
4) the backward equation;
5) the transition distribution;
6) the transition density function;
7) the transition moment generating function;

The main results of this part of BOOK 1, FOUNDATIONS I, involve those that regard solving the forward equations for the transition distribution and density functions that are of a known statistical form. These results are given in the fifth and sixth tables. This should make the purpose of the mathematics and mathematical working clear and give a frame of reference for the reader.

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Tables; First Order Continuous Diffusion

Table 1: Differential increment and infinitesimal parameters

Diffusion	**Differential increment** $dX(t) \mid X(t) = x$ **(abbreviated by** dX_t**)** $dX_t = \mu(x,t)dt + \sigma(x,t)dB_t$	**Infinitesimal mean** $\mu(x, t)$	**Infinitesimal std. deviation** $\sigma(x, t)$
Standard Brownian motion	$dX_t = dB_t$	0	1
Brownian motion	$dX_t = \mu dt + \sigma dB_t$	μ	σ
Geometric Brownian motion	$dX_t = \mu X_t dt + \sigma X_t dB_t$	μX_t	σX_t
Standard Ornstein-Uhlenbeck process	$dX_t = -\alpha X_t dt + \sigma dB_t$	$-\alpha X_t$	σ
Ornstein-Uhlenbeck process	$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$	$\alpha(\mu - X_t)$	σ
Geometric Ornstein-Uhlenbeck process	$dX_t = \alpha(\mu - \log X_t)X_t dt + \sigma X_t dB_t$	$\alpha(\mu - \log X_t)X_t$	σX_t
Eccentric square-root	$dX_t = -\alpha X_t dt + \sigma\sqrt{X_t}\, dB_t$	$-\alpha X_t$	$\sigma\sqrt{X_t}$
Three-parameter square-root	$dX_t = \alpha(\mu - X_t)dt + \sigma\sqrt{X_t}\, dB_t$	$\alpha(\mu - X_t)$	$\sigma\sqrt{X_t}$
Radial Ornstein-Uhlenbeck	$dX_t = \alpha\left(\frac{\mu}{X_t} - X_t\right)dt + \sigma dB_t$	$\alpha\left(\frac{\mu}{X_t} - X_t\right)$	σ

Table 2: Kernel of speed density and distribution implied by kernel

Diffusion	**Differential increment** $dX(t) \mid X(t) = x$ **(abbreviated by** dX_t**)** $dX_t = \mu(x,t)dt + \sigma(x,t)dB_t$	**Kernel of speed density**	**Distribution of kernel**
Standard Brownian motion	$dX_t = dB_t$	N/A	
Brownian motion	$dX_t = \mu dt + \sigma dB_t$	$e^{\mu x / \frac{1}{2}\sigma^2}$	
Geometric Brownian motion	$dX_t = \mu X_t dt + \sigma X_t dB_t$	$x^{\mu / \frac{1}{2}\sigma^2}$	
Standard Ornstein-Uhlenbeck process	$dX_t = -\alpha X_t dt + \sigma dB_t$	$e^{-\alpha x^2 / \sigma^2}$	Normal
Ornstein-Uhlenbeck process	$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$	$e^{-\frac{1}{2}\alpha(x^2 - 2\mu x)/\frac{1}{2}\sigma^2}$	Normal
Geometric Ornstein-Uhlenbeck process	$dX_t = \alpha(\mu - \log X_t)X_t dt + \sigma X_t dB_t$	$e^{-\frac{1}{2}\alpha(\log^2 x - 2\mu \log x)/\frac{1}{2}\sigma^2}$	Log-normal
Eccentric square-root	$dX_t = -\alpha X_t dt + \sigma\sqrt{X_t}\, dB_t$	$e^{-\alpha x / \frac{1}{2}\sigma^2}$	Exponential
Three-parameter square-root	$dX_t = \alpha(\mu - X_t)dt + \sigma\sqrt{X_t}\, dB_t$	$x^{\alpha\mu/\frac{1}{2}\sigma^2} e^{-\alpha x / \frac{1}{2}\sigma^2}$	Gamma (scaled chi-square)
Radial Ornstein-Uhlenbeck	$dX_t = \alpha\left(\frac{\mu}{X_t} - X_t\right)dt + \sigma dB_t$	$x^{2\alpha\mu/\sigma^2} e^{-\alpha x^2/\sigma^2}$	Scaled chi (including Maxwell and Rayleigh)

Table 3: Forward equations of first order Gaussian diffusion

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Diffusion	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,t) f_{X(t)\|X(t_0)}(x\|x_0)\right) - \frac{\partial}{\partial x}\left(\mu(x,t)\, f_{X(t)\|X(t_0)}(x\|x_0)\right)$
Standard Brownian motion	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f_{X(t)\|X(t_0)}(x\|x_0)$
Brownian motion	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} f_{X(t)\|X(t_0)}(x\|x_0) - \mu\frac{\partial}{\partial x} f_{X(t)\|X(t_0)}(x\|x_0)$
Geometric Brownian motion	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\left(x^2 f_{X(t)\|X(t_0)}(x\|x_0)\right) - \mu\frac{\partial}{\partial x}\left(x\, f_{X(t)\|X(t_0)}(x\|x_0)\right)$
Standard Ornstein-Uhlenbeck process	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} f_{X(t)\|X(t_0)}(x\|x_0) + \alpha\frac{\partial}{\partial x}\left(x\, f_{X(t)\|X(t_0)}(x\|x_0)\right)$
Ornstein-Uhlenbeck process	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} f_{X(t)\|X(t_0)}(x\|x_0) - \alpha\frac{\partial}{\partial x}\left((\mu-x)\, f_{X(t)\|X(t_0)}(x\|x_0)\right)$
Geometric Ornstein-Uhlenbeck process	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\left(x^2 f_{X(t)\|X(t_0)}(x\|x_0)\right) - \alpha\frac{\partial}{\partial x}\left((\mu-\log x)\, x\, f_{X(t)\|X(t_0)}(x\|x_0)\right)$
Eccentric square-root	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)\|X(t_0)}(x\|x_0)\right) + \alpha\frac{\partial}{\partial x}\left(x\, f_{X(t)\|X(t_0)}(x\|x_0)\right)$
Three-parameter square-root	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)\|X(t_0)}(x\|x_0)\right) - \alpha\frac{\partial}{\partial x}\left((\mu-x)\, f_{X(t)\|X(t_0)}(x\|x_0)\right)$
Radial Ornstein-Uhlenbeck	$\frac{\partial}{\partial t} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} f_{X(t)\|X(t_0)}(x\|x_0) - \alpha\frac{\partial}{\partial x}\left(\left(\frac{\mu}{x}-x\right) f_{X(t)\|X(t_0)}(x\|x_0)\right)$

Table 4: Backward equations of first order Gaussian diffusion

Diffusion	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2(x_0,t_0)\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) + \mu(x_0,t_0)\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$
Standard Brownian motion	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0)$
Brownian motion	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) + \mu\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$
Geometric Brownian motion	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2 x_0^2\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) + \mu x_0\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$
Standard Ornstein-Uhlenbeck process	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) - \alpha x_0\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$
Ornstein-Uhlenbeck process	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) + \alpha(\mu-x_0)\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$
Geometric Ornstein-Uhlenbeck process	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2 x_0^2\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) + \alpha(\mu-\log x_0)x_0\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$
Eccentric square-root	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2 x_0\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) - \alpha x_0\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$
Three-parameter square-root	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2 x_0\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) + \alpha(\mu-x_0)\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$
Radial Ornstein-Uhlenbeck	$-\frac{\partial}{\partial t_0} f_{X(t)\|X(t_0)}(x\|x_0) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x_0^2} f_{X(t)\|X(t_0)}(x\|x_0) + \alpha\left(\frac{\mu}{x_0}-x_0\right)\frac{\partial}{\partial x_0} f_{X(t)\|X(t_0)}(x\|x_0)$

Table 5: Backward equation of put and call option premia

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Diffusion	Equation of option premium
	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2(x_0,t_0)\frac{\partial^2 V}{\partial x_0^2} + \mu(x_0,t_0)\frac{\partial V}{\partial x_0}$
Standard Brownian motion	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\frac{\partial^2 V}{\partial x_0^2}$
Brownian motion	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x_0^2} + \mu\frac{\partial V}{\partial x_0}$
Geometric Brownian motion	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2 x_0^2\frac{\partial^2 V}{\partial x_0^2} + \mu x_0\frac{\partial V}{\partial x_0}$
Standard Ornstein-Uhlenbeck process	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x_0^2} - \alpha x_0\frac{\partial V}{\partial x_0}$
Ornstein-Uhlenbeck process	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x_0^2} + \alpha(\mu - x_0)\frac{\partial V}{\partial x_0}$
Geometric Ornstein-Uhlenbeck process	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2 x_0^2\frac{\partial^2 V}{\partial x_0^2} + \alpha(\mu - \log x_0)x_0\frac{\partial V}{\partial x_0}$
Eccentric square-root	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2 x_0\frac{\partial^2 V}{\partial x_0^2} - \alpha x_0\frac{\partial V}{\partial x_0}$
Three-parameter square-root	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2 x_0\frac{\partial^2 V}{\partial x_0^2} + \alpha(\mu - x_0)\frac{\partial V}{\partial x_0}$
Radial Ornstein-Uhlenbeck	$rV - \frac{\partial V}{\partial t_0} = \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x_0^2} + \alpha\left(\frac{\mu}{x_0} - x_0\right)\frac{\partial V}{\partial x_0}$

Table 6: Transition distributions of first order Gaussian diffusion

Diffusion	Differential increment $dX(t) \mid X(t) = x$ (abbreviated by dX_t)	Transition distribution $X(t) \mid X(0) = x_0$ (abbreviated by X_t)
Standard Brownian motion	$dX_t = dB_t$	$X_t \sim N(x_0, t)$
Brownian motion	$dX_t = \mu dt + \sigma dB_t$	$X_t \sim N(x_0 + \mu t, \sigma^2 t)$
Geometric Brownian motion	$dX_t = \mu X_t dt + \sigma X_t dB_t$	$X_t \sim \log N\left(\log x_0 + (\mu - \frac{1}{2}\sigma^2)t,\ \sigma^2 t\right)$
Standard Ornstein-Uhlenbeck	$dX_t = -\alpha X_t dt + \sigma dB_t$	$X_t \sim N\left\{x_0 e^{-\alpha t}, \sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\right\}$
Ornstein-Uhlenbeck	$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$	$X_t \sim N\left\{x_0 e^{-\alpha t} + \alpha\mu\left(\frac{1-e^{-\alpha t}}{\alpha}\right), \sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\right\}$
Geometric Ornstein-Uhlenbeck	$dX_t = \alpha(\mu - \log X_t)X_t dt + \sigma X_t dB_t$	$X_s \sim \log N\left\{e^{-\alpha t}\log x_0 + (\alpha\mu - \frac{1}{2}\sigma^2)\left(\frac{1-e^{-\alpha t}}{\alpha}\right),\ \sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\right\}$
Eccentric square-root	$dX_t = -\alpha X_t dt + \sigma\sqrt{X_t}\, dB_t$	$X_t \sim k\,\chi^2_0(x_0 e^{-\alpha t}/k)$ where $k = \frac{\sigma^2}{4\alpha}(1-e^{-\alpha t})$
Three-parameter square-root	$dX_t = \alpha(\mu - X_t)dt + \sigma\sqrt{X_t}\, dB_t$	$X_t \sim k\,\chi^2_{4\alpha\mu/\sigma^2}(x_0 e^{-\alpha t}/k)$ where $k = \frac{\sigma^2}{4\alpha}(1-e^{-\alpha t})$

Table 7: Transition density functions of first order Gaussian diffusion

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Diffusion	Density function of transition distribution $f_{X(t)\|X(0)}(x \mid x_0)$
Standard Brownian motion	$f_{X(t)\|X(0)}(x \mid x_0) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-x_0)^2}{2t} \right)$
Brownian motion	$f_{X(t)\|X(0)}(x \mid x_0) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{[x-x_0-\mu t]^2}{2\sigma^2 t} \right)$
Geometric Brownian motion	$f_{X(t)\|X(0)}(x \mid x_0) = \frac{1}{\sigma x\sqrt{2\pi t}} \exp\left(-\frac{[\log x - \log x_0 - (\mu - \sigma^2/2)t]^2}{2\sigma^2 t} \right)$
Standard Ornstein-Uhlenbeck process	$f_{X(t)\|X(0)}(x \mid x_0) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{(1-e^{-2\alpha t})/2\alpha}} \exp\left(-\frac{[x-x_0 e^{-\alpha t}]^2}{\sigma(1-e^{-2\alpha t})/\alpha} \right)$
Ornstein-Uhlenbeck process	$f_{X(t)\|X(0)}(x \mid x_0) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{(1-e^{-2\alpha t})/2\alpha}} \exp\left(-\frac{[x-x_0 e^{-\alpha t} - \mu(1-e^{-\alpha t})]^2}{\sigma^2(1-e^{-2\alpha t})/\alpha} \right)$
Geometric Ornstein-Uhlenbeck process	$f_{X(t)\|X(0)}(x \mid x_0) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{(1-e^{-2\alpha t})/2\alpha}} \exp\left(-\frac{[\log x - \log x_0 e^{-\alpha t} - (\mu - \frac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t})]^2}{\sigma^2(1-e^{-2\alpha t})/\alpha} \right)$
Eccentric square-root	No closed form expression
Three-parameter square-root	$f_{X(t)\|X(0)}(x \mid x_0) = \sum_{y=0}^{\infty} \exp(-\lambda/2) \frac{(\lambda/2)^y}{y!} \times \frac{x^{\nu/2+y-1}\exp(-x/2k)}{\Gamma(\nu/2+y)(2k)^{\nu/2+y}}$ $= \frac{1}{2k}\left(\frac{\sqrt{x}}{\sqrt{\lambda k}}\right)^{\nu/2-1} \exp\left(-\frac{\lambda + x/k}{2}\right) I_{\nu/2-1}\left(\frac{\sqrt{x}}{\sqrt{k/\lambda}}\right)$ with the parameterization[9] $\nu = 4\alpha\mu/\sigma^2$, $k = \frac{\sigma^2}{4\alpha}(1-e^{-\alpha t})$ and $\lambda = \frac{4x_0\alpha e^{-\alpha t}}{\sigma^2(1-e^{-\alpha t})}$
Radial Ornstein-Uhlenbeck	$f_{X(t)\|X(0)}(x \mid x_0) = 2\sum_{y=0}^{\infty} \exp(-\lambda/2) \frac{(\lambda/2)^y}{y!} \times \frac{x^{\nu+2y-1}\exp(-x^2/2k)}{\Gamma(\nu/2+y)(2k)^{\nu/2+y}}$ $= \frac{x}{k}\left(\frac{x}{\sqrt{\lambda k}}\right)^{\nu/2-1} \exp\left(-\frac{\lambda + x^2/k}{2}\right) I_{\nu/2-1}\left(\frac{x}{\sqrt{k/\lambda}}\right)$ with the parameterization $s = 2\sigma$, $a = 2\alpha$, and $m = \mu + \sigma^2/2\alpha$ and $\nu = 4am/s^2$, $k = \frac{s^2}{4a}(1-e^{-at})$ and $\lambda = x_0 e^{-at}/k = \frac{4x_0 a e^{-at}}{s^2(1-e^{-at})}$

Table 8: Moment generating function of transition distribution of first order Gaussian diffusion

[9] $I_\nu(x) = \sum_{y=0}^{\infty} \frac{(x/2)^y}{y!} \times \frac{(x/2)^{y+\nu}}{(y+\nu)!}$ is the modified Bessel function of the first kind

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Diffusion	**Moment generating function of transition distribution** $M_{X(t)\|X(0)}(\theta)$
Standard Brownian motion	$M_{X(t)\|X(0)}(\theta)=\exp\left(x_0\theta+\frac{1}{2}t\theta^2\right)$
Brownian motion	$M_{X(t)\|X(0)}(\theta)=\exp\left(x_0\theta+\mu t\theta+\frac{1}{2}t\theta^2\right)$
Geometric Brownian motion	$M_{X(t)\|X(0)}(\theta)=\sum_{k=0}^{\infty}\frac{\theta^k}{k!}\exp\left(k\log x_0+k(\mu-\frac{1}{2}\sigma^2)t+\frac{1}{2}\sigma^2tk^2\right)$
Standard Ornstein-Uhlenbeck process	$M_{X(t)\|X(0)}(\theta)=\exp\left(x_0e^{-\alpha t}\theta+\frac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\theta^2\right)$
Ornstein-Uhlenbeck process	$M_{X(t)\|X(0)}(\theta)=\exp\left(x_0e^{-\alpha t}\theta+\alpha\mu\left(\frac{1-e^{-\alpha t}}{\alpha}\right)\theta+\frac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\theta^2\right)$
Geometric Ornstein-Uhlenbeck process	$M_{X(t)\|X(0)}(\theta)=\sum_{k=0}^{\infty}\frac{\theta^k}{k!}\exp\left(x_0e^{-\alpha t}k+\alpha(\mu-\frac{1}{2}\sigma^2/\alpha)\left(\frac{1-e^{-\alpha t}}{\alpha}\right)k+\frac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)k^2\right)$
Eccentric square-root	$M_{X(T)\|X(t_0)}(\theta)=\exp\left(\frac{x_0e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right)$
Three-parameter square-root	$M_{X(t)\|X(0)}(\theta)=\left(1-\theta\sigma^2(1-e^{-\alpha t})/2\alpha\right)^{-\frac{2\alpha\mu}{\sigma^2}}\exp\left(\frac{x_0e^{-\alpha t}\theta}{1-\theta\sigma^2(1-e^{-\alpha t})/2\alpha}\right)$

Generalized Gaussian Distribution and Generalized Heat Equation

The standard heat equation is extended[10] by this thesis to include higher order diffusion effects and is referred to as the generalized heat equation. The generalized heat equations and its conjugate have been derived (and hence proved) by two alternative methods from first principles, by reference to the forward dynamics (differentiating the transition MGF) and by reference to the backward dynamics (expanding the Chapman-Kolmogorov equation). There is a focus on cumulants for skew and excess kurtosis in which case the standard heat equation (with scale) generalizes from $f_t=\frac{1}{2}\sigma^2 f_{xx}$ to $f_t=\frac{1}{4!}\kappa_4 f_{xxxx}-\frac{1}{3!}\kappa_3 f_{xxx}+\frac{1}{2}\sigma^2 f_{xx}$, where subscripts denote partial derivatives. The generalized heat equation to incorporate all higher order diffusion effects is

$$\frac{\partial f}{\partial t}=\sum_{j=2}^{\infty}(-1)^j\frac{1}{j!}\kappa_j\frac{\partial^j f}{\partial x^j}\qquad\text{with conjugate}\qquad-\frac{\partial f}{\partial s}=\sum_{j=2}^{\infty}\frac{1}{j!}\kappa_j\frac{\partial^j f}{\partial y^j}$$

The convergence of the generalized heat equation, such as $f_t=\frac{1}{4!}\kappa_4 f_{xxxx}-\frac{1}{3!}\kappa_3 f_{xxx}+\frac{1}{2}\sigma^2 f_{xx}$, to the standard heat equation $f_t=\frac{1}{2}\sigma^2 f_{xx}$, is a result of the Central Limit Theorem whereby the second order variation becomes dominant in relation to the higher order diffusion effects. Extending the conversions to higher orders for diffusions with generalized Gaussian bases, such as the generalized heat equation, transforms these equations in the transition density to the equivalent equations in the moment generating function.

$$\frac{\partial M}{\partial t}=\sum_{j=2}^{\infty}\frac{1}{j!}\kappa_j\theta^j M$$

The proper approach to include skew and excess kurtosis in the forward and backward equation of first order diffusion is that of the generalized heat equation, as well as its extension to diffusion of arbitrary order. The transition density function of any first order diffusion (and any basis distribution) can be written down in terms of the generalized heat equation postulated by this thesis, as well as it's conjugate, the backward equation and that of the option premium. Note that there is absolutely no reason for the density, with non-zero third and fourth

[10] Contribution of the author of this thesis, an equation for which I have called the generalized heat equation and is specifically defined in relation to the Central Limit Theorem for which the standard heat equation is the limiting form of the generalized heat. This choice of nomenclature is consistent with that of mathematical physics.

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order cumulants, to end up with a negative sign if it is normalized properly. The transition surface of the density function as the dependent variable of the generalized heat equation is similar to that for an ordinary Brownian motion but also has skew and features of the higher order statistical moments. A numerical algorithm to generate a *GenN* numerical distribution or simulated variate is then to numerically solve the PDE of the generalized heat equation. This can then yield the numerical density function or cumulative density function and simulated variables can then be obtained by either the accept-reject algorithm or by mapping simulated uniform $U(0,1)$ variables to the numerical *GenN* cumulative density function.

Any of the simulation code can be modified to simulate diffusion with non-Gaussian bases by altering the simulated vector of variates of the basis function. If one wants normal/Gaussian like variables with skew and excess, then this thesis recommends Fisher-Tippet (extreme-value) variables. These variables are logarithms of gamma variables, with centering and scaling, and it is further recommended by this thesis that these are preferable, as a theoretical precursor, before generating Generalized Gaussian variables by numerically generating a distribution by solving the generalized heat equation.

The results of the tables are obtained throughout various parts of the books of this thesis. These provide the main results of practical value at the implementation level. The various mathematics and derivations can be looked up as required, although one may have to become familiar with the content and the placing of the content throughout the thesis. The distribution of the diffusion basis is generalized Gaussian,

$$z_t \sim GenN\left(\mu, \frac{\sigma^2}{dt}, \frac{\kappa_3}{dt^2}, \frac{\kappa_4}{dt^3}\right)$$

where $\mu = \kappa_1$, $\sigma^2 = \kappa_2$, κ_3, and κ_4 are first, second, third, and fourth order infinitesimal differentials that are proportional to $(dt)^0$, $(dt)^1$, $(dt)^2$, and $(dt)^3$ respectively. For the multiplicative diffusions, the parameters that include the ito-like corrections, composed of Stirling numbers and factorials, replace each respective cumulant by

(1) $\mu - \frac{1}{2!}\sigma^2 + \frac{1}{3!}\kappa_3 - \frac{1}{4!}\kappa_4$

(2) $\frac{1}{2!}\sigma^2 - \frac{3}{3!}\kappa_3 + \frac{11}{4!}\kappa_4$

(3) $\frac{1}{3!}\kappa_3 - \frac{6}{4!}\kappa_4$

(4) κ_4

and note that these have the same maximum likelihood estimators when each infinitesimal parameter is being estimated by maximum likelihood. For example, when $dt = h$ the maximum likelihood estimators are

(1) $\mu - \frac{1}{2!}\sigma^2 + \frac{1}{3!}\kappa_3 - \frac{1}{4!}\kappa_4 = \overline{\Delta \log x_j} = \frac{1}{n}\sum \Delta \log x_j$

(2) $2! \times (\frac{1}{2!}\sigma^2 - \frac{3}{3!}\kappa_3 + \frac{11}{4!}\kappa_4) / h = \frac{1}{n}\sum \left(\Delta \log x_j - \overline{\Delta \log x_j}\right)^2$

(3) $3! \times (\frac{1}{3!}\kappa_3 - \frac{6}{4!}\kappa_4) / h^2 = \frac{1}{n}\sum \left(\Delta \log x_j - \overline{\Delta \log x_j}\right)^3$

(4) $4! \times \frac{1}{4!}\kappa_4 / h^3 = \frac{1}{n}\sum \left(\Delta \log x_j - \overline{\Delta \log x_j}\right)^4 - 3\left\{\frac{1}{n}\sum \left(\Delta \log x_j - \overline{\Delta \log x_j}\right)^2\right\}^2$

implying the corrections are simply reparameterizations, and as such, are null after estimation by maximum likelihood or moment matching. This is discussed in the relevant section arithmetic and geometric Brownian motion with a diffusion basis with non-zero skew and excess kurtosis,

Tables; Generalized Gaussian First Order Diffusion

Table 9: Distribution of forward transition distribution of first order generalized Gaussian diffusion

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	Distribution of forward transition distribution $X(T)$
Brownian motion	$X(T) \sim GenN\left(x_0 + \mu T, \sigma^2 T, \kappa_3 T, \kappa_4 T \right)$
Geometric Brownian motion	$X(T) \sim \log GenN\left(\log x_0 + \mu T, \sigma^2 T, \kappa_3 T, \kappa_4 T \right)$
Ornstein-Uhlenbeck process	$X(T) \sim GenN\left(x_0 e^{-\alpha T} + \mu \frac{1-e^{-\alpha T}}{\alpha}, \sigma^2 \frac{1-e^{-2\alpha T}}{2\alpha}, \kappa_3 \frac{1-e^{-3\alpha T}}{3\alpha}, \kappa_4 \frac{1-e^{-4\alpha T}}{4\alpha} \right)$
Geometric Ornstein-Uhlenbeck process	$X(T) \sim \log GenN\left(e^{-\alpha T} \log x_0 + \mu \frac{1-e^{-\alpha T}}{\alpha}, \sigma^2 \frac{1-e^{-2\alpha T}}{2\alpha}, \kappa_3 \frac{1-e^{-3\alpha T}}{3\alpha}, \kappa_4 \frac{1-e^{-4\alpha T}}{4\alpha} \right)$

Table 10: Moment generating function of first order generalized Gaussian diffusion

Moment generating function $M_{X(t)\|X(0)}(\theta)$
Brownian motion
$M_{X(T)\|X(0)}(\theta) = \exp\left(x_0\theta + \mu T\theta + \frac{1}{2!}\sigma^2 T\theta^2 + \frac{1}{3!}\kappa_3 T\theta^3 + \frac{1}{4!}\kappa_4 T\theta^4 \right)$
Geometric Brownian motion
$M_{X(T)\|X(0)}(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \exp\left(k\log x_0 + \mu T k + \frac{1}{2!}\sigma^2 T k^2 + \frac{1}{3!}\kappa_3 T k^3 + \frac{1}{4!}\kappa_4 T k^4 \right)$
Ornstein-Uhlenbeck process
$M_{X(T)\|X(0)}(\theta) = \exp\left(x_0 e^{-\alpha T}\theta + \mu \frac{1-e^{-\alpha T}}{\alpha}\theta + \frac{1}{2!}\sigma^2 \frac{1-e^{-2\alpha T}}{2\alpha}\theta^2 + \frac{1}{3!}\kappa_3 \frac{1-e^{-3\alpha T}}{3\alpha}\theta^3 + \frac{1}{4!}\kappa_4 \frac{1-e^{-4\alpha T}}{4\alpha}\theta^4 \right)$
Geometric Ornstein-Uhlenbeck process
$M_{X(T)\|X(0)}(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{k!} \exp\left(k e^{-\alpha T} \log x_0 + \mu \frac{1-e^{-\alpha T}}{\alpha} k + \frac{1}{2!}\sigma^2 \frac{1-e^{-2\alpha T}}{2\alpha} k^2 + \frac{1}{3!}\kappa_3 \frac{1-e^{-3\alpha T}}{3\alpha} k^3 + \frac{1}{4!}\kappa_4 \frac{1-e^{-4\alpha T}}{4\alpha} k^4 \right)$

Table 11: Forward equations of first order generalized Gaussian diffusion

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Forward equation

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) = -\frac{\partial}{\partial x} \mu(x,t) f_{X(t)|X(t_0)}(x \mid x_0)$$
$$+ \tfrac{1}{2!} \frac{\partial^2}{\partial x^2} \sigma^2(x,t) f_{X(t)|X(t_0)}(x \mid x_0) - \tfrac{1}{3!} \frac{\partial^3}{\partial x^3} \kappa_3(x,t) f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!} \frac{\partial^4}{\partial x^4} \kappa_4(x,t) f_{X(t)|X(t_0)}(x \mid x_0)$$

Brownian motion

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$= -\mu \frac{\partial}{\partial x} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{2!} \sigma^2 \frac{\partial^2}{\partial x^2} f_{X(t)|X(t_0)}(x \mid x_0) - \tfrac{1}{3!} \kappa_3 \frac{\partial^3}{\partial x^3} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!} \kappa_4 \frac{\partial^4}{\partial x^4} f_{X(t)|X(t_0)}(x \mid x_0)$$

Geometric Brownian motion

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$= -\mu \frac{\partial}{\partial x} x f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{2!} \sigma^2 \frac{\partial^2}{\partial x^2} x^2 f_{X(t)|X(t_0)}(x \mid x_0) - \tfrac{1}{3!} \kappa_3 \frac{\partial^3}{\partial x^3} x^3 f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!} \kappa_4 \frac{\partial^4}{\partial x^4} x^4 f_{X(t)|X(t_0)}(x \mid x_0)$$

Ornstein-Uhlenbeck process

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) = -\alpha \frac{\partial}{\partial x} (\mu/\alpha - x) f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{2!} \sigma^2 \frac{\partial^2}{\partial x^2} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$- \tfrac{1}{3!} \kappa_3 \frac{\partial^3}{\partial x^3} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!} \kappa_4 \frac{\partial^4}{\partial x^4} f_{X(t)|X(t_0)}(x \mid x_0)$$

Geometric Ornstein-Uhlenbeck process

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) = -\alpha \frac{\partial}{\partial x} (\mu/\alpha - \log x) x f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{2!} \sigma^2 \frac{\partial^2}{\partial x^2} x^2 f_{X(t)|X(t_0)}(x \mid x_0)$$
$$- \tfrac{1}{3!} \kappa_3 \frac{\partial^3}{\partial x^3} x^3 f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!} \kappa_4 \frac{\partial^4}{\partial x^4} x^4 f_{X(t)|X(t_0)}(x \mid x_0)$$

Table 12: equation of option premium for first order diffusion with generalized Gaussian basis

pde of option premium	$rV - \frac{\partial V}{\partial t_0} = \mu(x_0,t_0) \frac{\partial V}{\partial x_0} + \tfrac{1}{2!} \sigma^2(x_0,t_0) \frac{\partial^2 V}{\partial x_0^2} + \tfrac{1}{3!} \kappa_3(x_0,t_0) \frac{\partial^3 V}{\partial x_0^3} + \tfrac{1}{4!} \kappa_4(x_0,t_0) \frac{\partial^4 V}{\partial x_0^4}$
Brownian motion	$rV - \frac{\partial V}{\partial t_0} = \mu \frac{\partial V}{\partial x_0} + \tfrac{1}{2!} \sigma^2 \frac{\partial^2 V}{\partial x_0^2} + \tfrac{1}{3!} \kappa_3 \frac{\partial^3 V}{\partial x_0^3} + \tfrac{1}{4!} \kappa_4 \frac{\partial^4 V}{\partial x_0^4}$
Geometric Brownian motion	$rV - \frac{\partial V}{\partial t_0} = \mu x_0 \frac{\partial V}{\partial x_0} + \tfrac{1}{2!} \sigma^2 x_0^2 \frac{\partial^2 V}{\partial x_0^2} + \tfrac{1}{3!} \kappa_3 x_0^3 \frac{\partial^3 V}{\partial x_0^3} + \tfrac{1}{4!} \kappa_4 x_0^4 \frac{\partial^4 V}{\partial x_0^4}$
Ornstein-Uhlenbeck process	$rV - \frac{\partial V}{\partial t_0} = \alpha(\mu/\alpha - x_0) \frac{\partial V}{\partial x_0} + \tfrac{1}{2!} \sigma^2 \frac{\partial^2 V}{\partial x_0^2} + \tfrac{1}{3!} \kappa_3 \frac{\partial^3 V}{\partial x_0^3} + \tfrac{1}{4!} \kappa_4 \frac{\partial^4 V}{\partial x_0^4}$
Geometric Ornstein-Uhlenbeck process	$rV - \frac{\partial V}{\partial t_0} = \alpha(\mu/\alpha - \log x_0) x_0 \frac{\partial V}{\partial x_0} + \tfrac{1}{2!} \sigma^2 x_0^2 \frac{\partial^2 V}{\partial x_0^2} + \tfrac{1}{3!} \kappa_3 x_0^3 \frac{\partial^3 V}{\partial x_0^3} + \tfrac{1}{4!} \kappa_4 x_0^4 \frac{\partial^4 V}{\partial x_0^4}$

Table 13: Backward equations of first order generalized Gaussian diffusion

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Backward equation

$$-\frac{\partial}{\partial t_0} f_{X(t)|X(t_0)}(x \mid x_0) = \mu(x_0,t_0)\frac{\partial}{\partial x_0} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$+ \tfrac{1}{2!}\sigma^2(x_0,t_0)\frac{\partial^2}{\partial x_0^2} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{3!}\kappa_3(x_0,t_0)\frac{\partial^3}{\partial x_0^3} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!}\kappa_4(x_0,t_0)\frac{\partial^4}{\partial x_0^4} f_{X(t)|X(t_0)}(x \mid x_0)$$

Brownian motion

$$-\frac{\partial}{\partial t_0} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$= \mu\frac{\partial}{\partial x_0} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{2!}\sigma^2\frac{\partial^2}{\partial x_0^2} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{3!}\kappa_3\frac{\partial^3}{\partial x_0^3} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!}\kappa_4\frac{\partial^4}{\partial x_0^4} f_{X(t)|X(t_0)}(x \mid x_0)$$

Geometric Brownian motion

$$-\frac{\partial}{\partial t_0} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$= \mu x_0\frac{\partial}{\partial x_0} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{2!}\sigma^2 x_0^2\frac{\partial^2}{\partial x_0^2} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{3!}\kappa_3 x_0^3\frac{\partial^3}{\partial x_0^3} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!}\kappa_4 x_0^4\frac{\partial^4}{\partial x_0^4} f_{X(t)|X(t_0)}(x \mid x_0)$$

Ornstein-Uhlenbeck process

$$-\frac{\partial}{\partial t_0} f_{X(t)|X(t_0)}(x \mid x_0) = \alpha(\mu/\alpha - x_0)\frac{\partial}{\partial x_0} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$+ \tfrac{1}{2!}\sigma^2\frac{\partial^2}{\partial x_0^2} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{3!}\kappa_3\frac{\partial^3}{\partial x_0^3} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!}\kappa_4\frac{\partial^4}{\partial x_0^4} f_{X(t)|X(t_0)}(x \mid x_0)$$

Geometric Ornstein-Uhlenbeck process

$$-\frac{\partial}{\partial t_0} f_{X(t)|X(t_0)}(x \mid x_0) = \alpha(\mu/\alpha - \log x_0)x_0\frac{\partial}{\partial x_0} f_{X(t)|X(t_0)}(x \mid x_0)$$
$$+ \tfrac{1}{2!}\sigma^2 x_0^2\frac{\partial^2}{\partial x_0^2} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{3!}\kappa_3 x_0^3\frac{\partial^3}{\partial x_0^3} f_{X(t)|X(t_0)}(x \mid x_0) + \tfrac{1}{4!}\kappa_4 x_0^4\frac{\partial^4}{\partial x_0^4} f_{X(t)|X(t_0)}(x \mid x_0)$$

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STANDARD BROWNIAN MOTION

Introduction

Standard Brownian motion[11] is the integration, or cumulative infinitesimal summation, of a normed Gaussian Hilbert space, a continuous set of uncorrelated infinitesimal standard (zero mean and unit variance) normal distributions. This function is later referred exclusively to as the standardized diffusion basis and standard Brownian motion as the unit root integration of the standardized basis. Standard Brownian motion is denoted by B_t and has a single functional dependence, time t, which indexes the time location of the diffusion as well as measuring the total number of infinitesimal normal distributions that are convolved onto the backward variable of the diffusion for a given transition interval. This is the basic model of cumulative variation and provides the domain of attraction of the Central Limit Theorem for the aggregation of homogenous standardized random variables with finite variance. All processes of first order normal diffusion can be obtained by transforming and combining standard Brownian motions, where the transformations are either in the diffusion variable or the time variable. Since Brownian motion is the direct (unit root) integration of the diffusion basis, it is statistically fractal on all time scales. In particular, Brownian motion on the infinitesimal dt scale

$$dB_t = B_{dt} \sim N(0, dt)$$

is indistinguishable from Brownian motion on the non-infinitesimal scale, t

$$B_t \sim N(0, t)$$

after accounting for scale on the vertical axis. The infinitesimal parameters of standard Brownian motion are $\mu(x, t) = 0$ and $\sigma(x, t) = 1$. The stochastic differential equation is then $dX_t = dB_t$ subject to the Dirac distributed initial condition

$$\begin{aligned}\lim_{t\to 0} X(t) &\sim \delta(x_0) \\ &\sim \lim_{\varepsilon\to 0} U(x_0 - \varepsilon, x_0 + \varepsilon)\end{aligned}$$

It is usual to let standard Brownian motion start at the origin, $x_0 = 0$.

Integral Solution to Standard Brownian Motion

Standard Brownian motion is the basic diffusion and, in some regards, differential Brownian motion is simply a definition. The utility of the integral solution is the concept of integrating the continuous set of random variables of a continuum of differential Brownian motion,

$$dX_t = dB_t \qquad \text{where} \qquad dB_t \sim N(0, dt)$$

integrating

$$\int_0^t dX_y = \int_0^t dB_y$$

$$X_t - x_0 = B_t - B_0$$

or

$$X_t = x_0 + B_t$$

so that

$$\begin{aligned}X_t &\sim x_0 + N(0, t) \\ &\sim N(x_0, t)\end{aligned}$$

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[11] Brownian motion is also referred to as the Wiener-Bacheliar process
Feller, W., (1966) An Introduction to Probability Theory and Its Applications, Volume 2. John Wiley & Sons. Page 181.

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that is, a normal transition distribution with mean at the starting value x_0 and variance t.

The MGF of a Normal Distribution

The density function of a normal distribution $X \sim N(\mu, \sigma^2)$ is

$$f_{N(\mu,\sigma^2)}(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

Giving a moment generating function of

$$\begin{aligned} M_{N(\mu,\sigma^2)}(\theta) &= E\left(e^{\theta N(\mu,\sigma^2)}\right) \\ &= \int_{-\infty}^{\infty} e^{\theta y} f_{N(\mu,\sigma^2)}(y)dy \\ &= \int_{-\infty}^{\infty} e^{\theta y} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \\ &= e^{\mu\theta+\frac{1}{2}\sigma^2\theta^2} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu-\theta\sigma^2)^2}{2\sigma^2}} dy \\ &= e^{\mu\theta+\frac{1}{2}\sigma^2\theta^2} \int_{-\infty}^{\infty} f_{N(\mu+\theta\sigma^2,\sigma^2)}(y)dy \\ &= e^{\mu\theta+\frac{1}{2}\sigma^2\theta^2} \end{aligned}$$

That is,

$$M_{N(\mu,\sigma^2)}(\theta) = e^{\mu\theta+\frac{1}{2}\sigma^2\theta^2}$$

This result is fundamental to this entire thesis since this exposition deals almost exclusively with normal diffusion. The transition mgf of the standard Brownian motion is then

$$M_{X(T)|X(t_0)}(\theta) = \exp\left(x_0\theta + \tfrac{1}{2}(T-t_0)\theta^2\right)$$

Formation of the Forward and Backward Equations of Standard Brownian Motion

With the infinitesimal parameters $\mu(x, t) = 0$ and $\sigma(x, t) = 1$, the forward equation of standard Brownian motion is the standard heat equation $f_t = \frac{1}{2}\nabla^2 f$ where ∇^2 is the Laplacian or del squared, the dot of del and grad. This is somewhat trivial since this thesis concentrates almost exclusively on one dimensional diffusion. As such, the shorthand notation $f_t = \frac{1}{2}f_{xx}$ is probably more suitable.

A. Formation of the Forward Equation

Differentiating the mgf partially with respect to the forward time variable gives the forward equation in the mgf

$$\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta) = \tfrac{1}{2}\theta^2 M_{X(T)|X(t_0)}(\theta)$$

From the definition of the transition mgf

$$M_{X(T)|X(t_0)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0)dx$$

Differentiating partially with respect to the forward diffusion variable (by the product rule on the right)

$$\frac{\partial}{\partial x} M_{X(T)|X(t_0)}(\theta) = \theta\int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0)dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(T)|X(t_0)}(x \mid x_0)dx = 0$$

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and again

$$\frac{\partial^2}{\partial x^2} M_{X(T)|X(t_0)}(\theta) = -\theta^2 \int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0) dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) dx = 0$$

for

(1) $\theta M_{X(T)|X(t_0)}(\theta) - \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(T)|X(t_0)}(x \mid x_0) dx$

(2) $\theta^2 M_{X(T)|X(t_0)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) dx$

Hence the time derivative can be written in terms of the forward transition density as

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) dx = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) dx$$

which results in the forward equation of standard Brownian motion, the standard heat equation, in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0)$$

where $X(t)$ is the forward transition distribution of the Brownian motion from a starting value of $X(t_0) = x_0$ at time t_0 .

B. Formation of the Backward Equation

Differentiating the mgf partially with respect to the backward time variable

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = -\tfrac{1}{2} \theta^2 M_{X(T)|X(t_0)}(\theta)$$

Differentiating partially with respect to the backward diffusion variable

$$\frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) = \theta M_{X(T)|X(t_0)}(\theta)$$

and again

$$\frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta) = \theta^2 M_{X(T)|X(t_0)}(\theta)$$

Hence, the time derivative can be written as

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = -\tfrac{1}{2} \frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta)$$

which results the backward equation of standard Brownian motion, the Hermite conjugate of the standard heat equation, in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\tfrac{1}{2} \frac{\partial^2}{\partial x_0^2} f_{X(T)|X(t_0)}(x \mid x_0)$$

which is the Hermite conjugate, or formal adjoint, of the forward equation. For the basic process it is usually assumed that the starting time of the diffusion is time zero $t_0 = 0$.

Solution to the Forward Equation for Standard Brownian Motion

Transforming the forward equation in the transition density to that in the moment generating function by multiplying by an exponential factor in the forward diffusion variable and integrating over the range of the diffusion,

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$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{X(t)}(x)dx = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(t)}(x)dx$$

Substituting the results regarding the technique of converting derivatives in the forward diffusion variable to products of the moment generating function and the moment generating function parameter into the standard heat equation gives a first order linear differential equation in the moment generating function

$$\frac{d}{dt} M_{X(t)}(\theta) = \tfrac{1}{2}\theta^2 M_{X(t)}(\theta)$$

with initial condition of a mgf of a Dirac singularity at the backward variable

$$M_{X(t)}(\theta)\,|_{t=0} = M_{X(0)}(\theta) = E(e^{x_0\theta}) = \int_{-\infty}^{\infty} e^{\theta x}\delta_{x_0}(x)dx = e^{x_0\theta}$$

for a process starting at x_0 . Dividing both sides by $M_{X(t)}(\theta)$ gives a simple exact differential in the cumulant generating function,

$$\frac{d}{dt}\log M_{X(t)}(\theta) = \tfrac{1}{2}\theta^2$$

This is the heat equation in terms of the cumulant generating function, rather than the density, of the transition distribution and therefore describes the diffusion in terms of the dynamics of how the cumulants, rather than the density, evolve through time. In particular, this form of the equation describes a process of successive linear aggregation of the cumulants rather than success convolution of the infinitesimal density functions of the increments onto the aggregate diffusion transition density. The standard heat equation in the cumulant generating function considered here states mathematically that the single second order cumulant of the diffusion differentially aggregates at a constant rate. Multiplying the exact differential by dt and integrating over a dummy s

$$d\log M_{X(t)}(\theta) = \tfrac{1}{2}\theta^2 dt$$

$$\int_0^t d\log M_{X(s)}(\theta) = \tfrac{1}{2}\theta^2\int_0^t ds$$

$$\log M_{X(t)}(\theta) - \log M_{X(0)}(\theta) = \tfrac{1}{2}\theta^2 t$$

$$M_{X(t)}(\theta) = M_{X(0)}(\theta)e^{\frac{1}{2}\theta^2 t}$$

Substituting in the initial condition, $M_{X(0)}(\theta) = e^{x_0\theta}$, gives the moment generating function of the solution to the basic one-dimensional heat equation, the transition density function for standard Brownian motion with zero drift and unit volatility. This is

$$M_{X(t)}(\theta) = e^{x_0\theta + \frac{1}{2}t\theta^2}$$

which is the moment generating function of a normal distribution with variance t and centered at the initial value x_0 ,

$$X(t) \sim N(x_0, t)$$

For a process that starts at the origin, the initial value is zero and the transition distribution is zero mean normal with variance t . The transition density function, and the general solution to the heat equation $f_t = \frac{1}{2}\nabla^2 f$ is a time scaled normal,

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right) \qquad f_X \in \{(x,t); -\infty < x < \infty, 0 \le t < \infty\}$$

A zero drift Brownian motion without scale as this section relates will be denoted by $B(t)$ which is the integral of the right-hand-side of the stochastic differential equation, for the remainder of the discussion of continuous diffusions and the remaining books of this thesis. Standard Brownian motion, is parameter free so that it is fully specified without estimation or assigning parameter values. This is the basic model of cumulative standardized Gaussian noise. Recalling that the Gaussian distribution is the domain of attraction of the Central Limit Theorem, the standard heat equation is an equivalent representation of the Central Limit Theorem applied to continu-

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ous summations of finite variance random variables, this representation being a description of the differential equation of the kernel of the limiting distribution.

Simulation Algorithm

Discretizing the differential increment, by integrating over a span $h = \Delta t$ so that an interval t has t / h subintervals,

$$\Delta B_{k\Delta t} = \int_{k\Delta t}^{\Delta t + \Delta t} dB_t = \Delta z_t \sim N(0, \Delta t)$$

which is exact since the process has an infinitesimal unit-root and scales with the width of any increment. Summing the component integrals,

$$\begin{aligned} X_t &= x_0 + \sum_{j=\Delta t}^{t} \Delta z_j \\ &\sim x_0 + \sum_{j=1}^{t/\Delta t} N(0, \Delta t) \\ &\sim x_0 + N(0, \Delta t) + N(0, \Delta t) + N(0, \Delta t) + \ldots + N(0, \Delta t) \\ &\sim x_0 + \sqrt{\Delta t} \sum_{j=1}^{t/\Delta t} N(0,1) \end{aligned}$$

That is, generate t / h unit normal variates, calculate the vector of the cumulative sum, and scale the vector upwards by the square-root of the length of the increments by multiplying by a factor of $\sqrt{\Delta t}$. This gives a vector of observed points of a standard Brownian path observed at t / h points each spaced $h = \Delta t$ time units apart. The MATLAB code to do this is

```
x0=x0; h=h; n=n;                          %initialize diffusion parameters
z=randn(n,1).*sqrt(h); x=cumsum([x0;z]);  %simulate basis and sum
plot(x);                                  %plot
```

Simulated Paths of Standard Brownian Motion

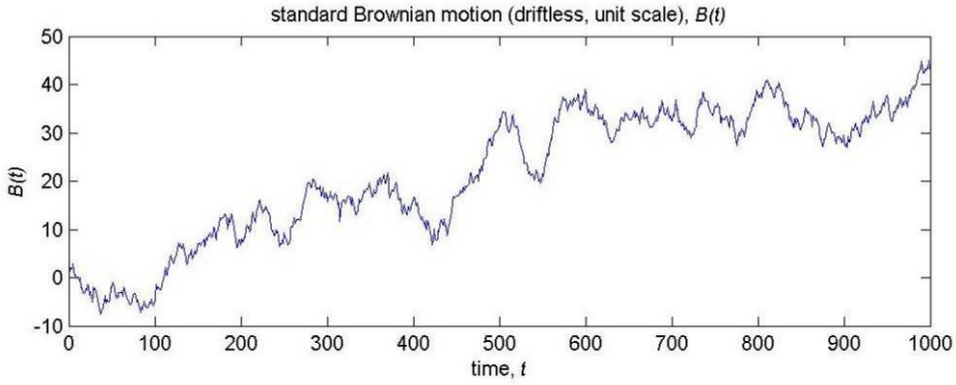


Figure 1: Simulated path of standard Brownian motion

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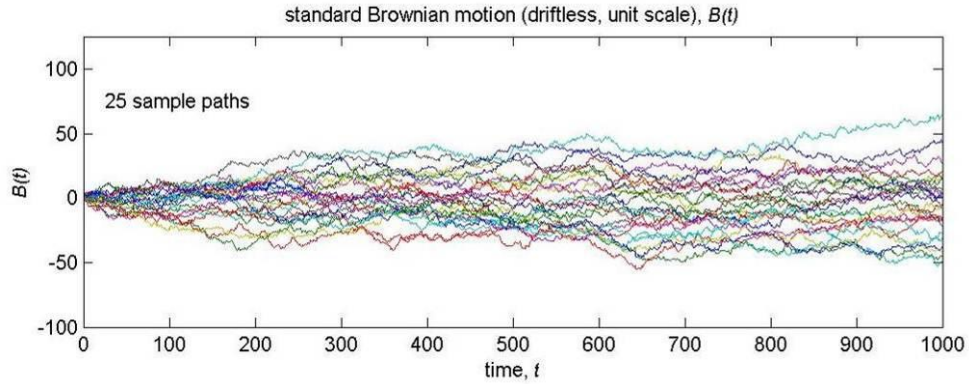


Figure 2: 25 simulated paths of standard Brownian motion

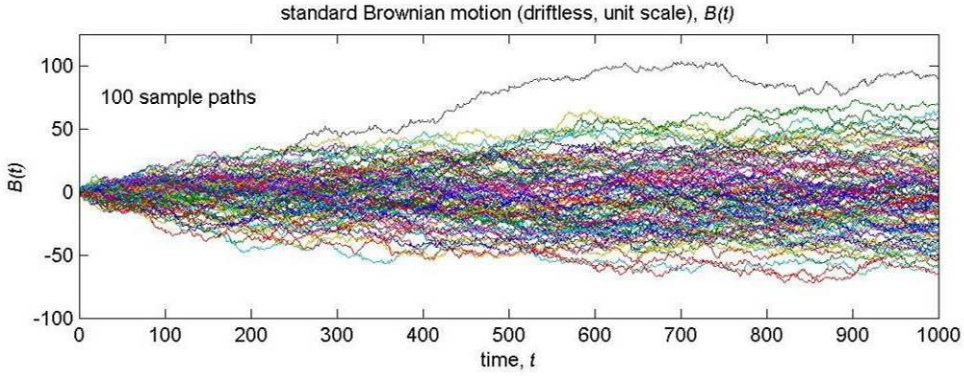


Figure 3: 100 simulated paths of standard Brownian motion

The MGF of an Inverse Squared Zero Mean Gaussian (Stable with Index ½)

An inverse square zero-mean Gaussian distribution $X \sim 1 / N(0,\sigma^2)^{\,2}$ is equivalent to an inverse Gaussian with infinite first parameter

$$X \sim IG\left(\frac{x}{0}, \frac{x}{\sigma^2}\right)$$

This distribution has the density function

$$f_X(x) = \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{1}{2}\frac{x^2}{t}}$$

For the moment generating function,

$$\begin{aligned} M_X(\theta) &= \int_0^\infty e^{\theta t} \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{1}{2}\frac{x^2}{t}} dt \\ &= \int_0^\infty \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{1}{2}\frac{x^2}{t} + \theta t} dt \\ &= e^{x\sqrt{-2\theta}} \int_0^\infty \frac{x}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{1}{2}\left(\frac{x + t\sqrt{-2\theta}}{\sqrt{t}}\right)^2} dt \end{aligned}$$

where the integrand is the density function of an inverse Gaussian distribution with parameters $-x / \sqrt{-2\theta}$ and x^2

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$$M_X(\theta) = e^{x\sqrt{-2\theta}} \int_0^\infty f_{IG(-x/\sqrt{-2\theta},x^2)}(t)dt$$

Since $\int_0^\infty f_{IG(-x/\sqrt{-2\theta},x^2)}(t)dt = 1$ the moment generating function is

$$M_X(\theta) = e^{x\sqrt{-2\theta}}$$

First Passage Time of Standard Brownian Motion

The forward equation is the standard heat equation in the density of the first passage time distribution, for a passage through the value x, the forward diffusion variable. Letting $T(x) \mid T(x_0)$ denote the conditional random variable measuring the time at which a standard Brownian motion first reaches the value x given that the diffusion is at the value x_0 now, which is the initial value and backward diffusion variable. The notation $T(x)$ is used by this thesis as the shorthand for the conditional variable. The notation for the density function is then $f_{T(x)}(t)$ as shorthand for $f_{T(x)\,|T(x_0)}(t)$. The forward equation can then be written concisely as

$$\frac{\partial f_{T(x)}(t)}{\partial t} = \frac{1}{2}\frac{\partial^2 f_{T(x)}(t)}{\partial x^2}$$

This equation is the heat equation with the roles of the time and diffusion variables reversed, and as such, is of first differential order in the distributional variable, now time, and second differential order in the evolution variable. This is in contrast to the heat equation of Brownian motion, which is an equation that is second order in the distributional variable (the diffusion variable) and first order in the evolution variable (time). The distribution is defined on all times greater than zero. Transforming the equation in the density to one in the moment generating function by multiplying both sides by an exponential factor in time and integrating over the range of the distribution, a possible infinite time horizon,

$$\int_0^\infty e^{\theta t}\frac{\partial f_{T(x)}(t)}{\partial t}dt = \frac{1}{2}\int_0^\infty e^{\theta t}\frac{\partial^2 f_{T(x)}(t)}{\partial x^2}dt$$

and by the generic result of obtaining multiplications in the moment generating function and the moment generating function parameter from derivatives in the density,

$$\int_0^\infty e^{\theta t}\frac{\partial f_{T(x)}(t)}{\partial t}dt = -\theta M_{T(x)}(\theta)$$

gives

$$-\theta M_{T(x)}(\theta) = \frac{1}{2}\frac{\partial^2}{\partial x^2}M_{T(x)}(\theta)$$

or

$$0 = \frac{1}{2}\frac{\partial^2}{\partial x^2}M_{T(x)}(\theta) + \theta M_{T(x)}(\theta)$$

The characteristic polynomial of the equation is $0 = \lambda^2 + 2\theta$ which has characteristic roots $\pm\sqrt{-2\theta}$. Factorizing the equation in differential operators gives

$$0 = e^{-x\sqrt{-2\theta}}\frac{d}{dx}e^{x\sqrt{-2\theta}}\left(e^{x\sqrt{-2\theta}}\frac{d}{dx}e^{-x\sqrt{-2\theta}}M_{T(x)}(\theta)\right)$$

or

$$0 = \frac{d}{dx}e^{2x\sqrt{-2\theta}}\left(\frac{d}{dx}e^{-x\sqrt{-2\theta}}M_{T(x)}(\theta)\right)$$

Integrating from the starting value of the diffusion, x_0,

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$$\int_{x_0}^{x} \frac{d}{dy} e^{2y\sqrt{-2\theta}} \left(\frac{d}{dy} e^{-y\sqrt{-2\theta}} M_{T(x)}(\theta) \right) dy = \int_{x_0}^{x} 0 dy$$

$$\left[e^{2y\sqrt{-2\theta}} \left(\frac{d}{dy} e^{-y\sqrt{-2\theta}} M_{T(x)}(\theta) \right) \right]_{x_0}^{x} = 0$$

$$e^{2x\sqrt{-2\theta}} \left(\frac{d}{dy} e^{-x\sqrt{-2\theta}} M_{T(x)}(\theta) \right) - \tfrac{1}{2} \frac{\partial^2}{\partial x_0^2} M_{T(x_0)}(\theta) - \theta M_{T(x_0)}(\theta) = 0$$

with the initial condition at the lower limit of integration

$$\tfrac{1}{2} \frac{\partial^2}{\partial x_0^2} M_{T(x_0)}(\theta) + \theta M_{T(x_0)}(\theta) = 0$$

by evaluating the original equation at the initial value x_0 and

$$M_{T(x_0)}(\theta) = \int_0^{\infty} e^{t\theta} f_{T(x)|T(x_0)}(t) dt = \int_0^{\infty} e^{t\theta} \delta_0(t) dt = e^{0\times\theta} = 1$$

since the time it takes to get from x_0 to the initial value of the diffusion x_0 is obviously zero and the density of the first passage time is a Dirac singularity. The equation reduces to

$$e^{2x\sqrt{-2\theta}} \left(\frac{d}{dx} e^{-x\sqrt{-2\theta}} M_{T(x)}(\theta) \right) = 0 \qquad \text{or} \qquad \frac{d}{dx} e^{-x\sqrt{-2\theta}} M_{T(x)}(\theta) = 0$$

which is an exact derivative. Integrating again from the starting value of the diffusion to the value of the diffusion variable for which the first passage time distribution relates, x ,

$$\int_{x_0}^{x} \left(\frac{d}{dy} e^{-y\sqrt{-2\theta}} M_{T(x)}(\theta) \right) dy = 0$$

or

$$e^{-x\sqrt{-2\theta}} M_{T(x)}(\theta) - e^{-x_0\sqrt{-2\theta}} M_{T(x_0)}(\theta) = 0$$

and with the initial condition $M_{T(x_0)}(\theta) = e^{t_0\theta} = 1$

$$e^{-x\sqrt{-2\theta}} M_{T(x)}(\theta) = e^{-x_0\sqrt{-2\theta}}$$

and finally, the moment generating function of the first passage time distribution of standard Brownian motion is

$$M_{T(x)}(\theta) = e^{(x-x_0)\sqrt{-2\theta}}$$

which is the moment generating function of a scaled inverse chi-square distribution on one degree of freedom and therefore

$$T(x) \sim inv\Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}(x-x_0)^2\right)$$
$$\sim IG\left(0, (x-x_0)^2\right)$$

This distribution is also an inverse gamma distribution with shape parameter ½ and scale parameter $(x - x_0)^2 / 2$ or stable distribution with index ½, and most intuitively, the inverse square of a zero mean normal distribution. This distribution has density function

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$$f_{T(x)|T(x_0)}(t) = \frac{x-x_0}{t\sqrt{t}}\phi\left(\frac{x-x_0}{\sqrt{t}}\right)$$
$$= \frac{x-x_0}{t^{3/2}\sqrt{2\pi}}\exp\left(-\frac{(x-x_0)^2}{2t}\right) \qquad f_{T_x}\ \forall\{(t,x): t\in[0,\infty), x\in[0,\infty)\}$$

which is the solution to the forward equation $f_t = \frac{1}{2} f_{xx}$ (the standard heat equation) for the time over state transition distribution. This solution can alternatively be derived by arguments based on intuitive reasoning regarding equal probabilities for equivalent sample spaces in the time and diffusion variables in relation to the maximum value of a Brownian path. The cumulative distribution function is

$$F_{T(x)|T(x_0)}(t) = 2\left(1-\Phi\left(\frac{x-x_0}{\sqrt{t}}\right)\right) \qquad \text{for } t \geq 0$$

Analysis of the MGF of the First Passage Time Distribution

From the moment generating function of the first passage time distribution for a zero drift Brownian motion

$$M_{T_a}(\theta) = \exp(a\sqrt{-2\theta}) = \exp(i\,a\sqrt{2\theta})$$

This function is a complex exponential. Derivatives of this function are also complex which implies that the moments are complex. The first derivative of the moment generating function is

$$M^{(1)}_{T_a}(\theta) = \frac{i\,a}{\sqrt{2\theta}}\exp(i\,a\sqrt{2\theta}) = \frac{i\,a}{\sqrt{2\theta}}cis(a\sqrt{2\theta}) = \frac{i\,a}{\sqrt{2\theta}}(\cos a\sqrt{2\theta} + i\sin a\sqrt{2\theta}) = \frac{a}{\sqrt{2\theta}}(i\cos a\sqrt{2\theta} - \sin a\sqrt{2\theta})$$

and has modulus

$$||M^{(1)}_{T_a}(\theta)|| = \frac{a}{\sqrt{2\theta}}$$

with the real and imaginary parts

$$\text{real}\, M^{(1)}_{T_a}(\theta) = -\frac{a}{\sqrt{2\theta}}\sin a\sqrt{2\theta} \qquad \text{and} \qquad \text{imag}\, M^{(1)}_{T_a}(\theta) = \frac{a}{\sqrt{2\theta}}\cos a\sqrt{2\theta}$$

Taking the limits by letting θ approach zero, the limit of the real part of the first moment is $-a^2$ while the limit of the complex part and the modulus of the first moment is infinite;

$$\lim_{\theta\to 0}\text{mod}\, M^{(1)}_{T_a}(\theta) = \infty \qquad \lim_{\theta\to 0}\text{real}\ M^{(1)}_{T_a}(\theta) = -a^2 \qquad \lim_{\theta\to 0}\text{imag}\ M^{(1)}_{T_a}(\theta) = \infty$$

Hence

$$\lim_{\theta\to 0} M^{(1)}_{T_a}(\theta) = -a^2 + i\times\infty$$

These results say that the real part of the first moment, the mean, of the first passage time is finite and nonzero although it is negative and this is not possible for variables relating to time. The complex part is infinite so the modulus is infinite. Since the integral of the first moment is divergent, the moment is infinite but this is not what the real part alone would imply, although the fact that it is less than zero indicates that there is a problem. Therefore, a real part of a moment is not sufficient to describe a moment if there is also an imaginary part. A vital part of future research, which has the potential to greatly improve the understanding of random variables and stochastic processes, relates to exactly how complex or imaginary parts to moments contribute to the propensity for large outcomes.

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BROWNIAN MOTION WITH DRIFT

Introduction

A Brownian motion is a first order unit root diffusion and is a standard Brownian motion whereby the basis function is translated and scaled by an infinitesimal mean μ and infinitesimal variance σ^2. Upon integration of the differential variable, the additional basis mean amounts to linear drift μt. The basis variance σ^2 is proportional to *dt* to keep the diffusion finite and in scale. Similarly to standard Brownian motion, a Brownian motion is the integration, or cumulative infinitesimal summation, of a Gaussian Hilbert space, the diffusion basis. Brownian motion differs from a standard Brownian motion only in that the basis components have a non-zero mean and are scaled. Brownian motion is a Central Limit Theorem for the aggregation of homogenous random variables with finite variance prior to standardization to standard Brownian motion, whereby the first two cumulants dominate the higher order variation as the diffusion cumulants aggregate linearly with time. Brownian motion, like the standardized counterpart, is statistically fractal on all time scales. The infinitesimal parameters are $\mu(x, t) = \mu$ and $\sigma(x, t) = \sigma$ and the stochastic differential equation is

$$dX_t = \mu dt + \sigma dB_t \quad \text{where } dB_t \sim N(0, dt)$$

subject to the Dirac distributed initial condition

$$\begin{aligned} \lim_{t \to 0} X(t) &\sim \delta(x_0) \\ &\sim \lim_{\varepsilon \to 0} U(x_0 - \varepsilon, x_0 + \varepsilon) \end{aligned}$$

Integral Solution to Arithmetic Brownian Motion

The utility of the integral solution is as a concept of integrating a continuous set of random variables of differential Brownian motion, now each having scale and non-zero mean, as well as returning the non-trivial transition distribution from which probabilities and transition statistics can be obtained.

$$dX_t = \mu dt + \sigma dB_t$$

$$\int_0^t dX_y = \int_0^t \left(\mu dy + \sigma dB_y \right)$$

$$X_t - x_0 = \left[\mu y + \sigma B_y \right]_0^t$$

or

$$X_t = x_0 + \mu t + \sigma B_t$$

so that

$$\begin{aligned} X_t &\sim x_0 + \mu t + \sigma N(0, t) \\ &\sim N(x_0 + \mu t, \sigma^2 t) \end{aligned}$$

noting that X_t = constant + deterministic function + standard Brownian motion

i.e.,

X_t = backward diffusion variable + linear drift + first order additive noise

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which is the decomposition[12] of the additive unit-root diffusion into a sum of the initial value of the diffusion, linear deterministic drift, and an additive Gaussian noise component. The initial value is also the backward variable of the Brownian motion. The transition mgf of Brownian motion is

$$M_{X(T)|X(t_0)}(\theta) = \exp\left(x_0\theta + \mu(T-t_0)\theta + \tfrac{1}{2}\sigma^2(T-t_0)\theta^2 \right)$$

Formation of the Forward and Backward Equations of Brownian Motion

A. Formation of the Forward Equation

Differentiating the mgf partially with respect to the forward time variable gives the forward equation in the mgf

$$\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta) = \left(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2 \right) M_{X(T)|X(t_0)}(\theta)$$

From the definition of the transition mgf

$$M_{X(T)|X(t_0)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0)dx$$

Differentiating partially with respect to the forward diffusion variable

$$\frac{\partial}{\partial x} M_{X(T)|X(t_0)}(\theta) = \theta \int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0)dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(T)|X(t_0)}(x \mid x_0)dx = 0$$

and again

$$\frac{\partial^2}{\partial x^2} M_{X(T)|X(t_0)}(\theta) = -\theta^2 \int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0)dx + \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0)dx = 0$$

for

(1) $\theta M_{X(T)|X(t_0)}(\theta) = -\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(T)|X(t_0)}(x \mid x_0)dx$

(2) $\theta^2 M_{X(T)|X(t_0)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0)dx$

Hence the time derivative can be written in terms of the transition density as

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0)dx = -\mu \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(T)|X(t_0)}(x \mid x_0)dx + \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0)dx$$

which results in the forward equation of Brownian motion, the heat equation with convection, in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) - \mu \frac{\partial}{\partial x} f_{X(T)|X(t_0)}(x \mid x_0)$$

Alternatively expressed with the diffusion side differentially factorized, referred to by the literature[13] as the canonical form,

---

[12] sometimes referred to by the literature as the Doob-Meyer decomposition of a Martingale. Such literature includes

i. Rogers, L. C. G. and Williams, D., (2000) *Diffusions, Markov Processes and Martingales: Volume 1 Foundations, 2nd Edition.* Cambridge. Cambridge University Press.

ii. Rogers, L. C. G. and Williams, D., (2000) *Diffusions, Markov Processes and Martingales: Volume 2 ito-calculus, 2nd Edition*. Cambridge. Cambridge University Press.

[13] Bhattacharya, R. N. and Waymire E. C., (1990) *Stochastic Processes with Applications, 2nd Edition.* John Wiley & Sons.

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$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(e^{\mu x/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x} e^{-\mu x/\frac{1}{2}\sigma^2} f_{X(T)|X(t_0)}(x \mid x_0) \right)$$

B. Formation of the Backward Equation

Differentiating the mgf partially with respect to the backward time variable

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = \left(-\mu\theta - \tfrac{1}{2}\sigma^2\theta^2 \right) M_{X(T)|X(t_0)}(\theta)$$

Differentiating the mgf partially with respect to the backward diffusion variable

$$\frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) = \theta M_{X(T)|X(t_0)}(\theta)$$

and again

$$\frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta) = \theta^2 M_{X(T)|X(t_0)}(\theta)$$

Hence, the time derivative can be written as

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = -\mu \frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) - \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta)$$

which results the backward equation of Brownian motion, the Hermite conjugate of the heat equation with convection, in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\mu \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0) - \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} f_{X(T)|X(t_0)}(x \mid x_0)$$

which can alternatively be expressed in the differentially factorized form

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 e^{-\mu x_0/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x_0}\left(e^{\mu x_0/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0) \right)$$

Solution to the Forward Equation for Brownian Motion: (by Transforming to the Standard Heat Equation)

Transforming the equation by centering and scaling the diffusion variable to standard Brownian motion through the standardization transform that leaves the Brownian time dependence in the variance

$$Y(t) = \frac{X(t) - x(0) - \mu t}{\sigma}$$

which then measures the total number of infinitesimal standard normals within the forward transition that have been convolved onto the Dirac delta function at the backward variable. The transformed forward diffusion variable is then

$$y = \frac{x - x_0 - \mu t}{\sigma}$$

which has the partial derivatives

(1) $\frac{\partial y}{\partial x} = \frac{1}{\sigma}$ **(2)** $\frac{\partial^2 y}{\partial x^2} = 0$ and **(3)** $\frac{\partial y}{\partial t} = -\frac{\mu}{\sigma}$

The Jacobian normalized transformed transition density is

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$$f_{X(t)}(x) = \frac{1}{\sigma} f_{Y(t)}(y)$$

which has the partial derivatives

(1) $$\frac{\partial}{\partial x} f_{X(t)}(x) = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left(\frac{1}{\sigma} f_{Y(t)}(y)\right) = \frac{1}{\sigma^2}\frac{\partial}{\partial y} f_{Y(t)}(y)$$

(2) $$\frac{\partial^2}{\partial x^2} f_{X(t)}(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f_{X(t)}(x)\right) = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left(\frac{1}{\sigma^2}\frac{\partial}{\partial y} f_{Y(t)}(y)\right) = \frac{1}{\sigma^3}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y)$$

(3) $$\frac{\partial}{\partial t} f_{X(t)}(x) = \frac{1}{\sigma}\frac{\partial}{\partial t} f_{Y(t)}(y) + \frac{1}{\sigma}\frac{\partial y}{\partial t}\frac{\partial}{\partial y} f_{Y(t)}(y) = \frac{1}{\sigma}\frac{\partial}{\partial t} f_{Y(t)}(y) - \frac{\mu}{\sigma^2}\frac{\partial}{\partial y} f_{Y(t)}(y)$$

Substituting the partial derivatives into the forward equation for the process $X(t)$ gives

$$\frac{1}{\sigma}\frac{\partial}{\partial t} f_{Y(t)}(y) - \frac{\mu}{\sigma^2}\frac{\partial}{\partial y} f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2 \times \frac{1}{\sigma^3}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y) - \mu \times \frac{1}{\sigma^2}\frac{\partial}{\partial y} f_{Y(t)}(y)$$

which upon rearrangement and cancelation of terms results in the standard heat equation as the forward equation for the standardized process $Y(t)$

$$\frac{\partial}{\partial t} f_{Y(t)}(y) = \tfrac{1}{2}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y)$$

This is the forward equation of standard Brownian motion. It was shown in the previous section that the solution to the standard heat equation is the density function of a zero mean time scaled normal distribution,

$$Y(t) = \frac{X(t) - x_0 - \mu t}{\sigma} \sim N(0, t)$$

Rescaling this distribution by σ and adding back the mean at time t by translating the diffusion distribution upwards by $x_0 + \mu t$ gives another time scaled normal distribution

$$X(t) \sim N\left(x_0 + \mu t, \sigma^2 t\right)$$

which is the transition distribution of Brownian motion. The solution to the forward equation is then the corresponding density function,

$$f_{X(t)|X(0)}(x \mid x_0) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{[x - x_0 - \mu t]^2}{2\sigma^2 t}\right) \qquad f_X \in \{(x,t); -\infty < x < \infty, 0 \le t < \infty\}$$

Transformation to the standard heat equation is the equivalent route to an ito-transform pair and provides an alternative to ito calculus. Transforming the forward equation is the only available method when the diffusion is non-Gaussian or when the order of the diffusion is greater than one.

Solution to the Forward Equation for Brownian Motion: (Solving the Equation Directly)

The converting forward equation of the density function $f_t = \frac{1}{2}\sigma^2 f_{xx} - \mu f_x$ to the forward equation of the moment generating function by multiplying by an exponential factor in the forward diffusion variable and integrating over the range of the diffusion,

$$\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial}{\partial t} f_{X(t)}(x)dx = \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x}\frac{\partial^2}{\partial x^2} f_{X(t)}(x)dx - \mu\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial}{\partial x} f_{X(t)}(x)dx$$

and by the earlier results concerning moment generating functions of derivatives of the density function, the forward equation of the moment generating function becomes $M_t = (\frac{1}{2}\sigma^2\theta^2 + \mu\theta)M$,

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$$\frac{\partial}{\partial t} M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 M_{X(t)}(\theta) + \mu\theta M_{X(t)}(\theta)$$

Dividing both sides by $M_{X(t)}(\theta)$ gives the simple exact differential $K_t = \tfrac{1}{2}\sigma^2\theta^2 + \mu\theta$ as the forward equation in the cumulant generating function

$$\frac{\partial}{\partial t} \log M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 + \mu\theta$$

which is saying that the time derivative of the first and second cumulants of the diffusion are constant (the others are zero) and therefore, upon integration, sum linearly with time. Integrating with respect to time and remembering that the initial condition is $M_{X(0)}(\theta) = E(e^{x_0\theta}) = e^{x_0\theta}$, i.e., $K_{X(0)}(\theta) = \log M_{X(0)}(\theta) = x_0\theta$, gives

$$\int_0^t d\log M_{X(y)}(\theta) = \int_0^t (\tfrac{1}{2}\sigma^2\theta^2 + \mu\theta)dy$$

$$\log M_{X(t)}(\theta) = \theta x_0 + \tfrac{1}{2}\sigma^2\theta^2 t + \mu\theta t$$

The exponential of which is

$$M_{X(t)}(\theta) = e^{x_0\theta + \mu t\theta + \frac{1}{2}\sigma^2 t\theta^2}$$

This is the moment generating function of a normal with mean $x_0 + \mu t$ and variance $\sigma^2 t$, so the transition distribution is

$$X(t) \sim N\left(x_0 + \mu t, \sigma^2 t\right)$$

and the forward equation for the density function, therefore, is satisfied by the associated normal density function with these parameters as per the alternative method of solution

$$f_{X(t)|X(0)}(x \mid x_0) = \frac{1}{\sigma\sqrt{2\pi t}} \exp\left\{-\frac{(x - x_0 - \mu t)^2}{2t\sigma^2}\right\} \qquad f_{X_t} \forall \{(x,t) : x \in (-\infty,\infty), t \in [0,\infty)\}$$

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The Transition Distribution of Brownian Motion

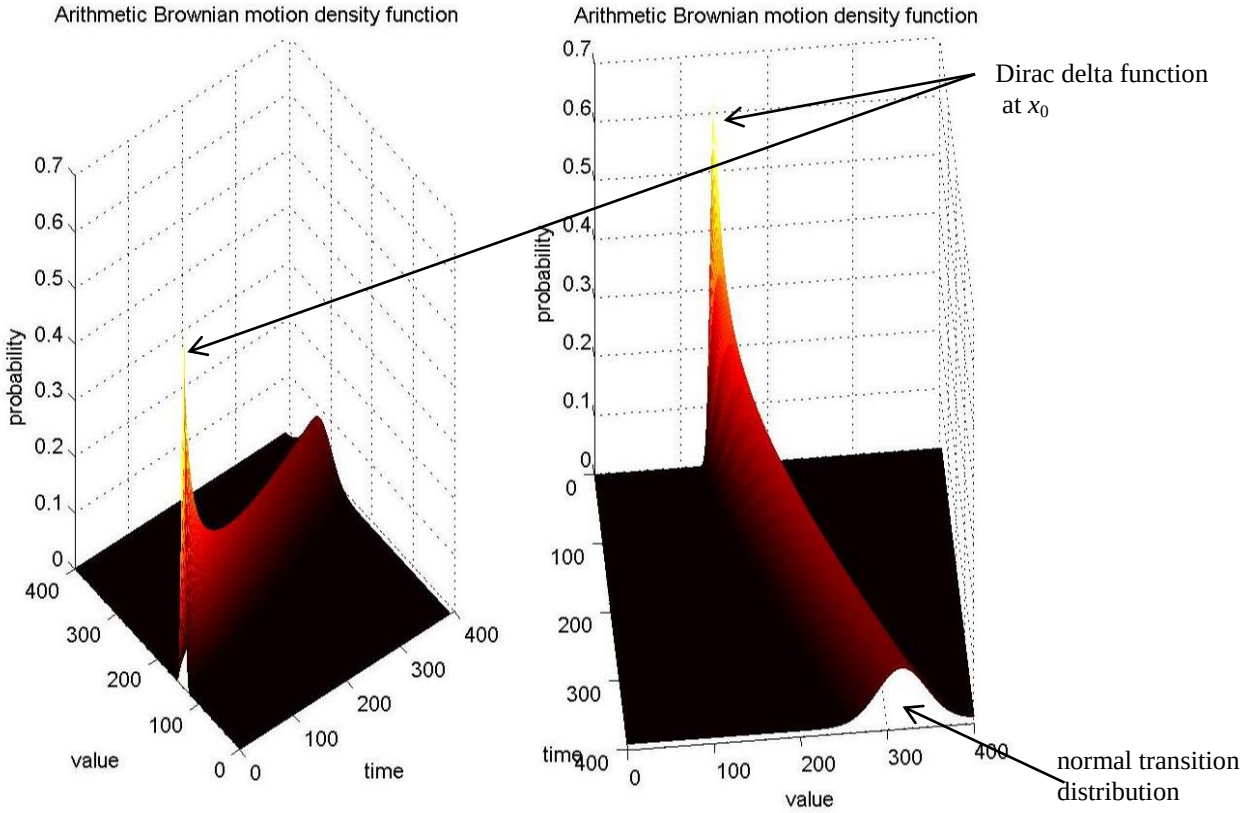


Figure 4: Arithmetic Brownian motion transition density surface

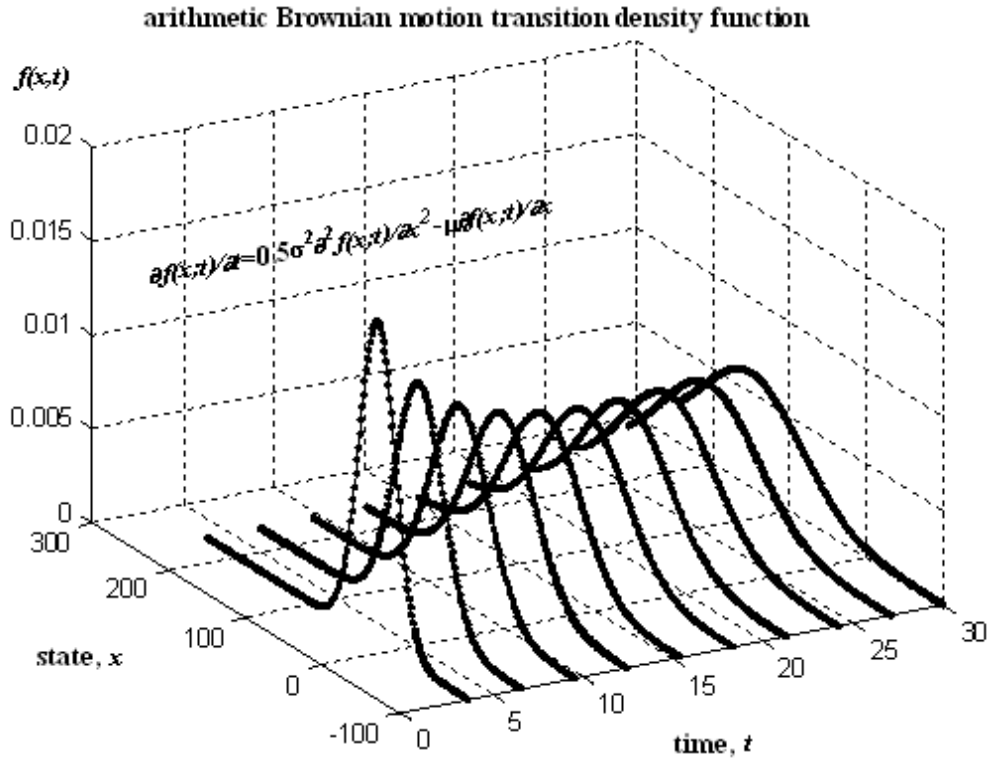


Figure 5: Successive arithmetic Brownian motion transition density curves

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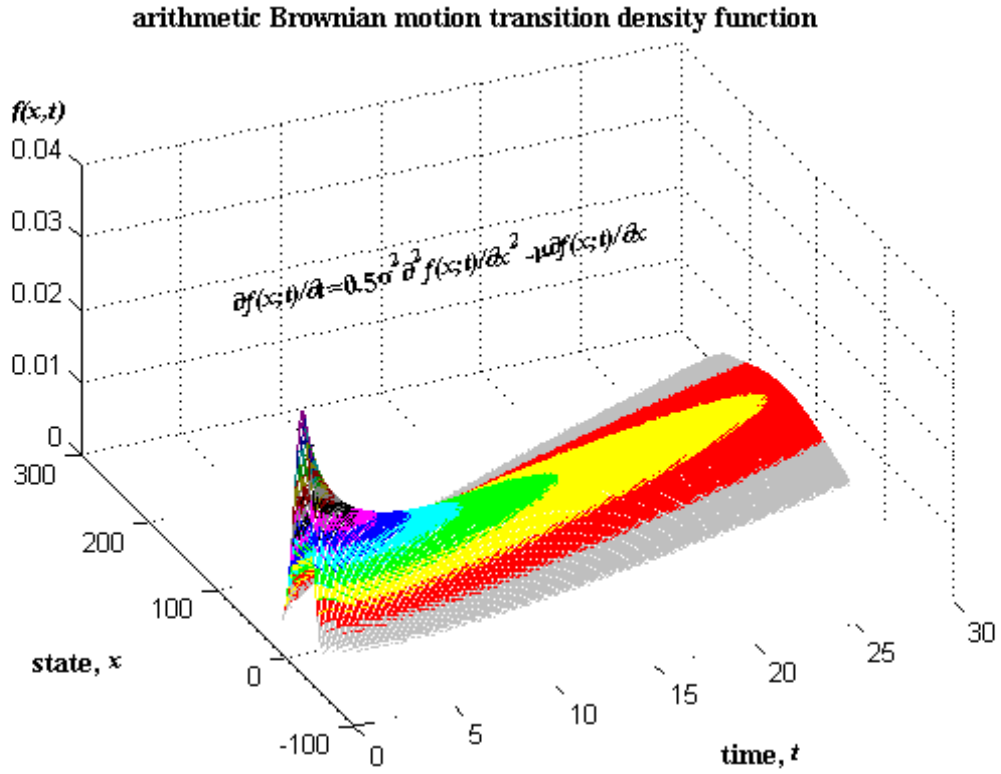


Figure 6: Arithmetic Brownian motion transition density surface with parameters (μ, σ) = (5,8)

The diagrams show the transition density surface of a Brownian motion $f_{X(t)|X(0)}(x\,|\,x_0)$, the general solution to the Kolmogorov forward equation as it evolves through time t , for the diffusion with infinitesimal parameters μ and σ^2 . The transition density function is an increasingly dispersed mesokurtic normal bell starting at a singularity at the initial value x_0 , and the center for which each bell is located rising at a constant linear rate of drift μ per unit time. This appears visually as a dispersing probability corridor. This dispersing corridor is generic to diffusion and the dynamics of the dependent variable of the heat equation. This is also a statement of the second law of thermodynamics, that the entropy (uncertainty as measured by expected log-likehood) of a closed system must only increase. This is evidenced by the monotone increasing diffusion entropy function. Brownian motion is characterized by independent additive increments and a probability distribution that is closed to the operation of convolving those increments. As such, the increasingly dispersed bell that results as the transition density extends through time results from the continuous cumulative addition, or convolution, of homogenous infinitesimal component bell curves and aggregation of probability. For each point in time and for each univariate bell, there are scalar valued statistical descriptors including the limited moments of undiscounted option premia. When the basis variance zeroes $\sigma \to 0$, this surface becomes a Dirac singularity traced out over the linear drift line, i.e., the diffusion equation becomes $X(t) \to x_0 + \mu t$ for which the transition density is

$$f_{X(t)|X(0)}(x\,|\,x_0) = \delta_{x_0+\mu t}(x)$$

Covariance and Joint Transition Distribution

The autocorrelation of the process at two successive points in time (s, t) has the general form of the covariance function for a process with independent increments,

$$Cov(X_S, X_t) = Cov(\mu s + \sigma B_s, \mu t + \sigma B_t) = \sigma^2 Cov(B_s.B_t) = \sigma^2(s \wedge t) = \sigma^2 \min(s,t) = s\sigma^2$$

Since the diffusion is Gaussian, the auto-joint transition distribution is specified by the marginal transition distributions and the auto joint covariance function. The auto-joint transition distribution is therefore bivariate normal;

$$[X_s, X_t]^T \sim BivN\left(\begin{bmatrix} x_0 + \mu s \\ x_0 + \mu t \end{bmatrix}, \begin{bmatrix} \sigma^2 s & \sigma^2(s \wedge t) \\ \sigma^2(s \wedge t) & \sigma^2 t \end{bmatrix}\right)$$

The auto-joint transition moment generating function is

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$$M_{X_s,X_t}(\theta_1,\theta_2) = \exp\left((x_0+\mu s)\theta_1 + \tfrac{1}{2}\sigma^2 s\theta_1^{\,2} + (x_0+\mu t)\theta_2 + \tfrac{1}{2}\sigma^2 t\theta_2^{\,2} + \sigma^2(s\wedge t)\theta_1\theta_2 \right)$$

The moment generating function of any two non-overlapping increments is the product of the two moment generating functions since non-overlapping increments are uncorrelated.

Further Statistical Description of Brownian Motion

From the transition distribution of drifting Brownian motion;

1) Quantiles

The quantiles are those obtained directly from scaling and shifting the inverse of the distribution function of a standard normal;

$$Q_{X_t}(p) = x_0 + \mu t + \sigma\sqrt{t}\,\Phi^{-1}(p)$$

From this function, the lower and upper quartiles of a drifting Brownian motion are

$$Q_{X_t}(0.25) = x_0 + \mu t - 0.6745\sigma\sqrt{t} \qquad \text{and} \qquad Q_{X_t}(0.75) = x_0 + \mu t + 0.6745\sigma\sqrt{t}\,)$$

while the inter-quartile range is

$$\begin{aligned} IQR(X_t) &= Q(0.75) - Q(0.25) \\ &= \left(x_0 + \mu t + 0.6745\,\sigma\sqrt{t}\right) - \left(x_0 + \mu t - 0.6745\,\sigma\sqrt{t}\right) \\ &= 1.3490\,\sigma\sqrt{t} \end{aligned}$$

For a 95% confidence band on the process at time t, the 2.5% and 97.5% percentiles are

$$Q(0.025) = x_0 + \mu t - 1.96\sigma\sqrt{t} \qquad \text{and} \qquad Q(0.975) = x_0 + \mu t + 1.96\sigma\sqrt{t}$$

The probability interval for a Brownian motion at time t with linear drift at rate μ is then

$$\begin{aligned} CI(X_t, 0.95) &= \left(Q(0.025), Q(0.975)\right) \\ &= x_0 + \mu t \pm 1.96\,\sigma\sqrt{t} \end{aligned}$$

2) Central Moments

The relevant normal calculations for the central moments with generic distribution $X \sim N(\mu, \sigma^2)$ are

$$\sigma_r(X) = \int_{-\infty}^{\infty} (x-\mu)^r \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \text{with } y = \frac{x-\mu}{\sigma} \text{ and } x = \mu + \sigma y \text{ as well as } \frac{dy}{dx} = \frac{1}{\sigma}$$

giving $\sigma_r(X) = \sigma^r \int_{-\infty}^{\infty} y^r \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = 0$ if r is odd, and if r is even,

$$\sigma_r(X) = \frac{\sigma^r 2^{(r-1)/2}}{\sqrt{2\pi}} \int_0^{\infty} u^{(r-1)/2} e^{-u} du = \sigma^r \frac{2^{(r+1)/2}}{\sqrt{2\pi}} \Gamma\left(\frac{r+1}{2}\right)$$

For Brownian motion $X_t \sim N(\mu t, \sigma^2 t)$ that has variance linearly related with time t,

$$\sigma_r(X_t) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \sigma^r t^{r/2} \dfrac{2^{(r+1)/2}}{\sqrt{2\pi}} \Gamma\left(\dfrac{r+1}{2}\right) & \text{if } r \text{ is even} \end{cases}$$

so that the first four non-zero central moments to order eight are

(1) $\sigma_2(X_t) = \sigma^2 t$, **(2)** $\sigma_4(X_t) = 3\sigma^4 t^2$, **(3)** $\sigma_6(X_t) = 15\sigma^6 t^3$, and **(4)** $\sigma_8(X_t) = 105\sigma^8 t^4$.

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3) Signal to Noise Ratio

After accounting for initial values, the signal to noise ratio of a Brownian motion $\mu t / \sigma^2 t = \mu / \sigma^2$ is time invariant and therefore constant. This is one of the important features of Brownian motion and first order unit-root diffusion.

4) Scaled Measures

Scaling the central moments of Brownian motion by the r^{th} power of the square root of the variance, $\sqrt{\sigma^{2r} t^r} = \sigma^r t^{r/2}$, leaves time invariant statistics peculiar to unit-root diffusion that are simply the central moments of a standard normal bell with unit scale,

$$\zeta_r(X_t) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \dfrac{2^{(r+1)/2}}{\sqrt{2\pi}} \Gamma\left(\dfrac{r+1}{2}\right) & \text{if } r \text{ is even} \end{cases}$$

This is related to the fact that a Brownian motion retains its statistical properties for any time horizon due to the unit-root and consequently does not have an ergodic distribution. The second, fourth, sixth, and eight standardized indices of shape are 1, 3, 15, and 105 respectively.

5) Raw Moments

The relevant normal calculations for the raw moments with generic distribution $X \sim N(\mu, \sigma^2)$ are

$$\mu_r(X) = \int_{-\infty}^{\infty} x^r \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad \text{with } y = \frac{x-\mu}{\sigma} \text{ and } x = \mu + \sigma y \text{ as well as } \frac{dy}{dx} = \frac{1}{\sigma}$$

giving $\mu_r(X) = \int_{-\infty}^{\infty} (\mu + \sigma y)^r \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$

so that

$$\begin{aligned}
\mu_2(X) &= \int_{-\infty}^{\infty} (\mu + \sigma y)^2 \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \int_{-\infty}^{\infty} (\mu^2 + 2\mu\sigma y + \sigma^2 y^2) \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \mu^2 E(Y^0) + 2\mu\sigma E(Y) + \sigma^2 E(Y^2) \\
&= \mu^2 \times 1 + 2\mu\sigma \times 0 + \sigma^2 \times 1 \\
&= \mu^2 + 2\mu\sigma + \sigma^2 \\
\mu_3(X) &= \int_{-\infty}^{\infty} (\mu + \sigma y)^3 \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \int_{-\infty}^{\infty} (\mu^3 + 3\mu^2\sigma y + 3\mu\sigma^2 y^2 + \sigma^3 y^3) \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \mu^3 E(Y^0) + 3\mu^2\sigma E(Y) + 3\mu\sigma^2 E(Y^2) + \sigma^3 E(Y^3) \\
&= \mu^3 \times 1 + 3\mu^2\sigma \times 0 + 3\mu\sigma^2 \times 1 + \sigma^3 \times 0 \\
&= \mu^3 + 3\mu\sigma^2 \\
\mu_4(X) &= \int_{-\infty}^{\infty} (\mu + \sigma y)^4 \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \int_{-\infty}^{\infty} (\mu^4 + 4\mu^3\sigma y + 6\mu^2\sigma^2 y^2 + 4\mu\sigma^3 y^3 + \sigma^4 y^4) \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\
&= \mu^4 E(Y^0) + 4\mu^3\sigma E(Y) + 6\mu^2\sigma^2 E(Y^2) + 4\mu\sigma^3 E(Y^3) + \sigma^4 y^4 E(Y^4) \\
&= \mu^4 \times 1 + 4\mu^3\sigma \times 0 + 6\mu^2\sigma^2 \times 1 + 4\mu\sigma^3 \times 0 + \sigma^4 \times 3 \\
&= \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4
\end{aligned}$$

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With the cumulants of the Brownian motion, μt and $\sigma^2 t$, these become

(1) $\mu_2(X_t) = (x_0 + \mu t)^2 + (2\mu\sigma + \sigma^2)t^2$
(2) $\mu_3(X_t) = (x_0 + \mu t)^3 + 3\mu\sigma^2 t^3$
(3) $\mu_4(X_t) = (x_0 + \mu t)^4 + (6\mu^2\sigma^2 + 3\sigma^2)t^4$

6) Uncertainty and Information – Diffusion Entropy

The entropy of a normal is $\log\sqrt{2\pi e\ \sigma^2}$. Hence, the entropy of Brownian motion is

$$H(X_t) = E[-\log f_{X_t}(X_t)] = \int_{-\infty}^{\infty} f_{X_t}(x) \log f_{X_t}(x) dx = \log\sqrt{2\pi\ e\ \sigma^2 t}$$

Uncertainty as measured by expected likelihood, entropy, depends only on the scale of the Brownian motion and the time between the forward and backward variables. Entropy of Brownian motion is independent of linear drift and the initial conditions and is a slowly varying function at infinity.

Simulation Algorithm

Considering that the solution to the stochastic differential increment $dX_t = \mu dt + \sigma dB_t$ is obtained by integration,

$$\begin{aligned} X_t &= x_0 + \mu t + \sigma\int_0^t dB_t \\ &= x_0 + \mu t + \sigma B_t \\ &\sim x_0 + \mu t + \sigma N(0,1) \\ &\sim N(x_0 + \mu t, \sigma^2 t) \end{aligned}$$

and applying the same principles in order to discretize the differential increment by integrating over a span $h = \Delta t$ so that an interval t has t / h subintervals, and then summing the component integrals,

$$\Delta X_t = \mu\Delta t + \sigma\Delta B_t$$

$$\begin{aligned} X_t &= \sum_{y=0}^{t/\Delta t} \Delta X_y \\ &= x_0 + \mu t + \sigma\sum_{j=0}^{n} \Delta z_j \end{aligned}$$

where $\Delta X_{k\Delta t} = \int_{k\Delta t}^{k\Delta t+\Delta t} dX_t$ so that

$$\Delta X_t \sim \mu\Delta t + \sigma N(0, \Delta t)$$

$$\begin{aligned} X_t &= \sum_{y=0}^{t/\Delta t} \Delta X_y \\ &\sim x_0 + \mu t + \sigma\sum_{j=0}^{t/\Delta t} N(0, \Delta t) \\ &\sim x_0 + \mu t + \sigma\sqrt{\Delta t}\sum_{j=0}^{n} N(0,1) \end{aligned}$$

That is, generate t / h increments with unit normal variates, calculate the vector of the cumulative sum of the increments, scale the vector upwards by multiplying by a factor of $\sigma\sqrt{\Delta t}$ and then add a vector of the linear drift starting at the initial value x_0 of the simulated diffusion. This gives a vector of observed points of a scaled drifting Brownian path. The MATLAB code to do this is

```
x0=x0; mu=μ; sigma=σ; h=h; n=n;          %initialize diffusion parameters
z=mu.*h+sigma.*randn(n,1).*sqrt(h); x=cumsum([x0;z]);          %simulate
plot(x);                                                          %plot
```

where `simulate` refers to generating and summing the basis.

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Simulated Arithmetic Brownian Motion

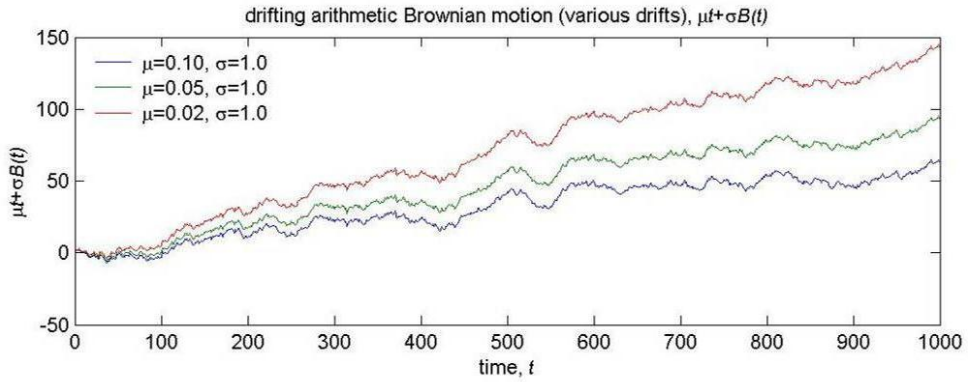


Figure 7: Single simulated path of drifting Brownian motion for various drifts

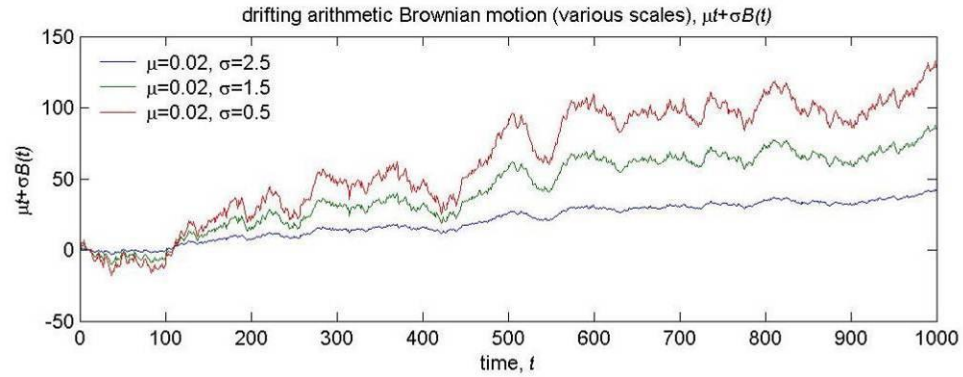


Figure 8: Single simulated path of drifting Brownian motion for various scales

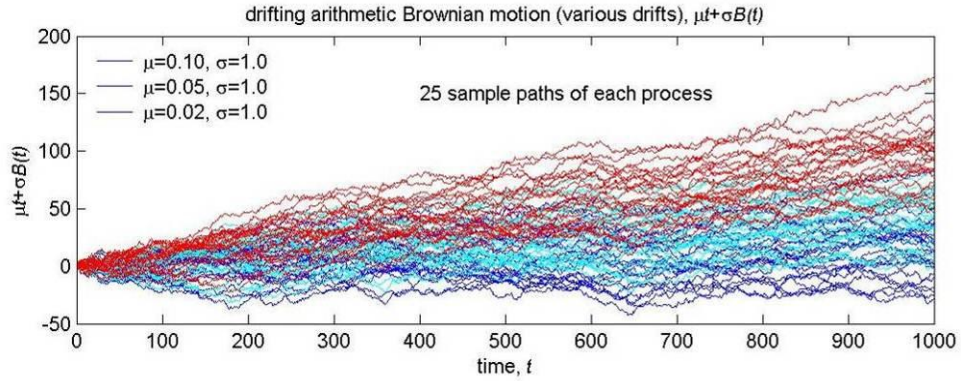


Figure 9: 25 simulated paths of drifting Brownian motion for various drifts

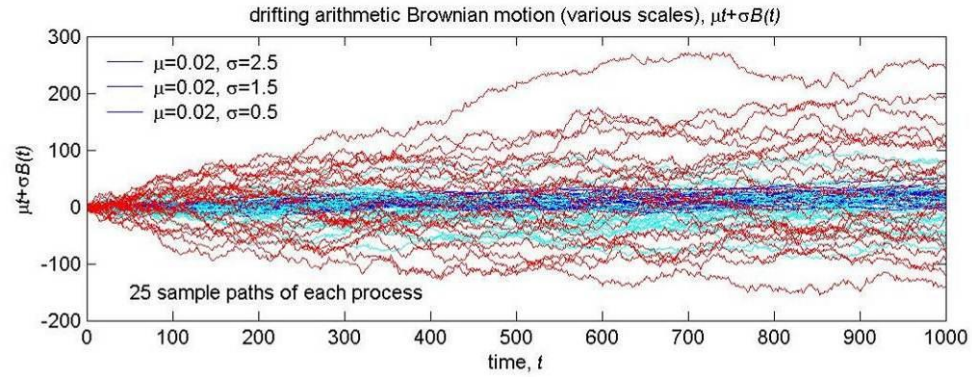


Figure 10: 25 simulated paths of drifting Brownian motion for various scales

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The MGF of an Inverse Gaussian

The density of an inverse Gaussian distribution with parameters a and b , $X \sim IG(a,b)$, is

$$f_T(t) = \frac{\sqrt{b}}{\sqrt{2\pi}} t^{-3/2} \exp\left\{ -\tfrac{1}{2}\frac{b}{a}\left(\frac{\sqrt{a}}{\sqrt{t}} - \frac{\sqrt{t}}{\sqrt{a}} \right)^2 \right\}$$

For the moment generating function,

$$M_{IG(a,b)}(\theta)$$

$$= \int_0^\infty e^{\theta t} \frac{\sqrt{b}}{\sqrt{2\pi}} t^{-3/2} e^{-\frac{1}{2}\frac{b}{a}\left(\frac{\sqrt{a}}{\sqrt{t}} - \frac{\sqrt{t}}{\sqrt{a}} \right)^2} dt$$

$$= \int_0^\infty \frac{\sqrt{b}}{\sqrt{2\pi}} t^{-3/2} \exp\left\{ \theta t - \tfrac{1}{2}\frac{b}{a}\left(\frac{\sqrt{a}}{\sqrt{t}} - \frac{\sqrt{t}}{\sqrt{a}} \right)^2 \right\} dt$$

$$= \exp\left\{ \frac{b}{a}\left(1 - \sqrt{1 - 2\theta a^2 / b} \right) \right\} \int_0^\infty \frac{\sqrt{b}}{\sqrt{2\pi}} t^{-3/2} \exp\left\{ -\frac{b}{a}\sqrt{1 - 2\theta a^2 / b}\left(\frac{\sqrt{a}}{\sqrt{t}(1 - 2\theta a^2 / b)^{1/4}} - \frac{\sqrt{t}(1 - 2\theta a^2 / b)^{1/4}}{\sqrt{a}} \right) \right\} dt$$

where the integrand is the density of an inverse Gaussian distribution with parameters $a / \sqrt{1 - 2\theta a^2 / b}$ and b and hence integrates to one

$$M_{IG(a,b)}(\theta) = \exp\left\{ \frac{b}{a}\left(1 - \sqrt{1 - 2\theta a^2 / b} \right) \right\} \int_0^\infty f_{IG(a/\sqrt{1-2\theta a^2/b}, b)}(t) dt \qquad \text{with } \int_0^\infty f_{IG(a/\sqrt{1-2\theta a^2/b}, b)}(t) dt = 1$$

so that the moment generating function of an inverse Gaussian distribution with parameters is a and b is

$$M_{IG(a,b)}(\theta) = \exp\left\{ \frac{b}{a}\left(1 - \sqrt{1 - 2\theta a^2 / b} \right) \right\}$$

First Passage Time of Drifting Brownian Motion

The forward equation of arithmetic Brownian motion for the transition density function of the time over state or first passage time distribution is

$$\frac{\partial f_{T(x)}(t)}{\partial t} = \tfrac{1}{2}\frac{\partial^2 f_{T(x)}(t)}{\partial x^2} - \mu \frac{\partial f_{T(x)}(t)}{\partial x}$$

Converting the forward partial differential equation for the transition density to an ordinary differential equation in the moment generating function by multiplying both sides by $e^{\theta t}$ and integrating over the range of the first passage time distribution,

$$\int_0^\infty e^{\theta t} \frac{\partial f_{T(x)}(t)}{\partial t} dt = \tfrac{1}{2}\int_0^\infty e^{\theta t} \frac{\partial^2 f_{T(x)}(t)}{\partial x^2} dt - \mu \int_0^\infty e^{\theta t} \frac{\partial f_{T(x)}(t)}{\partial x} dt$$

$$-\theta M_{T(x)}(\theta) = \tfrac{1}{2}\frac{\partial^2}{\partial x^2} M_{T(x)}(\theta) - \mu \frac{\partial}{\partial x} M_{T(x)}(\theta)$$

giving

$$0 = \tfrac{1}{2}\frac{\partial^2}{\partial x^2} M_{T(x)}(\theta) - \mu \frac{\partial}{\partial x} M_{T(x)}(\theta) + \theta M_{T(x)}(\theta)$$

which has characteristic polynomial $0 = \frac{1}{2}\sigma^2\lambda^2 - \mu\lambda + \theta$ and characteristic roots

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$$\lambda_1(\theta)=\frac{\mu-\sqrt{\mu^2-2\theta\sigma^2}}{\sigma^2} \qquad \text{and} \qquad \lambda_2(\theta)=\frac{\mu+\sqrt{\mu^2-2\theta\sigma^2}}{\sigma^2}$$

Factorizing the derivative operators of the equation,

$$0=e^{\lambda_2(\theta)x}\frac{d}{dx}e^{-\lambda_2(\theta)x}\left(e^{\lambda_1(\theta)x}\frac{d}{dx}e^{-\lambda_1(\theta)x}M_{T_x}(\theta)\right)$$

removing the normalizing exponential factor by division and proceeding in the same manner as standard Brownian motion by integrating from the starting value of the diffusion x_0 ,

$$\int_{x_0}^{x}\frac{d}{dy}e^{-\lambda_2(\theta)y}\left(e^{\lambda_1(\theta)y}\frac{d}{dy}e^{-\lambda_1(\theta)y}M_{T_y}(\theta)\right)dy=0$$

$$\left[\frac{d}{dy}e^{-\lambda_2(\theta)y}\left(e^{\lambda_1(\theta)y}\frac{d}{dy}e^{-\lambda_1(\theta)y}M_{T(y)}(\theta)\right)\right]_{x_0}^{x}=0$$

$$e^{-\lambda_2(\theta)x}\left(e^{\lambda_1(\theta)x}\frac{d}{dx}e^{-\lambda_1(\theta)x}M_{T(x)}(\theta)\right)-\left[\tfrac{1}{2}\frac{\partial^2}{\partial x_0^2}M_{T(x_0)}(\theta)-\mu\frac{\partial}{\partial x}M_{T(x_0)}(\theta)+\theta M_{T(x_0)}(\theta)\right]=0$$

with the initial conditions at the lower limit of integration

$$\tfrac{1}{2}\frac{\partial^2}{\partial x_0^2}M_{T(x_0)}(\theta)-\mu\frac{\partial}{\partial x}M_{T(x_0)}(\theta)+\theta M_{T(x_0)}(\theta)=0$$

and

$$M_{T(x_0)}(\theta)=\int_0^\infty e^{t\theta}f_{T(x)|T(x_0)}(t)dt=\int_0^\infty e^{t\theta}\delta_0(t)dt=e^{0\times\theta}=1$$

the time it takes to get where it is at the start, which is where it is now, is zero. The equation reduces to

$$e^{-\lambda_2(\theta)x}\left(e^{\lambda_1(\theta)x}\frac{d}{dx}e^{-\lambda_1(\theta)x}M_{T(x)}(\theta)\right)=0 \qquad \text{or} \qquad \frac{d}{dx}e^{-\lambda_1(\theta)x}M_{T(x)}(\theta)=0$$

which is an exact derivative. Integrating again from the starting value of the diffusion to the forward diffusion variable for which the first passage time distribution relates, x ,

$$\int_{x_0}^{x}\left(\frac{d}{dy}e^{-\lambda_1(\theta)y}M_{T(y)}(\theta)\right)dy=0$$

or

$$e^{-\lambda_1(\theta)x}M_{T(x)}(\theta)=e^{-\lambda_1(\theta)x_0}M_{T(x_0)}(\theta)$$

and with the initial condition $M_{T(x_0)}(\theta)=e^{t_0\theta}=1$

$$e^{-\lambda_1(\theta)x}M_{T(x)}(\theta)=e^{-\lambda_1(\theta)x_0}$$

Finally, the moment generating function of the first passage time distribution of standard Brownian motion is

$$M_{T(x)}(\theta)=e^{\lambda_1(\theta)(x-x_0)}$$

and by inserting the first root of the characteristic equation into the equation, the root with the negative sign so that the Polya criterion holds $\lim_{\theta\to\infty}\left\|M_{T(x)}(\theta)\right\|\to 0+i0$, the moment generating function becomes

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$$M_{T(x)}(\theta) = \exp\left((x - x_0) \frac{\mu - \sqrt{\mu^2 - 2\theta\sigma^2}}{\sigma^2} \right)$$

which is the moment generating function of an inverse Gaussian distribution with parameters $(x - x_0) / \mu$ and $(x - x_0)^2 / \sigma^2$,

$$\begin{aligned} T_x(x_0) &= T_{x - x_0} \\ &\sim IG\left(\frac{x - x_0}{\mu}, \frac{(x - x_0)^2}{\sigma^2} \right) \\ &\sim FPT(x - x_0, \mu, \sigma^2) \end{aligned}$$

and hence, the transition density of this distribution and the solution to the forward equation is

$$f_{T(x)|T(x_0)}(t) = \frac{x - x_0}{\sigma\sqrt{2\pi}} t^{-3/2} \exp\left(- \frac{[x - x_0 - \mu t]^2}{2\sigma^2 t} \right)$$

The cumulative survival distribution is given by

$$S_{T(x)|T(x_0)}(t) = \Phi\left(\frac{x - x_0 - \mu t}{\sigma\sqrt{t}} \right) - \exp\left(\frac{2\mu(x - x_0)}{\sigma^2} \right) \Phi\left(\frac{-x + x_0 - \mu t}{\sigma\sqrt{t}} \right)$$

and the distribution function is

$$F_{T(x)|T(x_0)}(t) = \Phi\left(\frac{-x + x_0 + \mu t}{\sigma\sqrt{t}} \right) + \exp\left(\frac{2\mu(x - x_0)}{\sigma^2} \right) \Phi\left(\frac{-x + x_0 - \mu t}{\sigma\sqrt{t}} \right)$$

where $\Phi(\cdot) = F_Y(\cdot)$ is the cumulative distribution function of a standard normal $Y \sim N(0,1)$. The technique for solving the forward equation for the first passage time distribution, for an arbitrary transformation of a Brownian motion, is to first transform the diffusion variable to the heat equation, leaving the convection term if it is present, and solving this for the first passage time distribution and then transforming back to the original diffusion variable. The diffusion variable is a parameter in the distribution of the first passage time and the density function and the transformation back to the original diffusion variable is not statistical. The Jacobian of the transformation of the original diffusion variable is required to get the transformed Kolmogorov equation to the correct form. The Jacobian is not required for transforming the diffusion variable back since the diffusion variable is a parameter in the distribution of the first passage time.

Limit Theorem of the First Passage Time of Brownian Motion

The cumulant generating function is

$$K_{T_{x-y}}(\theta) = (x - y) \frac{\mu - \sqrt{\mu^2 - 2\sigma^2\theta}}{\sigma^2}$$

and the first four derivatives of the cumulant generating function are

(1) $K^1_{T_{x-y}}(\theta) = \dfrac{x - y}{\sqrt{\mu^2 - 2\sigma^2\theta}}$

(2) $K^{(2)}_{T_{x-y}}(\theta) = \dfrac{\sigma^2(x - y)}{(\mu^2 - 2\sigma^2\theta)^{3/2}}$

(3) $K^{(3)}_{T_{x-y}}(\theta) = \dfrac{3\sigma^4(x - y)}{(\mu^2 - 2\sigma^2\theta)^{5/2}}$

(4) $K^{(4)}_{T_{x-y}}(\theta) = \dfrac{15\sigma^6(x - y)}{(\mu^2 - 2\sigma^2\theta)^{7/2}}$

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which give the first four cumulants, the mean, variance, skewness, and excess kurtosis

(1) $\kappa_1(T_{x-y}) = (x-y)/\mu$

(2) $\kappa_2(T_{x-y}) = \sigma^2(x-y)/\mu^3$

(3) $\kappa_3(T_{x-y}) = 3\sigma^4(x-y)/\mu^5$

(4) $\kappa_4(T_{x-y}) = 15\sigma^6(x-y)/\mu^7$

The coefficient of skewness is

$$\zeta_3(T_{x-y}) = \frac{\kappa_3(T_{x-y})}{\kappa_2^{3/2}(T_{x-y})} = \frac{3\sigma}{\sqrt{\mu}\sqrt{x-y}}$$

and the coefficient of kurtosis is

$$\zeta_4(T_{x-y}) = 3 + \frac{\kappa_4(T_{x-y})}{\kappa_2^2(T_{x-y})} = 3 + \frac{15\sigma^2}{\mu(x-y)}$$

As the first passage time for which the process variable measures $x-y$ becomes large, the standardized variable becomes a zero skew mesokurtic bell, $\zeta_3(T_{x-y}) \to 0$ and $\zeta_4(T_{x-y}) \to 3$ as $x-y \to \infty$ and is hence normal. Therefore, the first passage time process, the process of a Brownian motion over the time dimension, is an asymptotic normal process. Further, since the mean and variance of the distribution over the time dimension is proportional to the width of the increment in the state dimension, $x-y$, and all higher order standardized cumulants diminish in importance since the r^{th} order cumulant is of order $(x-y)^{1-r/2}$, it is also an asymptotic Brownian motion,

$$T_{x-y} \sim N\left(\frac{x-y}{\mu}, \frac{\sigma^2(x-y)^2}{\mu^3}\right)$$

as $x-y$ becomes large. This is an example of the Central Limit Theorem, for which Brownian motion and the heat equation is the domain of attraction as asserted by this thesis. Because the probability density of the inverse Gaussian distribution is the solution to the backward equation for time over state and the fact that the distribution of the density is referred to in this context as the first passage time distribution, it is useful to define an alternative distributional notation, *FPT*. This distributional notation is parameterized in a way consistent with the forward transition distribution of the diffusion and more amenable to distributional work,

$$\begin{aligned} T_{a-x} &\sim IG\left(\frac{a-x}{\mu}, \frac{(a-x)^2}{\sigma^2}\right) \\ &\sim FPT(a-x, \mu, \sigma^2) \end{aligned}$$

Sums of such variables, if independent, correspond to sums of the first argument in the parameter set of the *FPT* parameterized distribution,

$$\begin{aligned} \sum_{j=1}^{n} T(a_j) &\sim IG\left(\frac{1}{\mu}\sum_{j=1}^{n} a_j, \frac{1}{\sigma^2}\left(\sum_{j=1}^{n} a_j\right)^2\right) \\ &\sim FPT\left(\sum_{j=1}^{n} a_j, \mu, \sigma^2\right) \end{aligned}$$

PROPORTIONAL BROWNIAN MOTION

Introduction

A proportional Brownian motion is a first order unit root exponential diffusion and is called geometric Brownian motion since it forms a group that is closed to geometric averaging. Hence, a diffusion X_t is a geometric Brownian motion if the ratio of successive values are log-normally distributed, with logarithmic variance equal to the width of the time increment over which any ratio applies. The logarithm of the successive ratios, or equivalently, the first differences of the logarithms of the diffusion are normally distributed with variance equal to the width of

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the time increment and forms an ordinary additive Brownian motion, say Y_t. In the context of modelling prices, the price relativities are log-normal so that the returns are normal, i.e., returns are Brownian motion. This implies an exponential transformation from an ordinary Brownian motion to geometric Brownian motion,

$$X_t = e^{Y_t}$$

This is the basic model of multiplicatively aggregated Gaussian variables and can be referred to as compound interest with first order multiplicative Gaussian variation. The infinitesimal parameters of geometric Brownian motion are proportional to the current value of the diffusion at any point, $\mu(x,t) = \mu x$ and $\sigma(x,t) = \sigma x$. which is why it is called proportional Brownian motion. The stochastic differential equation is

$$dX_t = \mu X_t dt + \sigma X_t dz_t$$

where $dB_t \sim N(0,dt)$ and $X_0 \sim \delta_{x(0)}(x)$. The logarithm of the diffusion variable transforms the diffusion to an ordinary additive Brownian motion. Using the change of variable technique from the stochastic calculus of first order diffusion, resulting in the Ito-correction for quadratic variation, results in the following two pairs of transformations;

Additive-Multiplicative Transformation Pairs for Brownian Motion

Geometric Brownian Motion to Arithmetic Brownian Motion

$dX_t = \mu X_t dt + \sigma X_t dB_t$ where $dX_t \sim N\left(\mu X_t dt, \sigma^2 X_t dt\right)$

$d\log X_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t$ where $d\log X_t \sim N\left((\mu - \frac{1}{2}\sigma^2)dt, \sigma^2 dt\right)$

Arithmetic Brownian Motion to Geometric Brownian Motion

$dY_t = \mu dt + \sigma dB_t$ where $dY_t \sim N\left(\mu dt, \sigma^2 dt\right)$

$de^{Y_t} = (\mu + \frac{1}{2}\sigma^2)e^{Y_t}dt + \sigma e^{Y_t}dB_t$ where[14] $de^{Y_t} \sim N\left((\mu + \frac{1}{2}\sigma^2)e^{Y_t}dt, \sigma^2 e^{Y_t}dt\right)$

Alternatively, transforming the forward or backward equations by the techniques of transforming dependent variables of partial differential equations give the equivalent results. The diffusion side of the partial differential equation for the additive arithmetic process, which is the heat equation $f_t = \frac{1}{2}\sigma^2 f_{xx}$ or the heat equation with drift, $f_t = \frac{1}{2}\sigma^2 f_{xx} - \mu f_x$, is an ordinary differential equation with constant coefficients while the diffusion side of the partial differential equation for the multiplicative geometric process is a Euler-Cauchy differential equation.

Integral Solution to Proportional Brownian Motion

From the ito transformation of $dX_t = \mu X_t dt + \sigma X_t dB_t$ to $d\log X_t = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dB_t$ by transforming the diffusion variable to logarithms by either the ito-calculus or transforming the forward equation, the differential in logarithms is exact and the integration proceeds as per the Brownian motion

$$\int_0^t d\log X_y = \int_0^t \left((\mu - \tfrac{1}{2}\sigma^2)dy + \sigma dB_y\right)$$

$$\log X_t - \log x_0 = \left[(\mu - \tfrac{1}{2}\sigma^2)y + \sigma B_y\right]_0^t$$

or

$$\log X_t = \log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma B_t$$

[14] Remembering that the distribution of the stochastic increment is defined in a conditional sense, a conditional random variable for the transition interval from t to $t + dt$; $dX(t) \mid X(t)$, $d\log X(t) \mid X(t)$, $dY(t) \mid Y(t)$, and $d\exp Y(t) \mid Y(t)$, although the conditional notation is omitted as shorthand.

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or

$$X_t = \exp\left(\log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma B_t\right)$$

so that

$$\begin{aligned}\log X_t &\sim \log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma N(0,t) \\ &\sim N\left(\log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t,\, \sigma^2 t\right)\end{aligned}$$

that is,

$$\begin{aligned}X_t &\sim \log N\left(\log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t,\, \sigma^2 t\right) \\ &\sim x_0 e^{(\mu - \frac{1}{2}\sigma^2)t} e^{\sigma N(0,1)}\end{aligned}$$

noting that X_t = constant × deterministic function × exponential standard Brownian motion

i.e.,

X_t = backward diffusion variable × exponential drift × first order multiplicative noise

which is the decomposition[15] of the multiplicative unit-root diffusion into a product of the initial value of the process (a fixed dollar sum, say), a deterministic compounding factor, and an exponential multiplicative noise component.

The MGF of a Log-Normal Distribution and Proportional Brownian Motion

The moment generating function of an exponentiated distribution is the mgf of the underlying distribution (the log of the exponential) with a Poisson distributed mgf parameter with Poisson mean at the mgf parameter

$$M_X(\theta) = e^{\theta} M_{\log X}(\Lambda) \qquad \text{where} \qquad \Lambda \sim P(\theta)$$

that is,

$$M_X(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j!} M_{\log X}(\theta)$$

This result holds for any variable that has been exponentiated, as a result of the definition of the mgf as an expectation of an exponential function and, alternatively, as a Taylor series expansion of an exponential function with additional coefficients containing the distribution's raw moments. Since a log-normal distribution

$$X \sim \log N(\mu, \sigma^2)$$

is the result of an exponential transformation of a normal distribution, $\log X \sim N(\mu, \sigma^2)$, which has mgf

$$M_{\log X}(\theta) = \exp(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2)$$

the mgf of a normal variable with a Poisson distributed mgf parameter gives the mgf of a log-normal variable

$$M_X(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j!} \exp\left(\mu j + \tfrac{1}{2}\sigma^2 j^2\right)$$

---

[15] This is sometimes referred to by the literature as the Doob-Meyer decomposition of an exponential Martingale. Such literature includes

i. Rogers, L. C. G. and Williams, D., (2000) *Diffusions, Markov Processes and Martingales: Volume 1 Foundations, 2nd Edition.* Cambridge. Cambridge University Press.

ii. Rogers, L. C. G. and Williams, D., (2000) *Diffusions, Markov Processes and Martingales: Volume 2 ito-calculus, 2nd Edition*. Cambridge. Cambridge University Press.

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In the context of a diffusion variable, the mgf of an exponentiated diffusion variable is a Poisson weighted sum of the mgf of the underlying logged variable

$$\begin{aligned}
M_{X(T)|X(t_0)}(\theta) &= E\left(e^{\theta X(T)} \mid X(t_0)\right) \\
&= E\left(e^{\theta e^{\log X(T)}} \mid X(t_0)\right) \\
&= E\left(1+\theta e^{\log X(T)} + \tfrac{1}{2}\theta^2 e^{2\log X(T)} + \tfrac{1}{3!}\theta^3 e^{3\log X(T)} + \ldots \mid X(t_0)\right) \\
&= 1+\theta E\left(e^{\log X(T)} \mid X(t_0)\right) + \tfrac{1}{2}\theta^2 E\left(e^{2\log X(T)} \mid X(t_0)\right) + \tfrac{1}{3!}\theta^3 E\left(e^{3\log X(T)} \mid X(t_0)\right) + \ldots \\
&= 1+\theta M_{\log X(T)|\log X(t_0)}(1) + \tfrac{1}{2}\theta^2 M_{\log X(T)|\log X(t_0)}(2) + \tfrac{1}{3!} M_{\log X(T)|\log X(t_0)}(3) + \ldots \\
&= \sum_{j=0}^{\infty} \frac{\theta^j}{j!} M_{\log X(T)|\log X(t_0)}(j)
\end{aligned}$$

Since the logged variable is normal,

$$\begin{aligned}
&\log X(T) \mid \log X(t_0) \\
&\quad \sim N\left(\log x_0 + \mu(T-t_0), \sigma^2(T-t_0)\right)
\end{aligned}$$

the mgf of the logged variable is

$$M_{\log X(T)|\log X(t_0)}(\theta) = \exp\left(\theta \log x_0 + \mu(T-t_0)\theta + \tfrac{1}{2}\sigma^2(T-t_0)\theta^2\right)$$

and the transition mgf of the proportional Brownian motion is consequently

$$M_{X(T)|X(t_0)}(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j!} \exp\left(j \log x_0 + \mu(T-t_0) j + \tfrac{1}{2}\sigma^2(T-t_0) j^2\right)$$

This function is used to form the forward equation and the backward equation of geometric Brownian motion, and is to obtained from solving the forward equation transformed from the transition density to that in the transition mgf.

Formation of the Forward and Backward Equations of Proportional Brownian Motion

A. Formation of the Forward Equation

Method 1

Differentiating the mgf partially with respect to the forward time variable gives the forward equation in the mgf

$$\begin{aligned}
\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta) &= \sum_{j=0}^{\infty} \left(\mu j + \tfrac{1}{2}\sigma^2 j^2\right) \frac{\theta^j}{j!} \exp\left(j \log x_0 + \mu(T-t_0) j + \tfrac{1}{2}\sigma^2(T-t_0) j^2\right) \\
&= \mu \sum_{j=0}^{\infty} (j) \frac{\theta^j}{j!} \exp\left(j \log x_0 + \mu(T-t_0) j + \tfrac{1}{2}\sigma^2(T-t_0) j^2\right) \\
&\qquad + \tfrac{1}{2}\sigma^2 \sum_{j=0}^{\infty} \left(j^2\right) \frac{\theta^j}{j!} \exp\left(j \log x_0 + \mu(T-t_0) j + \tfrac{1}{2}\sigma^2(T-t_0) j^2\right) \\
&= (\mu + \tfrac{1}{2}\sigma^2)\theta \sum_{j=0}^{\infty} (j) \frac{\theta^{j-1}}{j!} \exp\left(j \log x_0 + \mu(T-t_0) j + \tfrac{1}{2}\sigma^2(T-t_0) j^2\right) \\
&\qquad + \tfrac{1}{2}\sigma^2\theta^2 \sum_{j=0}^{\infty} \left(j(j-1)\right) \frac{\theta^{j-2}}{j!} \exp\left(j \log x_0 + \mu(T-t_0) j + \tfrac{1}{2}\sigma^2(T-t_0) j^2\right)
\end{aligned}$$

Differentiating partially with respect to the mgf parameter

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$$\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)=\sum\nolimits_{j=0}^{\infty}\ (\ j\ )\frac{\theta^{j-1}}{j!}\exp\Big(j\log x_0+\mu(T-t_0)j+\tfrac{1}{2}\sigma^2(T-t_0)j^2\Big)$$

and again

$$\frac{\partial^2}{\partial\theta^2}M_{X(T)|X(t_0)}(\theta)=\sum\nolimits_{j=0}^{\infty}\ \big(j(j-1)\big)\frac{\theta^{j-2}}{j!}\exp\Big(j\log x_0+\mu(T-t_0)j+\tfrac{1}{2}\sigma^2(T-t_0)j^2\Big)$$

Hence, the time derivative can be written for the forward equation in the transition mgf

$$\frac{\partial}{\partial T}M_{X(T)|X(t_0)}(\theta)=(\mu+\tfrac{1}{2}\sigma^2)\,\theta\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)+\tfrac{1}{2}\sigma^2\theta^2\frac{\partial^2}{\partial\theta^2}M_{X(T)|X(t_0)}(\theta)$$

Differentiating the mgf partially once and then twice with respect to both the forward diffusion variable and the mgf parameter, i.e.,

$$\frac{\partial^2}{\partial x\partial\theta}M_{X(T)|X(t_0)}(\theta)\qquad\text{and}\qquad\frac{\partial^4}{\partial x^2\partial\theta^2}M_{X(T)|X(t_0)}(\theta)$$

for

(1) $\theta\dfrac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)=-\displaystyle\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial}{\partial x}x\,f_{X(T)|X(t_0)}(x\,|\,x_0)dx$

(2) $\theta^2\dfrac{\partial^2}{\partial\theta^2}M_{X(T)|X(t_0)}(\theta)=\displaystyle\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial^2}{\partial x^2}x^2\,f_{X(T)|X(t_0)}(x\,|\,x_0)dx$

The forward equation can be written in terms of the transition density as

$$\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial}{\partial T}f_{X(T)|X(t_0)}(x\,|\,x_0)dx=(\mu+\tfrac{1}{2}\sigma^2)\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial}{\partial x}x\,f_{X(T)|X(t_0)}(x\,|\,x_0)dx+\tfrac{1}{2}\sigma^2\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial^2}{\partial x^2}x^2\,f_{X(T)|X(t_0)}(x\,|\,x_0)dx$$

which results in the forward equation of proportional Brownian motion in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial T}f_{X(T)|X(t_0)}(x\,|\,x_0)=\tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\Big(x^2\,f_{X(T)|X(t_0)}(x\,|\,x_0)\Big)-\mu\frac{\partial}{\partial x}\Big(x\,f_{X(T)|X(t_0)}(x\,|\,x_0)\Big)$$

Indifferentially factorized form, this is

$$\frac{\partial}{\partial T}f_{X(T)|X(t_0)}(x\,|\,x_0)=\tfrac{1}{2}\sigma^2\frac{\partial}{\partial x}\left(x^{\mu/\frac{1}{2}\sigma^2}\frac{\partial}{\partial x}x^{-\mu/\frac{1}{2}\sigma^2}\Big(x^2\,f_{X(T)|X(t_0)}(x\,|\,x_0)\Big)\right)$$

Method 2

The logarithm of the diffusion variable is a Brownian motion with transition mgf

$$M_{\log X(T)|\log X(t_0)}(\theta)=\exp\Big(\log x_0\theta+\mu(T-t_0)\theta+\tfrac{1}{2}\sigma^2(T-t_0)\theta^2\Big)$$

From the mathematics of Brownian motion, the forward equation of the underlying additive Brownian motion is

$$\frac{\partial}{\partial T}f_{\log X(T)|\log X(t_0)}(\log x\,|\,\log x_0)=\left(\tfrac{1}{2}\sigma^2\frac{\partial^2}{(\partial\log x)^2}-\mu\frac{\partial}{\partial\log x}\right)f_{\log X(T)|\log X(t_0)}(\log x\,|\,\log x_0)$$

Transforming back to the original variable with

$$f_{\log X(T)|\log X(t_0)}(\log x\,|\,\log x_0)=x\,f_{X(T)|X(t_0)}(x\,|\,x_0)$$

and the transformed partial derivatives

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$$\frac{\partial}{\partial \log x} = \frac{\partial x}{\partial \log x}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x}$$

and

$$\frac{\partial^2}{(\partial \log x)^2} = \frac{\partial x}{\partial \log x}\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial \log x}\frac{\partial}{\partial x}\right) = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) = x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} = x\frac{\partial^2}{\partial x^2}x - x\frac{\partial}{\partial x}$$

since

$$x\frac{\partial^2}{\partial x^2}x\,g(x) = x\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}x\,g(x)\right) = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}g(x) + g(x)\right) = x^2\frac{\partial^2}{\partial x^2}g(x) + 2x\frac{\partial}{\partial x}g(x)$$

The forward equation becomes

$$\begin{aligned} x\frac{\partial}{\partial T}f_{X(T)|X(t_0)}(x\,|\,x_0) &= \left(\tfrac{1}{2}\sigma^2 x\frac{\partial^2}{\partial x^2}x - \tfrac{1}{2}\sigma^2 x\frac{\partial}{\partial x} - \mu x\frac{\partial}{\partial x}\right)x\,f_{X(T)|X(t_0)}(x\,|\,x_0) \\ &= x\left(\tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}x - (\mu + \tfrac{1}{2}\sigma^2)\frac{\partial}{\partial x}\right)x\,f_{X(T)|X(t_0)}(x\,|\,x_0) \end{aligned}$$

which, upon cancelation of the Jacobian is the forward equation of a proportional Brownian motion

$$\frac{\partial}{\partial T}f_{X(T)|X(t_0)}(x\,|\,x_0) = \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}x^2 f_{X(T)|X(t_0)}(x\,|\,x_0) - (\mu + \tfrac{1}{2}\sigma^2)\frac{\partial}{\partial x}x\,f_{X(T)|X(t_0)}(x\,|\,x_0)$$

B. Formation of the Backward Equation

Method 1

$$M_{X(T)|X(t_0)}(\theta) = \sum_{j=0}^{\infty}\frac{\theta^j}{j!}\exp\left(j\log x_0 + \mu(T-t_0)j + \tfrac{1}{2}\sigma^2(T-t_0)j^2\right)$$

Differentiating the mgf partially with respect to the backward time variable

$$\begin{aligned} \frac{\partial}{\partial t_0}M_{X(T)|X(t_0)}(\theta) &= -\sum_{j=0}^{\infty}\left(\mu\,j + \tfrac{1}{2}\sigma^2 j^2\right)\frac{\theta^j}{j!}\exp\left(j\log x_0 + \mu(T-t_0)j + \tfrac{1}{2}\sigma^2(T-t_0)j^2\right) \\ &= -\mu\sum_{j=0}^{\infty}(\,j\,)\frac{\theta^j}{j!}\exp\left(j\log x_0 + \mu(T-t_0)j + \tfrac{1}{2}\sigma^2(T-t_0)j^2\right) \\ &\quad -\tfrac{1}{2}\sigma^2\sum_{j=0}^{\infty}\left(j^2\right)\frac{\theta^j}{j!}\exp\left(j\log x_0 + \mu(T-t_0)j + \tfrac{1}{2}\sigma^2(T-t_0)j^2\right) \end{aligned}$$

Differentiating the mgf partially with respect to the backward diffusion variable

$$\frac{\partial}{\partial x_0}M_{X(T)|X(t_0)}(\theta) = \sum_{j=0}^{\infty}\left(\frac{j}{x_0}\right)\frac{\theta^j}{j!}\exp\left(j\log x_0 + \mu(T-t_0)j + \tfrac{1}{2}\sigma^2(T-t_0)j^2\right)$$

i.e.,

$$x_0\frac{\partial}{\partial x_0}M_{X(T)|X(t_0)}(\theta) = \sum_{j=0}^{\infty}j\frac{\theta^j}{j!}\exp\left(j\log x_0 + \mu(T-t_0)j + \tfrac{1}{2}\sigma^2(T-t_0)j^2\right)$$

and again

$$\frac{\partial^2}{\partial x_0^2}M_{X(T)|X(t_0)}(\theta) = \sum_{j=0}^{\infty}\left\{\left(\frac{j}{x_0}\right)^2 - \frac{j}{x_0^2}\right\}\frac{\theta^j}{j!}\exp\left(j\log x_0 + \mu(T-t_0)j + \tfrac{1}{2}\sigma^2(T-t_0)j^2\right)$$

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i.e.,

$$x_0^2 \frac{\partial^2}{\partial x_0^2} = \sum\nolimits_{j=0}^{\infty} \left(j^2 - j\right) \frac{\theta^j}{j!} \exp\left(j \log x_0 + \mu(T - t_0)\, j + \tfrac{1}{2}\sigma^2 (T - t_0)\, j^2\right)$$

and

$$x_0^2 \frac{\partial^2}{\partial x_0^2} + x_0 \frac{\partial}{\partial x_0} = \sum\nolimits_{j=0}^{\infty} \left(j^2\right) \frac{\theta^j}{j!} \exp\left(j \log x_0 + \mu(T - t_0)\, j + \tfrac{1}{2}\sigma^2 (T - t_0)\, j^2\right)$$

Hence, the time derivative can be written as

$$\begin{aligned} \frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) &= -\mu\, x_0 \frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) - \tfrac{1}{2}\sigma^2 \left(x_0^2 \frac{\partial^2}{\partial x_0^2} + x_0 \frac{\partial}{\partial x_0}\right) M_{X(T)|X(t_0)}(\theta) \\ &= -\left(\mu + \tfrac{1}{2}\sigma^2\right) x_0 \frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) - \tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta) \end{aligned}$$

which results the backward equation of proportional Brownian motion in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\left(\mu + \tfrac{1}{2}\sigma^2\right) x_0 \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0) - \tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{\partial x_0^2} f_{X(T)|X(t_0)}(x \mid x_0)$$

and is a partial differential equation for which the derivative side of the diffusion variable forms a Euler–Cauchy equation. This method is simpler than transforming the backward equation of the underlying additive Brownian motion.

In differentially factorized form this is

$$-\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2\, x_0^{2-\mu/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x_0}\left(x_0^{\mu/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0)\right)$$

Method 2

From the mathematics of Brownian motion, the backward equations of the underlying additive Brownian motion is

$$\frac{\partial}{\partial t_0} f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0) = -\left(\mu \frac{\partial}{\partial \log x_0} + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{(\partial \log x_0)^2}\right) f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0)$$

Transforming back to the original variable with

$$f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0) = x\, f_{X(T)|X(t_0)}(x \mid x_0)$$

and

$$\frac{\partial}{\partial \log x_0} = \frac{\partial x_0}{\partial \log x_0} \frac{\partial}{\partial x_0} = x_0 \frac{\partial}{\partial x_0} \qquad \frac{\partial^2}{(\partial \log x_0)^2} = x_0 \frac{\partial}{\partial x_0}\left(x_0 \frac{\partial}{\partial x_0}\right) = x_0^2 \frac{\partial^2}{\partial x_0^2} + x_0 \frac{\partial}{\partial x_0}$$

$$\frac{\partial}{\partial t_0} x\, f_{X(T)|X(t_0)}(x \mid x_0) = -\left(\mu x_0 \frac{\partial}{\partial x_0} + \tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{\partial x_0^2} + \tfrac{1}{2}\sigma^2 x_0 \frac{\partial}{\partial x_0}\right) x\, f_{X(T)|X(t_0)}(x \mid x_0)$$

and cancelling the Jacobian for the backward equation of proportional Brownian motion

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\left((\mu + \tfrac{1}{2}\sigma^2) x_0 \frac{\partial}{\partial x_0} + \tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{\partial x_0^2}\right) f_{X(T)|X(t_0)}(x \mid x_0)$$

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Solution to the Forward Equation for Proportional Brownian Motion: (by Transforming to the Standard Heat Equation)

From the forward equation

$$\frac{\partial}{\partial t} f_{X(t)}(x) = \tfrac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2 x^2 f_{X(t)}(x)] - \frac{\partial}{\partial x}[\mu x f_{X(t)}(x)]$$

transforming the diffusion variable with a logarithmic transformation

$$Y(t) = \log X(t)$$

so that the independent variable of the equation becomes $y = \log x$ which has derivatives in the forward diffusion variable

$$\frac{dy}{dx} = \frac{1}{x} \text{ and } \frac{d^2 y}{dx^2} = -\frac{1}{x^2}$$

and a zero time derivative

$$\frac{dy}{dt} = 0$$

The Jacobian normalized transformation of the transition density $f_{X(t)}$ is

$$f_{Y(t)}(y) = x f_{X(t)}(x)$$

which gives the following relevant partial derivatives;

1) $$\frac{\partial}{\partial x}[x f_{X(t)}(x)] = \frac{\partial}{\partial y} f_{Y(t)}(y)\frac{\partial y}{\partial x} = \frac{1}{x}\frac{\partial}{\partial y} f_{Y(t)}(y)$$

2) $$\frac{\partial}{\partial x}[x^2 f_{X(t)}(x)] = \frac{\partial}{\partial x}[x f_{Y(t)}(x)] = f_{Y(t)}(y) + x\frac{\partial}{\partial y} f_{Y(t)}(y)\frac{\partial y}{\partial x} = f_{Y(t)}(y) + \frac{\partial}{\partial y} f_{Y(t)}(y)$$

3) $$\frac{\partial^2}{\partial x^2}[x^2 f_{X(t)}(x)] = \frac{\partial}{\partial y} f_{Y(t)}(x)\frac{\partial y}{\partial x} + \frac{\partial^2}{\partial y^2} f_{Y(t)}(x)\frac{\partial y}{\partial x} = \frac{1}{x}\frac{\partial}{\partial y} f_{Y(t)}(y) + \frac{1}{x}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y)$$

4) $$\frac{\partial}{\partial t} f_{X(t)}(x) = \frac{1}{x}\frac{\partial}{\partial t} f_{Y(t)}(y) + \frac{1}{x}\frac{\partial}{\partial y} f_{Y(t)}(y)\frac{\partial y}{\partial t} = \frac{1}{x}\frac{\partial}{\partial t} f_{Y(t)}(y)$$

Substituting these derivatives into the forward equation gives

$$\frac{1}{x}\frac{\partial}{\partial t} f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2\left[\frac{1}{x}\frac{\partial}{\partial y} f_{Y(t)}(y) + \frac{1}{x}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y)\right] - \mu\frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\partial}{\partial y} f_{Y(t)}(y)\right]$$

and rearranging terms into the appropriate format

$$\frac{\partial}{\partial t} f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2} f_{Y(t)}(y) - \left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial y} f_{Y(t)}(y)$$

which is the forward equation for arithmetic Brownian motion with a modified infinitesimal mean. The modified infinitesimal mean corresponds to the ito correction. The further standardizing transformation that removes the drift and scale, $U(t) = [Y(t) - y_0 - (\mu - \frac{1}{2}\sigma^2)\,t\,] / \sigma$, then reduces this equation to the standard heat equation $f_t = \frac{1}{2} f_{xx}$ with the transition distribution $U(t) \sim N(0,t)$. Re-standardizing from standard Brownian motion $U(t)$ to the original underlying additive diffusion $Y(t)$ variable with the transformation $Y(t) = y_0 + (\mu - \frac{1}{2}\sigma^2)\,t + \sigma U(t)$ gives the logarithm of the diffusion

$$Y(t) = \log X(t) \sim N\left(y_0 + (\mu - \tfrac{1}{2}\sigma^2)t,\ \sigma^2 t\right)$$

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Transforming the random variable back to the original diffusion variable by exponentiating $X(t) = e^{Y(t)}$ gives a log-normal variable,

$$X(t) \sim \log N\left(\log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t,\ \sigma^2 t\right)$$

This distribution, with cumulants which are linear time functions, implies that the diffusion increments are not independent but the successive ratios are. The density of this distribution is the general solution to the forward equation

$$f_{X(t)}(x \mid x_0) = \frac{1}{\sigma x\sqrt{2\pi t}} \exp\left(-\frac{[\log x - \log x_0 - (\mu - \sigma^2/2)t]^2}{2\sigma^2 t}\right)$$

Solution to the Forward Equation for Proportional Brownian Motion: (Solving the Equation Directly)

From the forward equation

$$\frac{\partial}{\partial t} f_{X(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}[x^2 f_{X(t)}(x)] - \mu \frac{\partial}{\partial x}[x\ f_{X(t)}(x)]$$

multiplying by an exponential factor in the forward diffusion variable and integrating

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{X(t)}(x)dx = \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2}[x^2 f_{X(t)}(x)]dx - \mu \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x}[x\ f_{X(t)}(x)]dx$$

and applying integration by parts twice on the first term of the right-hand-side and once on the second term gives the forward equation for the transition distribution moment generating function

$$\frac{\partial}{\partial t} M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 \frac{\partial^2}{\partial \theta^2} M_{X(t)}(\theta) + \mu\theta \frac{\partial}{\partial \theta} M_{X(t)}(\theta)$$

This is a partial differential equation of first order in time and second order in the parameter of the generating function. Removing the time dimension and the time derivative by defining $\lambda(\theta; s) = L\{M_{X(t)}(\theta)\}$ as the Laplace transform of the moment generating function and taking the Laplace transform of each side

$$s\lambda(\theta; s) - M_{X(0)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 \frac{\partial^2}{\partial \theta^2}\lambda(\theta; s) + \mu\theta \frac{\partial}{\partial \theta}\lambda(\theta; s)$$

$$-e^{\theta x_0} = \tfrac{1}{2}\sigma^2\theta^2 \frac{\partial^2}{\partial \theta^2}\lambda(\theta; s) + \mu\theta \frac{\partial}{\partial \theta}\lambda(\theta; s) - s\lambda(\theta; s)$$

which is a Euler-Cauchy equation of order two. The general solution to an Euler-Cauchy equation is a power function of the independent variable. Letting $\lambda = \theta^n$ the derivatives of which being $\lambda_\theta = n\,\theta^{n-1}$ and $\lambda_{\theta\theta} = n(n-1)\,\theta^{n-2}$. Substituting these into the complementary equation gives the characteristic equation,

$$0 = \tfrac{1}{2}\sigma^2\theta^2 n(n-1)\theta^{n-2} + \mu\theta n\theta^{n-1} - s\theta^n \qquad \text{or} \qquad 0 = \tfrac{1}{2}\sigma^2 n(n-1) + \mu n - s$$

giving the characteristic roots, which include the ito-correction, of

$$n_{1,2} = -\frac{(\mu - \tfrac{1}{2}\sigma^2) \pm \sqrt{(\mu - \tfrac{1}{2}\sigma^2)^2 + 2\sigma^2 s}}{\sigma^2}$$

Differentially factorizing the Euler-Cauchy equation in the power function of the characteristic roots

$$-e^{\theta x_0} = \tfrac{1}{2}\sigma^2\theta^{n_1+1} \frac{\partial}{\partial \theta}\theta^{-n_1} \times \left(\theta^{n_2+1} \frac{\partial}{\partial \theta}\theta^{-n_2}\lambda(\theta; s)\right)$$

and expressing the exact nested derivatives in terms of a power series

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$$\frac{\partial}{\partial\theta}\theta^{-n_1}\times\left(\theta^{n_2+1}\frac{\partial}{\partial\theta}\theta^{-n_2}\lambda(\theta;s)\right)=-\frac{1}{\frac{1}{2}\sigma^2}\theta^{-n_1-1}e^{\theta x_0}=-\frac{1}{\frac{1}{2}\sigma^2}\left(\theta^{-n_1-1}+x_0\theta^{-n_1}+\tfrac{1}{2}x_0^2\theta^{-n_1+1}+\tfrac{1}{3!}x_0^3\theta^{-n_1+2}+\ldots\right)$$

and proceeding with the first integration gives

$$\theta^{n_2-n_1+1}\frac{\partial}{\partial\theta}\theta^{-n_2}\lambda(\theta;s)=-\frac{1}{\frac{1}{2}\sigma^2}\left(-\frac{\theta^{-n_1}}{n_1}-\frac{x_0\theta^{-n_1+1}}{n_1-1}-\frac{\frac{1}{2}x_0^2\theta^{-n_1+2}}{n_1-2}-\frac{\frac{1}{3!}x_0^3\theta^{-n_1+3}}{n_1-3}+\ldots\right)$$

or

$$\frac{\partial}{\partial\theta}\theta^{-n_2}\lambda(\theta;s)=-\frac{1}{\frac{1}{2}\sigma^2}\left(-\frac{\theta^{-n_2-1}}{n_1}-\frac{x_0\theta^{-n_2}}{n_1-1}-\frac{\frac{1}{2}x_0^2\theta^{-n_2+1}}{n_1-2}-\frac{\frac{1}{3!}x_0^3\theta^{-n_2+2}}{n_1-3}+\ldots\right)$$

and the second integration

$$\theta^{-n_2}\lambda(\theta;s)=-\frac{1}{\frac{1}{2}\sigma^2}\left(\frac{\theta^{-n_2}}{n_1n_2}+\frac{x_0\theta^{-n_2+1}}{(n_1-1)(n_2-2)}+\frac{\frac{1}{2}x_0^2\theta^{-n_2+2}}{(n_1-2)(n_2-2)}+\frac{\frac{1}{3!}x_0^3\theta^{-n_2+3}}{(n_1-3)(n_2-3)}+\ldots\right)$$

or

$$\begin{aligned}\lambda(\theta;s)&=-\frac{1}{\frac{1}{2}\sigma^2}\left(\frac{\frac{1}{0!}}{n_1n_2}+\frac{\frac{1}{1!}x_0\theta}{(n_1-1)(n_2-2)}+\frac{\frac{1}{2!}x_0^2\theta^2}{(n_1-2)(n_2-2)}+\frac{\frac{1}{3!}x_0^3\theta^3}{(n_1-3)(n_2-3)}+\ldots\right)\\&=-\frac{1}{\frac{1}{2}\sigma^2}\sum_{k=0}^{\infty}\frac{x_0^k\theta^k}{k!(n_1-k)(n_2-k)}\end{aligned}$$

Noting that $-\frac{1}{2}\sigma^2(n_1-k)(n_2-k)=s-(\mu-\frac{1}{2}\sigma^2)k-\frac{1}{2}\sigma^2k^2$, the Laplace transform of the moment generating function is

$$\lambda(\theta;s)=\sum_{k=0}^{\infty}\frac{x_0^k\theta^k}{k!\left(s-(\mu-\frac{1}{2}\sigma^2)k-\frac{1}{2}\sigma^2k^2\right)}$$

Inverting the Laplace transform and reintroducing the time domain $M_{X(t)}(\theta)=L^{-1}\{\lambda(\theta;s)\}$ gives the moment generating function of the diffusion variable and the solution to the forward equation in the moment generating function

$$M_{X(t)}(\theta)=\sum_{k=0}^{\infty}L^{-1}\left\{\frac{x_0^k\theta^k}{k!}\left(s-(\mu-\tfrac{1}{2}\sigma^2)k-\tfrac{1}{2}\sigma^2k^2\right)^{-1}\right\}=\sum_{k=0}^{\infty}\frac{\theta^k}{k!}x_0^k\exp\left((\mu-\tfrac{1}{2}\sigma^2)tk+\tfrac{1}{2}\sigma^2tk^2\right)$$

This is the moment generating function of the transition distribution of a geometric Brownian motion starting at x_0 which is a log-normal variable with log-mean $\log x_0+(\mu-\frac{1}{2}\sigma^2)t$ and log variance σ^2t

$$X_t\sim\log N\left(\log x_0+(\mu-\tfrac{1}{2}\sigma^2)t,\,\sigma^2t\right)$$

The solution to the forward equation is therefore the density function

$$f_{X(t)}(x\mid x_0)=\frac{1}{x\sigma\sqrt{t}\sqrt{2\pi}}\exp\left(-\frac{[\log x-\log x_0-(\mu-\frac{1}{2}\sigma^2)t]^2}{2\sigma^2t}\right)$$

The standard distribution function for the interval [0,*x*] is

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$$F_{X(t)|X(p)}(x \mid x_0) = \int_{-\infty}^{x} \frac{1}{y\sigma\sqrt{2\pi t}} \exp\left\{-\frac{[\log x - \log x_0 - (\mu - \frac{1}{2}\sigma^2)t]}{2t\sigma^2}\right\} dy = \Phi\left(\frac{\log x - \log x_0 - (\mu - \frac{1}{2}\sigma^2)t}{\sigma}\right)$$

for $\{(x,t) : x \in (0,\infty), t \in [0,\infty)\}$

The Transition Distribution

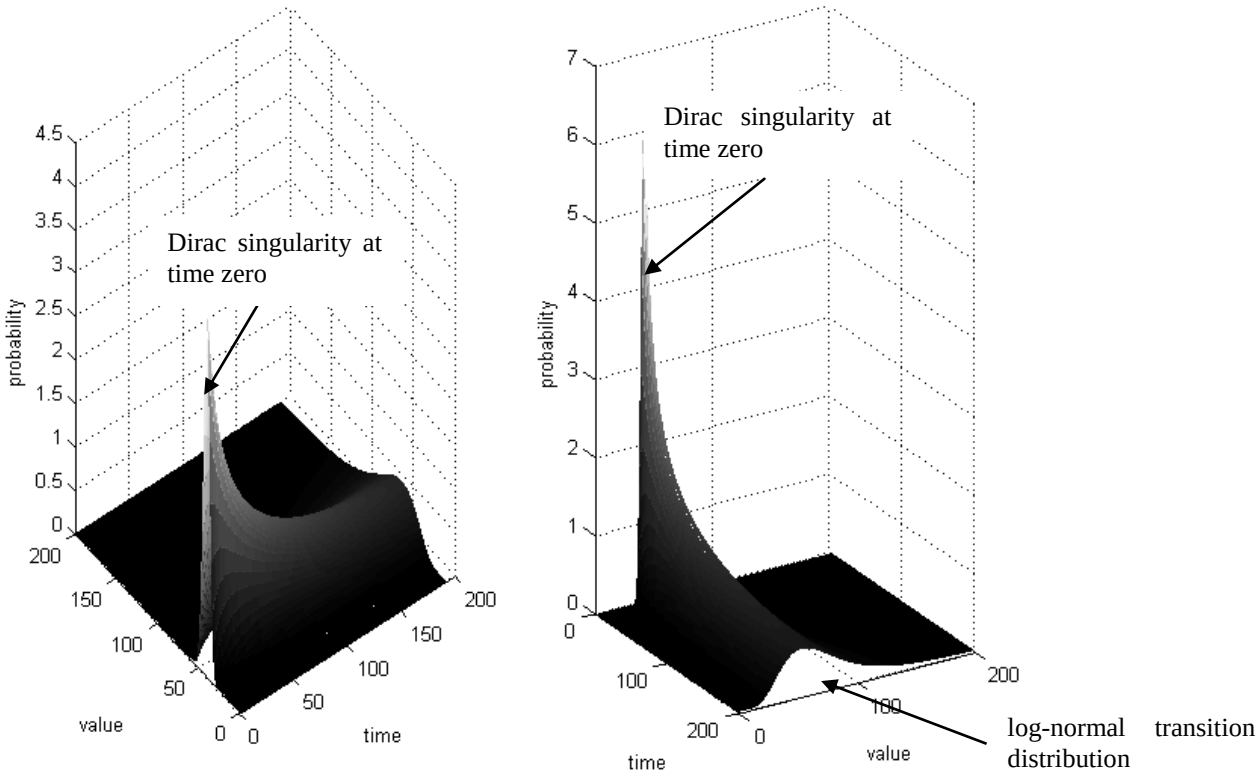


Figure 11: Geometric Brownian motion transition density surface

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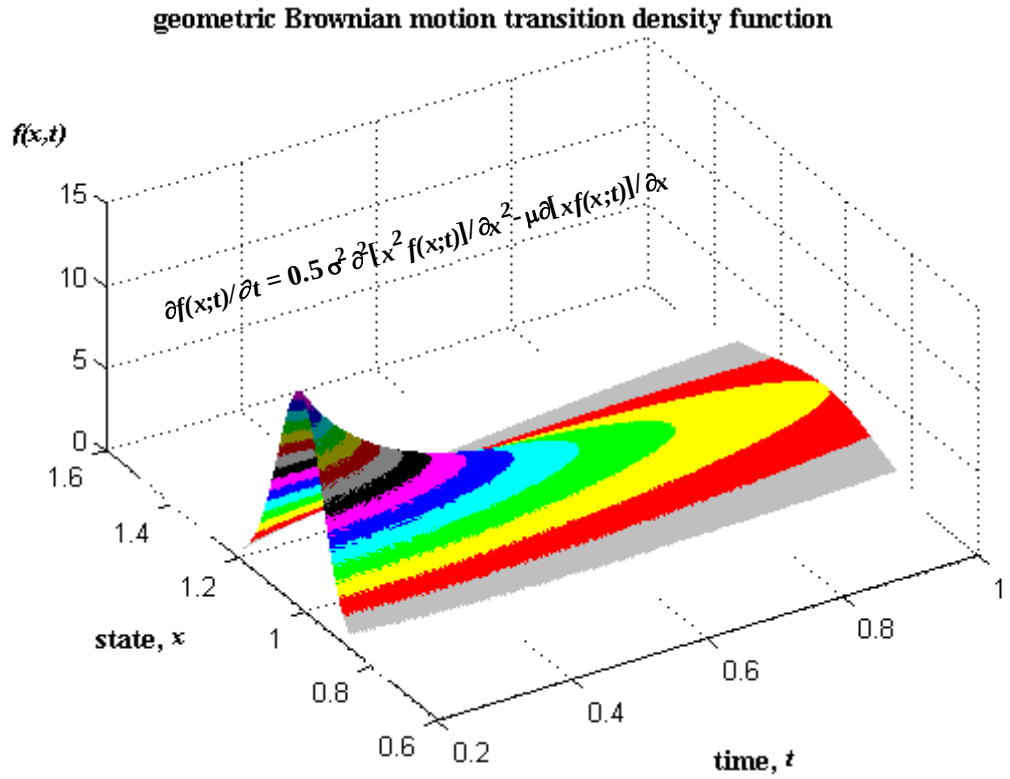


Figure 12: Geometric Brownian motion transition density surface with parameters (μ, σ) = (0.10, 0.15)

The diagrams show the transition density surface of a geometric Brownian motion, the general solution to (the dependent variable of) the Kolmogorov forward equation as it evolves over time for the diffusion with infinitesimal parameters μx and $\sigma^2 x^2$. The graphics relate to a geometric Brownian motion with parameters μ = 0.10 and σ = 0.15 . The ito-corrected parameter is $\mu - \frac{1}{2}\sigma^2$ = 0.08875. The transition density function is an increasingly dispersed log-normal, mesokurtic on logs, exponential bell starting at a singularity at the initial value x_0 . The center for which each exponential bell is located increases at a constant exponential rate of log drift per unit time and exp(μt) to time *t*. The surface takes the form of the dispersing probability corridor generic to diffusion. Geometric Brownian motion is characterized by independent multiplicative ratios and a probability distribution that is closed to the operation of random variable multiplication (but not summation), that is, under log-convolution. As such, the increasingly dispersed exponential bell that results as the density function of the geometric Brownian motion extends through time results from the continuous cumulative multiplication, or log-convolution, of homogeneous infinitesimal component log-normal exponential bell curves. For each point in time, or for each univariate exponential bell, there are scalar valued statistical descriptors including the limited moments of undiscounted option premia. When the basis variance zeroes $\sigma \to 0$, this surface becomes a Dirac singularity traced out over the exponential drift line, i.e., the diffusion equations becomes $X(t) \to x_0 \exp(\mu t)$ for which the transition density is

$$f_{X(t)|X(0)}(x \mid x_0) = \delta_{x_0 \exp(\mu t)}(x)$$

Mean and Variance of Proportional Brownian Motion

From the moment generating function of the transition distribution at time *t*

$$M_{\log N(\log x_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t)}(\theta) = \sum_{r=0}^{\infty} \frac{(\theta x_0 e^{(\mu - \frac{1}{2}\sigma^2)t})^r}{r!} M_{N(0,1)}(r\sigma\sqrt{t})$$

$$= \sum_{r=0}^{\infty} \frac{(\theta x_0 e^{(\mu - \frac{1}{2}\sigma^2)t})^r}{r!} e^{\frac{1}{2}\sigma^2 t r^2}$$

$$= \sum_{r=0}^{\infty} \frac{\theta^r}{r!} x_0^r e^{(\mu - \frac{1}{2}\sigma^2)t r + \frac{1}{2}\sigma^2 t r^2}$$

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the coefficients of $\theta^r / r!$ in the second and third terms in the infinite sum give the mean and variance of the process at time t

$$E(X_t) = \exp\left(\log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \tfrac{1}{2}\sigma^2 t\right) = x_0 e^{\mu t}$$

and

$$Var(X_t) = \exp\left(2\log x_0 + 2(\mu - \tfrac{1}{2}\sigma^2)t + 2\sigma^2 t\right) - \exp\left(2\log x_0 + 2(\mu - \tfrac{1}{2}\sigma^2)t + \sigma^2 t\right) = {x_0}^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$$

Covariance of Proportional Brownian Motion

The general raw (auto) joint moment of a geometric Brownian motion at times s and t is

$$E({X_t}^{\theta_1} {X_s}^{\theta_2}) = E(e^{\theta_1 \log X_t + \theta_2 \log X_s}) = M_{\log X_t, \log X_s}(\theta_1, \theta_2)$$

which is the (auto) joint moment generating function of the underlying arithmetic Brownian motion. This is, after the ito-correction

$$M_{\log X_t, \log X_s}(\theta_1, \theta_2) = \exp\left\{\theta_1 \log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t\theta_1 + \theta_2 \log x_0 + (\mu - \tfrac{1}{2}\sigma^2)s\theta_2 + \left(\tfrac{1}{2}\sigma^2 t\,\theta_1^2 + \tfrac{1}{2}\sigma^2 s\theta_2^2 + \sigma^2(s \wedge t)\theta_1\theta_2\right)\right\}$$

where $(s \wedge t) = \min(s, t)$. The first raw (auto) joint moment is

$$\begin{aligned} E(X_t X_s) &= M_{\log X_t, \log X_s}(1,1) \\ &= \exp\left(2\log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t + (\mu - \tfrac{1}{2}\sigma^2)s + \left(\tfrac{1}{2}\sigma^2 t + \tfrac{1}{2}\sigma^2 s + \sigma^2(s \wedge t)\right)\right) \\ &= x_0^2 \exp\left(\mu(t+s) + \sigma^2(s \wedge t)\right) \end{aligned}$$

The individual raw moments are

$$E(X_t) = M_{\log X_t, \log X_s}(1,0) = \exp(\log x_0 + \mu t) = x_0 \exp(\mu t)$$

and

$$E(X_s) = M_{\log X_t, \log X_s}(0,1) = \exp(\log x_0 + \mu s) = x_0 \exp(\mu s)$$

giving the (auto) covariance of the geometric Brownian motion

$$\begin{aligned} Cov(X_t, X_s) &= M_{\log X_t, \log X_s}(1,1) - M_{\log X_t, \log X_s}(1,0) M_{\log X_t, \log X_s}(0,1) \\ &= x_0^2 \exp\left(\mu(t+s) + \sigma^2(s \wedge t)\right) - x_0^2 \exp\left(\mu(t+s)\right) \\ &= x_0^2\, \mathrm{e}^{\mu(t+s)}\left(\mathrm{e}^{\sigma^2(s \wedge t)} - 1\right) \end{aligned}$$

Further Statistical Description of Proportional Brownian Motion

These quantities do not include the ito-correction. The ito correction is included by reparameterizing μ to $\mu - \frac{1}{2}\sigma^2$ in the various statistical functions or replacing μ in the forward and backward equations with $\mu + \frac{1}{2}\sigma^2$.

1) Quantiles

The quantile function of a geometric Brownian motion is given by

$$Q(p) = x_0 \exp\{\mu t + \sigma\sqrt{t}\,\Phi^{-1}(p)\}$$

From this function, the lower and upper quartiles of a geometric Brownian motion are

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$$Q(0.25) = x_0 \exp(\mu t - 0.6745\sigma\sqrt{t}) \quad \text{and} \quad Q(0.75) = x_0 \exp(\mu t + 0.6745\sigma\sqrt{t})$$

while the inter-quartile range is

$$\begin{aligned} IQR(X_t) &= Q(0.75) - Q(0.25) \\ &= x_0 \exp\left(\mu t + 0.6745\,\sigma\sqrt{t}\right) - x_0 \exp\left(\mu t - 0.6745\,\sigma\sqrt{t}\right) \\ &= x_0 e^{\mu t}\left(e^{0.6745\,\sigma\sqrt{t}} - e^{-0.6745\,\sigma\sqrt{t}}\right) \\ &= 2x_0 e^{\mu t} \sinh(0.6745\,\sigma\sqrt{t}) \end{aligned}$$

For a 95% confidence band on the process at time *t*, the 2.5% and 97.5% percentiles are

$$Q(0.025) = x_0 \exp(\mu t - 1.96\sigma\sqrt{t}) \quad \text{and} \quad Q(0.975) = x_0 \exp(\mu t + 1.96\sigma\sqrt{t})$$

The probability interval for a geometric Brownian motion at time *t* is then

$$\begin{aligned} CI(X_t, 0.95) &= \left(Q(0.025), Q(0.975)\right) \\ &= x_0 \exp\left(\mu t \pm 1.96\,\sigma\sqrt{t}\right) \end{aligned}$$

The width of this interval is

$$x_0\, \mathrm{e}^{\mu t}\left(e^{1.96\sigma\sqrt{t}} - e^{-1.96\sigma\sqrt{t}}\right) = 2x_0\, \mathrm{e}^{\mu t} \sinh(1.96\,\sigma\sqrt{t})$$

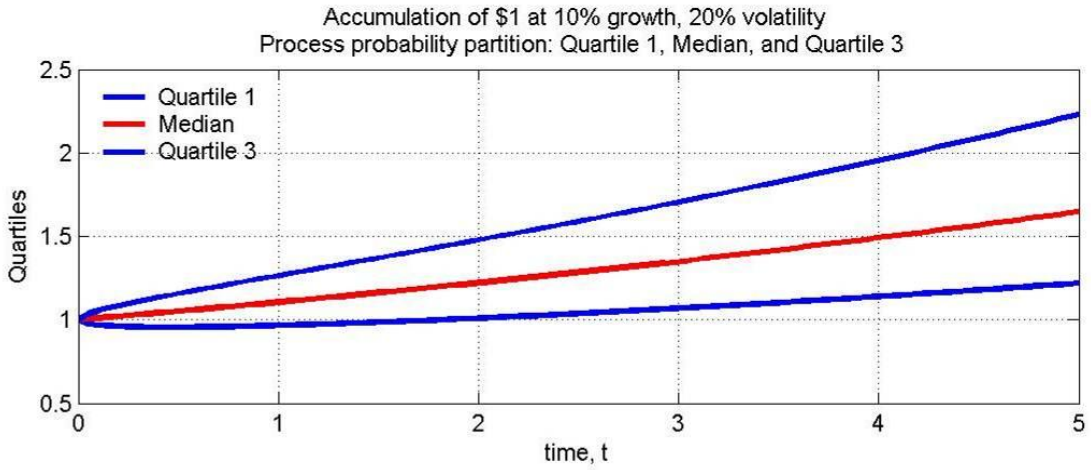


Figure 13: Lower quartile, median, and upper quartile of geometric Brownian motion

2) Raw Moments

The first four raw moments directly from the moment generating function of the underlying normal are

$$\begin{aligned} \mu_1(X_t) &= x_0 e^{\mu t + \frac{1}{2}\sigma^2 t} \\ \mu_2(X_t) &= x_0^2 e^{2\mu t + 2\sigma^2 t} \\ \mu_3(X_t) &= x_0^3 e^{3\mu t + 4\frac{1}{2}\sigma^2 t} \\ \mu_4(X_t) &= x_0^4 e^{4\mu t + 8\sigma^2 t} \end{aligned}$$

The raw moment of order *r* is directly proportional to the r^{th} power of the first, the mean of the diffusion and the accumulation function of compound interest. Splitting $\mu_r(X_t)$ into two factors, the r^{th} power of the mean $\mu_1{}^r(X_t) = E(X_t)^r$ and the remaining factor, gives

$$E(X_t^r) = x_0^r e^{\mu t r + \frac{1}{2}\sigma^2 t r^2 - \frac{1}{2}\sigma^2 t r + \frac{1}{2}\sigma^2 t r} = (x_0 e^{\mu t + \frac{1}{2}\sigma^2 t})^r\, e^{\frac{1}{2}\sigma^2 t r^2 - \frac{1}{2}\sigma^2 t r} = (x_0 e^{\mu t + \frac{1}{2}\sigma^2 t})^r\, e^{\frac{1}{2}r(r-1)\sigma^2 t} = E(X_t)^r\, e^{\frac{1}{2}r(r-1)\sigma^2 t}$$

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that is

$$E(X_t^r) = \mu_1^r(X_t)e^{\frac{1}{2}r^{(2)}\sigma^2 t} \qquad \text{for } r = 1,2,3,\ldots$$

The raw moments can then be expressed as

$$\mu_1(X_t) = x_0 e^{\mu t + \frac{1}{2}\sigma^2 t}$$
$$\mu_2(X_t) = \mu_1(X_t)^2 e^{\sigma^2 t}$$
$$\mu_3(X_t) = \mu_1(X_t)^3 e^{3\sigma^2 t}$$
$$\mu_4(X_t) = \mu_1(X_t)^4 e^{6\sigma^2 t}$$

3) Central Moments

The central moments are also proportional to ascending powers of the mean. Ignoring the ito-correction,

$$\sigma_2(X_t) = Var(X_t) = \mu_2(X_t) - \mu_1^2(X_t) = x_0^2 e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1) = \mu_1(X_t)^2(e^{\sigma^2 t} - 1)$$

$$\sigma_3(X_t) = Skew(X_t) = \mu_3(X_t) - 3\mu_2(X_t)\mu_1(X_t) + 2\mu_1^3(X_t)$$
$$= x_0^3 e^{3\mu t + \frac{3}{2}\sigma^2 t}(e^{3\sigma^2 t} - 3e^{\sigma^2 t} + 2) = \mu_1(X_t)^3(e^{3\sigma^2 t} - 3e^{\sigma^2 t} + 2)$$
$$\sigma_4(X_t) = Kurt(X_t) = \mu_4(X_t) - 4\mu_3(X_t)\mu_1(X_t) + 6\mu_2(X_t)\mu_1^2(X_t) - 3\mu_1^4(X_t)$$
$$= x_0^4 e^{4\mu t + 2\sigma^2 t}(e^{6\sigma^2 t} - 4e^{3\sigma^2 t} + 6e^{\sigma^2 t} - 3) = \mu_1(X_t)^4(e^{6\sigma^2 t} - 4e^{3\sigma^2 t} + 6e^{\sigma^2 t} - 3)$$

The general expression of the r^{th} central moment, also expressed as a function of the mean of the process,

$$\sigma_r(X_t) = E[\{X_t - \mu_1(X_t)\}^r] = \mu_1^r(X_t)\sum_{j=0}^{r}(-1)^j \times {}^rC_j e^{\frac{1}{2}j(j-1)\sigma^2 t} = \mu_1^r(X_t)\sum_{j=0}^{r}(-1)^j \times \frac{r^{(j)}}{j!}e^{\frac{1}{2}j^{(2)}\sigma^2 t}$$

where $r^{(j)}$ is the descending factorial from integer r limited to j terms.

When calculated numerically, the appropriate root according to the order of the central moments can make the measures comparable across order and in the same units as the variable, for example, if the variable is in dollars, the cube root of skewness is also in dollars as per the standard deviation. The relevant roots are

$$\sqrt{\sigma_2(X_t)} = \sqrt{Var(X_t)} = \mu_1(X_t) \times \sqrt{e^{\sigma^2 t} - 1}$$
$$\sqrt[3]{\sigma_3(X_t)} = \sqrt[3]{Skew(X_t)} = \mu_1(X_t) \times \sqrt[3]{e^{3\sigma^2 t} - 3e^{\sigma^2 t} + 2}$$
$$\sqrt[4]{\sigma_4(X_t)} = \sqrt[4]{Kurt(X_t)} = \mu_1(X_t) \times \sqrt[4]{e^{6\sigma^2 t} - 4e^{3\sigma^2 t} + 6e^{\sigma^2 t} - 3}$$

4) Signal to Noise Ratio

The variance of a geometric Brownian motion is proportional to the square of its mean

$$\sigma_2(X_t) = Var(X_t) = [\mu_1(X_t)]^2(e^{\sigma^2 t} - 1)$$

so that the standard deviation is proportional to its mean

$$\sqrt{\sigma_2(X_t)} = SD(X_t) = \mu_1(X_t)\sqrt{e^{\sigma^2 t} - 1}$$

During the early periods of a geometric Brownian motion, the mean is greater than the standard deviation and the process is very close to an ordinary Brownian motion. The ratio of the mean to the standard deviation or variance is monotonic decreasing so that a geometric Brownian motion has a monotonic decreasing signal to noise ratio. This is in contrast to an ordinary Brownian motion for which the ratio of the mean to variance is constant. For any point in time past

$$t = \frac{1}{\sigma_2(X_t)}\log\left[1 + \left(\sqrt{\sigma_2(X_t)} \Big/ \mu_1(X_t)\right)^2\right] = \frac{1}{\sigma_2(X_t)}\log\left[1 + c.e.Var(X_t)^2\right] = \frac{1}{\sigma_2(X_t)}\log\left[\mu_2(X_t)/\mu_1(X_t)^2\right]$$

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the standard deviation of a geometric Brownian motion is greater than the mean and the signal to noise ratio is less than one.

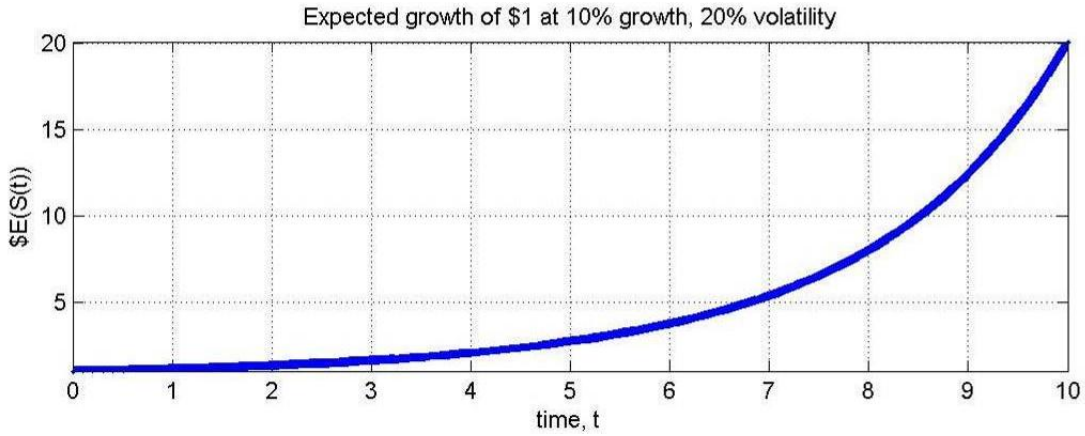


Figure 14: Expected value of geometric Brownian motion

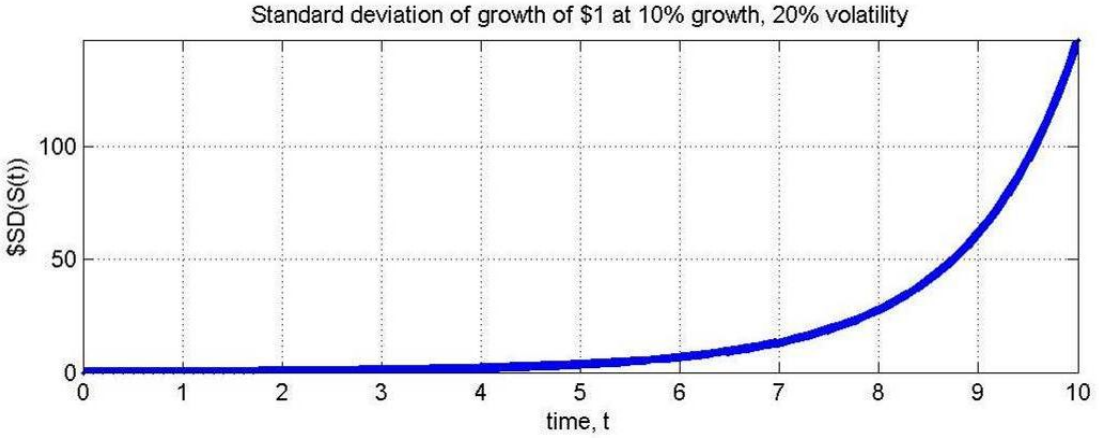


Figure 15: Standard deviation of proportional Brownian motion

Note that the first graph, the mean of the transition distribution of geometric Brownian motion, is the continuously compounded geometric accumulation factor from compound interest theory.

5) Scaled Measures

The standardized indices of distributional shape are the standardized central moments. $\zeta_r(X_t)$ are the r^{th} central moments scaled by the appropriate power of the standard deviation, the 3^{rd} being the coefficient of skewness ζ_3, and the 4^{th} being the coefficient of kurtosis ζ_4. These are

$$\zeta_3(X_t) = \frac{Skew(X_t)}{\sqrt{Var(X_t)^3}} = \frac{e^{3\sigma^2 t} - 3e^{\sigma^2 t} + 2}{(e^{\sigma^2 t} - 1)^{3/2}}$$

$$\zeta_4(X_t) = \frac{Kurt(X_t)}{Var(X_t)^2} = \frac{e^{6\sigma^2 t} - 4e^{3\sigma^2 t} + 6e^{\sigma^2 t} - 3}{(e^{\sigma^2 t} - 1)^2}$$

and the general formula for the r^{th} standardized central moment is

$$\zeta_r(X_t) = E\left(\left[\frac{X_t - \mu_1(X_t)}{\sqrt{\sigma_2(X_t)}}\right]^r\right) = \frac{\sum_{j=0}^{r} (-1)^j \times {}^rC_j \; e^{\frac{1}{2}j(j-1)\sigma^2 t}}{(e^{\sigma^2 t} - 1)^{r/2}}$$

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The r^{th} roots of these indices of shape, with the signum retained, give numerical measures of shape that are independent of both location and scale are unit invariant in the sample domain space, and are comparable across orders of the moments. These are simply,

$$|\zeta_r(X_t)|^{1/r}\ sign[\zeta_r(X_t)]$$

and since, the log-normal is right skew, the signum can be omitted. The cube root and the fourth root of the coefficient of skewness and kurtosis are

$$\zeta_3(X_t)^{1/3} = \frac{\sqrt[3]{Skew(X_t)}}{\sqrt{Var(X_t)}} = \frac{\sqrt[3]{e^{3\sigma^2 t} - 3e^{\sigma^2 t} + 2}}{\sqrt{e^{\sigma^2 t} - 1}}$$

and

$$\zeta_4(X_t)^{1/4} = \frac{\sqrt[4]{Kurt(X_t)}}{\sqrt{Var(X_t)}} = \frac{\sqrt[4]{e^{6\sigma^2 t} - 4e^{3\sigma^2 t} + 6e^{\sigma^2 t} - 3}}{\sqrt{e^{\sigma^2 t} - 1}}$$

6) Uncertainty and Information – Diffusion Entropy

The entropy of a normal is $\log\sqrt{2\,\pi e\ \sigma^2}$ and the entropy of an exponentiated distribution is the sum of the mean and the entropy of the distribution being exponentiated. The entropy of a log-normal is therefore $\mu + \log\sqrt{2\,\pi e\ \sigma^2}$ and the entropy of geometric Brownian motion is

$$H(X_t) = E[-\log f_{X_t}(X_t)] = \int_{-\infty}^{\infty} f_{X_t}(x) \log f_{X_t}(x) dx = \log x_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \log\sqrt{2\pi\ e\ \sigma^2 t}$$

Uncertainty as measured by expected likelihood, entropy, depends on both the cumulants of the underlying additive Brownian motion as well as scale and the time between the forward and backward variables. This is in contrast to Brownian motion, which is independent of linear drift and the backward diffusion variable.

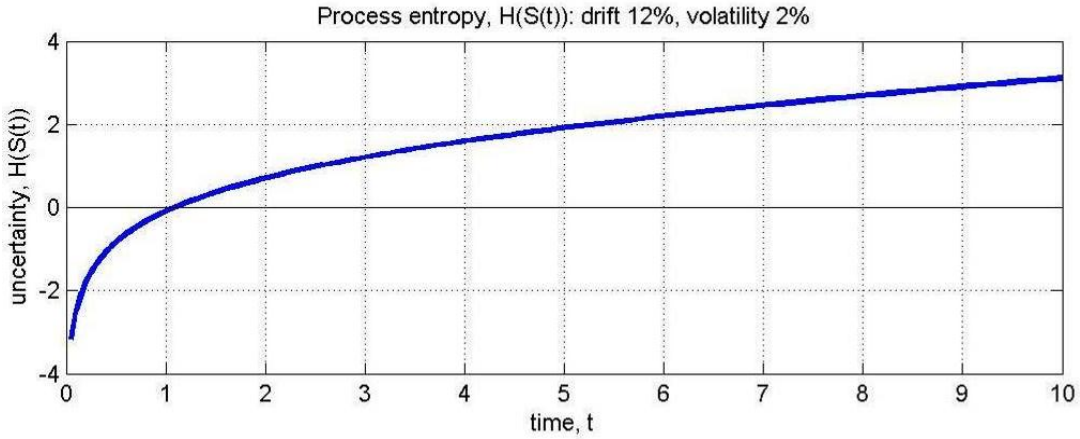


Figure 16: Entropy of proportional Brownian motion

7) Raw Moments of Excess Function (Call Option Expiry)

The undiscounted value of a call option, the first moment of the location shifted limited distribution (the excess function), that of an undiscounted premium, is

$$E[(X_t - k) \vee 0] = \int_k^{\infty} (x - k) f_{X_t}(x) dx$$
$$= \mu_1(X_t)\overline{F}_Y\{y(k) - \sigma\sqrt{ht})\} - k\overline{F}_Y\{y(k)\}$$
$$= x_0 e^{\mu ht + \frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k) - \sigma\sqrt{ht})\} - k\overline{F}_Y\{y(k)\}$$

Continuing for higher raw moments, by using the results of theorems regarding moment generating functions of limited normals and raw moments of limited log-normals, the second raw moment of the excess function is

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$$E[(X_t-k)^2 \vee 0] = \int_k^\infty (x-k)^2 f_{X_t}(x)dx$$
$$= \mu_2(X_t)\overline{F}_Y\{y(k)-2\sigma\sqrt{ht})\} - 2\mu_1(X_t)k\ \overline{F}_Y\{y(k)-\sigma\sqrt{ht})\} + k^2\overline{F}_Y\{y(k)\}$$
$$= x_0^2 e^{2\mu ht+2\sigma^2 ht}\overline{F}_Y\{y(k)-2\sigma\sqrt{ht})\} - 2x_0 k\, e^{\mu ht+\frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k)-\sigma\sqrt{ht})\} + k^2\overline{F}_Y\{y(k)\}$$

noting that the variance of the option payoff is

$$Var[(X_t-k)\vee 0] = E[(X_t-k)^2\vee 0] - E^2[(X_t-k)\vee 0]$$

third raw moment is

$$E[(X_t-k)^3 \vee 0] = \int_k^\infty (x-k)^3 f_{X_t}(x)dx$$
$$= \mu_3(X_t)\overline{F}_Y\{y(k)-3\sigma\sqrt{ht})\} - 3\mu_2(X_t)k\ \overline{F}_Y\{y(k)-2\sigma\sqrt{ht})\} + 3\mu_1(X_t)k^2\overline{F}_Y\{y(k)-\sigma\sqrt{ht})\} - k^3\overline{F}_Y\{y(k)\}$$
$$= x_0^3 e^{3\mu ht+4\frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k)-3\sigma\sqrt{ht})\} - 3x_0^2 k\ e^{2\mu ht+2\sigma^2 ht}\overline{F}_Y\{y(k)-2\sigma\sqrt{ht})\}$$
$$+ 3x_0 k^2\, e^{\mu ht+\frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k)-\sigma\sqrt{ht})\} - k^3\overline{F}_Y\{y(k)\}$$

and the fourth raw moment

$$E[(X_t-k)^4 \vee 0] = \int_k^\infty (x-k)^4 f_{X_t}(x)dx$$
$$= \mu_4(X_t)\overline{F}_Y\{y(k)-4\sigma\sqrt{ht})\} - 4\mu_3(X_t)k\overline{F}_Y\{y(k)-3\sigma\sqrt{ht})\}$$
$$+ 6\mu_2(X_t)k^2\overline{F}_Y\{y(k)-2\sigma\sqrt{ht})\} - 4\mu_1(X_t)k^3\overline{F}_Y\{y(k)-\sigma\sqrt{ht})\}$$
$$- k^4\overline{F}_Y\{y(k)\}$$
$$= x_0^4 e^{4\mu ht+8\sigma^2 ht}\overline{F}_Y\{y(k)-4\sigma\sqrt{ht})\} - 4x_0^3 k\ e^{3\mu ht+4\frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k)-3\sigma\sqrt{ht})\}$$
$$+ 6x_0^2 k^2 e^{2\mu ht+2\sigma^2 ht}\overline{F}_Y\{y(k)-2\sigma\sqrt{ht})\} - 4x_0 k^3 e^{\mu ht+\frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k)-\sigma\sqrt{ht})\}$$
$$- k^4\overline{F}_Y\{y(k)\}$$

The general r^{th} raw moment of the excess function is

$$E[(X_t-k)^r \vee 0] = \int_k^\infty (x-k)^r f_{X_t}(x)dx$$
$$= \sum_{j=0}^r (-1)^{r-j}\binom{r}{j}\mu_j(X_t)\ k^{r-j}\overline{F}_Y\{y(k)-j\sigma\sqrt{ht})\}$$
$$= \sum_{j=0}^r (-1)^{r-j}\binom{r}{j}x_0^j k^{r-j} e^{j\mu ht+\frac{1}{2}j^2\sigma^2 ht}\overline{F}_Y\{y(k)-j\sigma\sqrt{ht})\}$$

which is in the r^{th} power of the units of the variable. To see this, observe that both the initial value x_0 and the exercise value are both dollar quantities. Each term contains a product of powers of the initial value and the exercise value, the sum of the indices of each equalling the order of the moment, r. Hence the units are dollars raised to the r^{th} power. The magnitude of the limited moment depends on the relation between x_0 and k. The bigger x_0 is in relation to k, the nearer the expectation is to $E(X_t^r)$, which is the value of the expectation for which k is zero. The moment, for any order, is therefore an increasing function of x_0 and a decreasing function of k. The variance of the option expiry is

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$$
\begin{aligned}
&Var[(X_t - k) \vee 0] \\
&= E[(X_t - k)^2 \vee 0] - E^2[(X_t - k) \vee 0] \\
&= \mu_2(X_t)\overline{F}_Y\{y(k) - 2\sigma\sqrt{ht})\} - 2\mu_1(X_t)\overline{F}_Y\{y(k) - \sigma\sqrt{ht})\} + k^2\overline{F}_Y\{y(k)\} \\
&\quad - \left[\mu_1(X_t)\overline{F}_Y\{y(k) - \sigma\sqrt{ht})\} - k\overline{F}_Y\{y(k)\}\right]^2 \\
&= x_0^2 e^{2\mu ht + 2\sigma^2 ht}\overline{F}_Y\{y(k) - 2\sigma\sqrt{ht})\} - 2x_0 k\, e^{\mu ht + \frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k) - \sigma\sqrt{ht})\} + k^2\overline{F}_Y\{y(k)\} \\
&\quad - \left[x_0 e^{\mu ht + \frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k) - \sigma\sqrt{ht})\} - k\overline{F}_Y\{y(k)\}\right]^2 \\
&= x_0^2 e^{2\mu ht + \sigma^2 ht}\left[e^{\sigma^2 ht}\overline{F}_Y\{y(k) - 2\sigma\sqrt{ht})\} - \overline{F}_Y\{y(k) - \sigma\sqrt{ht})\}^2\right] \\
&\quad - 2x_0 k\, e^{\mu ht + \frac{1}{2}\sigma^2 ht}\overline{F}_Y\{y(k) - \sigma\sqrt{ht})\}\left[1 - \overline{F}_Y\{y(k)\}\right] \\
&\quad + k^2\overline{F}_Y\{y(k)\}\left[1 - \overline{F}_Y\{y(k)\}\right]
\end{aligned}
$$

Higher central moments of the excess function on a geometric Brownian motion are similar but unwieldy and can be programmed using the formulae

$$E[(X_t - k)^r \vee 0] = \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} x_0^r k^{r-j} e^{j\mu ht + \frac{1}{2}j^2\sigma^2 ht}\overline{F}_Y\{y(k) - j\sigma\sqrt{ht})\}$$

and $\sigma_n = \sum_{j=0}^{n} {}^nC_j (-1)^j \mu_j \mu^{n-j}$ where $\mu_j = E[(X_t - k)^j \vee 0]$

Algebraic analysis of skew and kurtosis is omitted due to the unwieldiness of the formulae.

8) Raw Moments of Shortfall Function (Put Option Expiry)

The undiscounted value of a put option, the first moment of the reflected and location shifted limited distribution (the residual function), is

$$
\begin{aligned}
&E[(k - X_t) \vee 0] = \int_0^k (k - x) f_{X_t}(x)dx \\
&= k\ F_Y\{y(k)\} - \mu_1(X_t)F_Y\{y(k) - \sigma\sqrt{ht})\} \\
&= k\ F_Y\{y(k)\} - x_0 e^{\mu ht + \frac{1}{2}\sigma^2 ht}F_Y\{y(k) - \sigma\sqrt{ht})\}
\end{aligned}
$$

Continuing for higher raw moments of the shortfall function, the second limited raw moment of log-normals, the second is

$$
\begin{aligned}
&E[(k - X_t)^2 \vee 0] = \int_0^k (k - x)^2 f_{X_t}(x)dx \\
&= k^2 F_Y\{y(k)\} - 2\mu_1(X_t)k\ F_Y\{y(k) - \sigma\sqrt{ht})\} + \mu_2(X_t)F_Y\{y(k) - 2\sigma\sqrt{ht})\} \\
&= k^2 F_Y\{y(k)\} - 2x_0 k\, e^{\mu ht + \frac{1}{2}\sigma^2 ht}F_Y\{y(k) - \sigma\sqrt{ht})\} + x_0^2 e^{2\mu ht + 2\sigma^2 ht}F_Y\{y(k) - 2\sigma\sqrt{ht})\}
\end{aligned}
$$

and third raw moment

$$
\begin{aligned}
&E[(k - X_t)^3 \vee 0] = \int_0^k (k - x)^3 f_{X_t}(x)dx \\
&= k^3 F_Y\{y(k)\} - 3\mu_1(X_t)k^2 F_Y\{y(k) - \sigma\sqrt{ht})\} + 3\mu_2(X_t)k\ F_Y\{y(k) - 2\sigma\sqrt{ht})\} - \mu_3(X_t)F_Y\{y(k) - 3\sigma\sqrt{ht})\} \\
&= k^3 F_Y\{y(k)\} - 3x_0 k^2\, e^{\mu ht + \frac{1}{2}\sigma^2 ht}F_Y\{y(k) - \sigma\sqrt{ht})\} \\
&\quad + 3x_0^2 k\ e^{2\mu ht + 2\sigma^2 ht}F_Y\{y(k) - 2\sigma\sqrt{ht})\} - x_0^3 e^{3\mu ht + 4\frac{1}{2}\sigma^2 ht}F_Y\{y(k) - 3\sigma\sqrt{ht})\}
\end{aligned}
$$

and raw fourth

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$$E[(k - X_t)^4 \vee 0] = \int_0^k (k - x)^4 f_{X_t}(x)dx$$
$$= k^4 F_Y\{y(k)\} - 4\mu_1(X_t)k^3 F_Y\{y(k) - \sigma\sqrt{ht})\} + 6\mu_2(X_t)k^2 F_Y\{y(k) - 2\sigma\sqrt{ht})\}$$
$$- 4\mu_3(X_t)k\ F_Y\{y(k) - 3\sigma\sqrt{ht})\} + \mu_4(X_t)F_Y\{y(k) - 4\sigma\sqrt{ht})\}$$
$$= k^4 F_Y\{y(k)\} - 4x_0 k^3 e^{\mu ht + \frac{1}{2}\sigma^2 ht} F_Y\{y(k) - \sigma\sqrt{ht})\} + 6x_0^2 k^2 e^{2\mu ht + 2\sigma^2 ht} F_Y\{y(k) - 2\sigma\sqrt{ht})\}$$
$$- 4x_0^3 k\ e^{3\mu ht + 4\frac{1}{2}\sigma^2 ht} F_Y\{y(k) - 3\sigma\sqrt{ht})\} + x_0^4 e^{4\mu ht + 8\sigma^2 ht} F_Y\{y(k) - 4\sigma\sqrt{ht})\}$$

The general r^{th} raw moment of the shortfall function is

$$E[(k - X_t)^r \vee 0] = \int_0^k (k - x)^r f_{X_t}(x)dx$$
$$= \sum_{j=0}^r (-1)^j \binom{r}{j} \mu_j(X_t)\ k^{r-j} F_Y\{y(k) - j\sigma\sqrt{ht})\}$$
$$= \sum_{j=0}^r (-1)^j \binom{r}{j} x_0^j k^{r-j} e^{j\mu ht + \frac{1}{2}j^2\sigma^2 ht} F_Y\{y(k) - j\sigma\sqrt{ht})\}$$

which is, like that for the excess function, in the r^{th} power of the units of the variable. The magnitude of the shortfall limited moment depends on the relation between x_0 and k. The smaller x_0 is in relation to k, the nearer the expectation is to $E[(-X_t)^r]$, which is the value of the expectation for which k is zero. The moment is therefore a decreasing function of x_0 and an increasing function of k. The undiscounted variance of a put option expiry is

$$Var[(k - X_t) \vee 0]$$
$$= E[(k - X_t)^2 \vee 0] - E^2[(k - X_t) \vee 0]$$
$$= k^2 F_Y\{y(k)\} - 2\mu_1(X_t)F_Y\{y(k) - \sigma\sqrt{ht})\} + \mu_2(X_t)F_Y\{y(k) - 2\sigma\sqrt{ht})\}$$
$$- \left[kF_Y\{y(k)\} - \mu_1(X_t)F_Y\{y(k) - \sigma\sqrt{ht})\}\right]^2$$
$$= k^2 F_Y\{y(k)\} - 2x_0 k\, e^{\mu ht + \frac{1}{2}\sigma^2 ht} F_Y\{y(k) - \sigma\sqrt{ht})\} + x_0^2 e^{2\mu ht + 2\sigma^2 ht} F_Y\{y(k) - 2\sigma\sqrt{ht})\}$$
$$- \left[kF_Y\{y(k)\} - x_0 e^{\mu ht + \frac{1}{2}\sigma^2 ht} F_Y\{y(k) - \sigma\sqrt{ht})\}\right]^2$$
$$= k^2 F_Y\{y(k)\}\left[1 - F_Y\{y(k)\}\right]$$
$$- 2x_0 k\, e^{\mu ht + \frac{1}{2}\sigma^2 ht} F_Y\{y(k) - \sigma\sqrt{ht})\}\left[1 - F_Y\{y(k)\}\right]$$
$$+ x_0^2 e^{2\mu ht + \sigma^2 ht}\left[e^{\sigma^2 ht} F_Y\{y(k) - 2\sigma\sqrt{ht})\} - F_Y\{y(k) - \sigma\sqrt{ht})\}^2\right]$$

Higher central moments are similar but unwieldy and can be programmed using the formulae

$$E[(k - X_t)^r \vee 0] = \sum_{j=0}^r (-1)^j \binom{r}{j} x_0^j k^{r-j} e^{j\mu ht + \frac{1}{2}j^2\sigma^2 ht} F_Y\{y(k) - j\sigma\sqrt{ht})\}$$

and $\sigma_n = \sum_{j=0}^n {}^nC_j (-1)^j \mu_j \mu^{n-j}$ where $\mu_j = E[(k - X_t)^j \vee 0]$

Again, algebraic analysis of skew and kurtosis is omitted due to the unwieldiness of the formulae. The numerical evaluations may prove to be very useful in an applied setting.

Simulation Algorithm

Generate a path of an ordinary Brownian motion in log-parameters (log mean and log standard deviation) and exponentiate the components of the path and plot these ordinates against a mantissa vector indexing time. Streamlined in MATLAB, this is

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```
x0=x0; mu=μ; sigma=σ; h=h; n=n;          %initialize diffusion parameters
z=mu.*h+sigma.*randn(n,1).*sqrt(h); x=exp(cumsum([x0;z]));        %simulate
plot(x);                                                            %plot
```

Simulated Proportional Brownian Motion

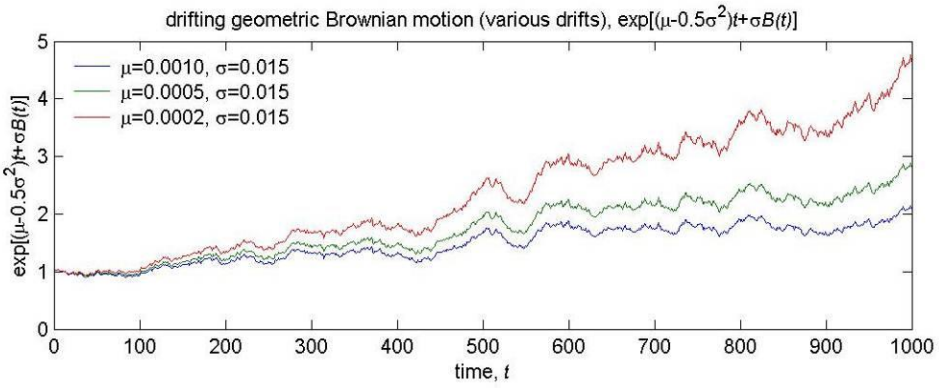


Figure 17: Single simulated path of drifting geometric Brownian motion for various drifts

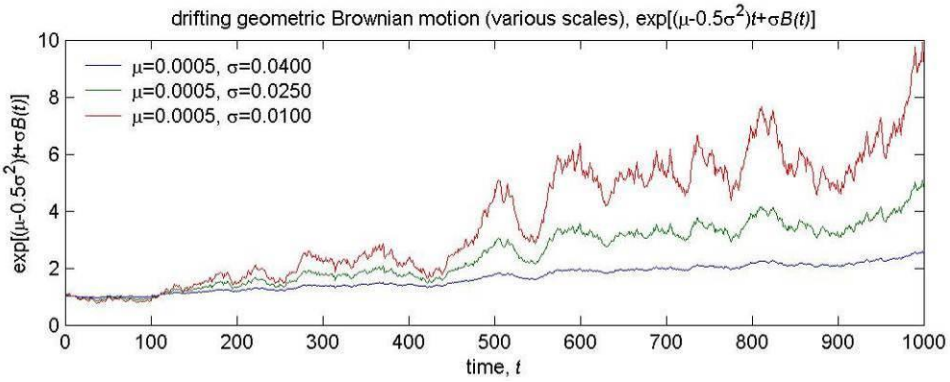


Figure 18: Single simulated path of drifting geometric Brownian motion for various scales

MODULATED BROWNIAN MOTION WITH ZERO BASIS MEAN (THE ORNSTEIN-UHLENBECK PROCESS)

Introduction and Discussion

The Ornstein-Uhlenbeck process[16] is modulated Brownian motion. Like Brownian motion, the Ornstein-Uhlenbeck process is additive and has a normal transition distribution. The infinitesimal convolutions are modulated by a time correlation parameter whereby the diffusion is defined as an exponential integration of a zero mean diffusion basis. This time correlation parameter is the infinitesimal characteristic root of the first order diffusion. If this root is zero, the exponential factor is null and the process reverts to a scaled standard Brownian motion, $dXt = \sigma dBt$. This is equivalent to a discrete unit-root. If the reversionary pull is towards zero, the process is recurrent. If it is pushing the process away from zero it is transient. Scaled standard Brownian motion is the dividing line between recurrence and transience. In both cases the process is stationary for $|\alpha| < 1$. A change of sign of the characteristic root results in a change in the nature of the diffusion from a recurrent process to a transient process or from a transient process to a recurrent process. The Ornstein-Uhlenbeck diffusion considered here are considered by default to be recurrent, in which case the modulating diffusion root is synonymously re-

[16] The Ornstein-Uhlenbeck process is the term used by Feller, Cox and Miller (although not Bartlett or Bailey) and for this reason this thesis has referred to it as such; the original paper is cited in Cox and Miller
Uhlenbeck, G. E. and Ornstein, L. S., (1930) *On the Theory of Brownian Motion. Phys. Rev.* No. 36, pp. 823-841.
It is also nomenclature of mathematical physics.

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ferred to as a moderated root. Depending on the sign of α, the further the process is from zero, the stronger the reversionary force towards, or away from, the origin.

This is the standard model of first order additive normal diffusion for which Brownian motion is a special case. The Ornstein-Uhlenbeck behaves differently on different time scales. As such, the process is not statistically fractal. The infinitesimal parameters are $\mu(x, t) = -\alpha x$ and $\sigma(x, t) = \sigma$ and the stochastic differential increment is

$$dX_t = -\alpha X_t dt + \sigma dB_t \qquad \text{where} \qquad dB_t \sim N(0, dt)$$

The Ornstein-Uhlenbeck process can be also interpreted as a Brownian motion with time dilation. This time dilation can either take the form of time expansion or time compression, depending on the sign of the characteristic root, and equivalently, whether the diffusion is recurrent or transient. This fact is exploited in order to derive the integral expression of the Ornstein-Uhlenbeck process in terms of standard Brownian motion that is free from the integral operator. When the characteristic root is zero, there is no time dilation in the summation of the diffusion basis and the process is a standard (scaled) Brownian motion. When the time dilation results in a slowing of the aggregation, the process has the physical interpretation of a Brownian motion with viscous friction and is alternatively referred to as damped Brownian motion.

Integral Solution to Modulated Brownian Motion With Zero Basis Mean

Unlike Brownian motion which solves by direct integration since all terms are exact differentials due to the unit-root, the integral solution of the Ornstein-Uhlenbeck process proceeds by an application of an integrating factor. This factor removes the characteristic root and consequently introduces time correlation into the integration of the diffusion basis. From the stochastic differential,

$$dX_t = -\alpha X_t dt + \sigma dB_t$$

$$dX_t + \alpha X_t dt = \sigma dB_t$$

which is linear and first order in the diffusion variable with respect to time. The integrating factor is

$IF(t) = \exp\left(\int \alpha dt\right) = e^{\alpha t}$; multiplying both sides by this factor and factorizing the diffusion variable of the equation into an exact differential,

$$e^{\alpha t}(dX_t + \alpha X_t dt) = \sigma e^{\alpha t} dB_t$$

$$d(e^{\alpha t} X_t) = \sigma e^{\alpha t} dB_t$$

$$\frac{d}{dt}(e^{\alpha t} X_t) = \sigma e^{\alpha t} \frac{dB_t}{dt}$$

The solution of this equation can proceed by two possible avenues; either both sides can be integrated and left in integral form whereby the right-hand-side is left as an exponentially weighted integral of differential Brownian motion. Alternatively, the integration can proceed with a change of variable so that the time integrals on both sides are transformed to integrals of exact differentials and are hence integrable. The solution is then free from integral operators but the Brownian motion is dependent on a modified function of time. The transition distribution of the Ornstein-Uhlenbeck process is then obtained directly from inspection. By the first avenue, integrating over time from time zero to time t

$$\int_0^t \frac{d}{dy}(e^{\alpha y} X_y) dy = \sigma \int_0^t e^{\alpha y} \frac{dB_y}{dy} dy$$

$$e^{\alpha t} X_t = x_0 + \sigma \int_0^t e^{\alpha y} dB_y$$

or

$$X_t = x_0 e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha y} dB_y$$

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which has utility in visually contrasting the

(1) $\int_0^t dB_y$ of a Brownian motion and **(2)** $e^{-\alpha t}\int_0^t e^{\alpha y}dB_y$ of a Ornstein-Uhlenbeck process

the former being a special case of the latter, both sometimes providing the definitions of the respective processes. This is looked at in greater depth in BOOK 3. A further application of parts to the integral term gives

$$X_t = x_0 e^{-\alpha t} + \sigma B_t - \alpha\sigma e^{-\alpha t}\int_0^t e^{\alpha y}B_y dy$$

which is a Brownian motion reduced by a fraction of the exponential integration of the path of the Brownian motion, this fractional reduction being proportional to the infinitesimal characteristic root of the diffusion, α. From this expression, it is seen quite clearly that the process reverts to a Brownian motion when α is set to zero – the term containing the exponentially integrated path zeroes leaving only the Brownian term σB_t and the diffusion initial value x_0. By the alternative avenue to solve the differentially factorized equation $d(e^{\alpha t}X_t) = \sigma e^{\alpha t}dB_t$, transforming from linear time with a change of variable t to $s = (e^{2\alpha t} - 1)/2\alpha$, the derivative of which is $ds = e^{2\alpha t}dt$, i.e., $ds = (1 + 2\alpha)dt$. The integrating factor in the new time variable is $e^{\alpha t} = \sqrt{1 + 2\alpha s}$ and $t(s) = \log(1 + 2\alpha s)/2\alpha$, and the equation

$$\frac{d}{dt}(e^{\alpha t}X_t) = \sigma e^{\alpha t}\frac{dB_t}{dt}$$

becomes

$$\frac{ds}{dt}\times\frac{d}{ds}(X_{t(s)}\sqrt{1+2\alpha s}) = \sigma\sqrt{1+2\alpha s}\,\frac{dB_t}{dt}$$

or

$$\frac{d}{ds}(X_{t(s)}\sqrt{1+2\alpha s}) = \sigma\sqrt{1+2\alpha s}\,\frac{dt}{ds}\times\frac{dB_t}{dt}$$

which requires distribution theory rather than ito-calculus

$$dB_t \sim N(0, dt)$$

$$\frac{dB_t}{ds} \sim N\left(0, \frac{1}{ds}\times\frac{dt}{ds}\right)$$
$$\sim N\left(0, \frac{1}{ds}\times\frac{1}{1+2\alpha s}\right)$$

$$\sqrt{1+2\alpha s}\,\frac{dB_t}{ds} \sim N\left(0, \frac{1}{ds}\times\frac{1}{1+2\alpha s}\right)$$
$$\sim N\left(0, (1+2\alpha s)\times\frac{1}{ds}\times\frac{1}{1+2\alpha s}\right)$$
$$\sim N\left(0, \frac{1}{ds}\right)$$
$$\sim \frac{1}{ds}N(0, ds)$$

that is,

$$\sqrt{1+2\alpha s}\,\frac{dB_t}{ds} = \frac{dB_s}{ds}$$

so that

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$$\frac{d}{ds}(X_{t(s)}\sqrt{1+2\alpha s}) = \sigma\frac{dB_s}{dt}$$

Integrating,

$$\int_0^s \frac{d}{dy}(X_{t(s)}\sqrt{1+2\alpha y})dy = \sigma\int_0^s \frac{dB_y}{dy}dy$$

$$X_{t(s)}\sqrt{1+2\alpha s} - x_{t(0)} = \sigma B_s - \sigma B_0$$

or

$$X_{t(s)} = \frac{x_0}{\sqrt{1+2\alpha s}} + \frac{\sigma}{\sqrt{1+2\alpha s}}B_s$$

which has the transition distribution in the exponentially transformed time variable of

$$X_{t(s)} \sim N\left(\frac{x_0}{\sqrt{1+2\alpha s}}, \frac{\sigma^2 s}{1+2\alpha s}\right)$$

Changing back to linear time by noting that $1/\sqrt{1+2\alpha s} = e^{-\alpha t}$, gives the integral solution

$$X_t = x_0 e^{-\alpha t} + \sigma e^{-\alpha t} B_{(e^{2\alpha t}-1)/2\alpha}$$

and further noting that $B_{(e^{2\alpha t}-1)/2\alpha} \sim N\left(0, \frac{e^{2\alpha t}-1}{2\alpha}\right)$ gives the transition distribution

$$X_t \sim N\left(x_0 e^{-\alpha t}, \sigma^2\frac{1-e^{-2\alpha t}}{2\alpha}\right)$$

which is the transition distribution for which the density function is now also obtained by solving the forward equation. The transition distribution for a transient Ornstein-Ulhenbeck diffusion is

$$X_t \sim N\left(x_0 e^{\alpha t}, \sigma^2\frac{e^{2\alpha t}-1}{2\alpha}\right)$$

The mgf of the transition distribution, inclusive of forward and backward variables $X(T)|X(t_0) = x_0$, is

$$M_{X(T)|X(t_0)}(\theta) = \exp\left(x_0 e^{-\alpha(T-t_0)}\theta + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha(T-t_0)}}{2\alpha}\right)\theta^2\right)$$

The Forward and Backward Equations for Modulated Brownian Motion

A. Formation of the Forward Equation

Modulating the mgf parameter

$$M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) = \exp\left(x_0\theta + \tfrac{1}{2}\sigma^2\left(\frac{e^{2\alpha(T-t_0)}-1}{2\alpha}\right)\theta^2\right)$$

Differentiating the resulting mgf partially with respect to the forward time variable gives the forward equation in the mgf

$$\frac{\partial}{\partial T}M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) + \frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)})\frac{d}{dT}\theta e^{\alpha(T-t_0)} = \left(\tfrac{1}{2}e^{2\alpha(T-t_0)}\sigma^2\theta^2\right)M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)})$$

i.e.,

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$$\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) + \alpha e^{\alpha(T-t_0)} \theta \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) = \tfrac{1}{2} e^{2\alpha(T-t_0)} \sigma^2 \theta^2 M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)})$$

and in the unmodulated mgf parameter

$$\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta) + \alpha\theta \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta) = \tfrac{1}{2} \sigma^2 \theta^2 M_{X(T)|X(t_0)}(\theta)$$

From the definition of the transition mgf

$$M_{X(T)|X(t_0)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0) dx$$

Differentiating the mgf partially with respect to the mgf parameter and the forward diffusion variable, and then twice with respect to forward diffusion variable, i.e.,

$$\frac{\partial^2}{\partial x \partial \theta} M_{X(T)|X(t_0)}(\theta) \qquad \text{and} \qquad \frac{\partial^2}{\partial x^2} M_{X(T)|X(t_0)}(\theta)$$

for

(1) $\theta \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta) = -\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x\, f_{X(T)|X(t_0)}(x \mid x_0) dx$

(2) $\theta^2 M_{X(T)|X(t_0)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) dx$

Hence the forward equation can be written in terms of the transition density as

$$\frac{\partial}{\partial T} \int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0) dx - \alpha \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x\, f_{X(T)|X(t_0)}(x \mid x_0) dx = \tfrac{1}{2} \sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) dx$$

and upon inversion of the mgf results in the forward equation of standard modulated Brownian motion, the modulated heat equation, in the transition density

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) + \alpha \frac{\partial}{\partial x} \left(x\, f_{X(T)|X(t_0)}(x \mid x_0) \right)$$

In differentially factorized form, the forward equation is expressible as

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2} \sigma^2 \frac{\partial}{\partial x} \left(e^{-\alpha x^2/\sigma^2} \frac{\partial}{\partial x} e^{\alpha x^2/\sigma^2} f_{X(T)|X(t_0)}(x \mid x_0) \right)$$

B. Formation of the Backward Equation

Differentiating the mgf partially with respect to the backward time variable

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = \left(\alpha x_0 e^{-\alpha(T-t_0)} \theta - \tfrac{1}{2} \sigma^2 e^{-2\alpha(T-t_0)} \theta^2 \right) M_{X(T)|X(t_0)}(\theta)$$

Differentiating partially with respect to the backward diffusion variable

$$\frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) = e^{-\alpha(T-t_0)} \theta M_{X(T)|X(t_0)}(\theta)$$

and again

$$\frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta) = e^{-2\alpha(T-t_0)} \theta^2 M_{X(T)|X(t_0)}(\theta)$$

Hence, the time derivative can be written as

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$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = \left(\alpha x_0 \frac{\partial}{\partial x_0} - \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} \right) M_{X(T)|X(t_0)}(\theta)$$

which results the backward equation of standard modulated Brownian motion (OU), the Hermite conjugate of the modulated heat equation, in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} f_{X(T)|X(t_0)}(x \mid x_0) + \alpha x_0 \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0)$$

With the diffusion side factorized, this is expressible as

$$-\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 e^{\alpha x_0^2/\sigma^2} \frac{\partial}{\partial x_0} \left(e^{\alpha x_0^2/\sigma^2} \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0) \right)$$

Solution to the Forward Equation of Modulated Brownian Motion: (Transforming to the Heat Equation)

The forward equation of the Ornstein-Uhlenbeck process mean reverting to zero, the exponential integration of a zero mean diffusion basis, is

$$\frac{\partial}{\partial t} f_{X(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{X(t)}(x) + \alpha \frac{\partial}{\partial x}\left(x\, f_{X(t)}(x) \right)$$

Transforming the diffusion variable $X(t)$ to $Y(t) = e^{\alpha t}X(t)$; The transformed independent forward diffusion variable is $y = e^{\alpha t}x$ giving the Jacobian of the transformation

$$J = \frac{dx}{dy} = e^{-\alpha t}$$

The normalized density function of the transformed diffusion variable is

$$f_{Y(t)}(y) = f_{X(t)}(x) \mid J \mid = e^{-\alpha t} f_{X(t)}(x)$$

which also implies $f_{X(t)}(x) = e^{\alpha t} f_{Y(t)}(y)$ and the relation $x\, f_{X(t)}(x) = y\, f_{Y(t)}(y)$. The relevant partial derivatives of the transform are

$$\frac{\partial y}{\partial x} = e^{\alpha t} \quad \text{and} \qquad \frac{\partial y}{\partial t} = \alpha e^{\alpha t} x = \alpha e^{\alpha t}(y e^{-\alpha t}) = \alpha y$$

The transformed partial derivatives of the forward equation are

(1) $$\frac{\partial}{\partial x}[x f_{X(t)}(x)] = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}[y f_{Y(t)}(y)] = e^{\alpha t}\frac{\partial}{\partial y}[y f_{Y(t)}(y)] = e^{\alpha t} f_{Y(t)}(y) + e^{\alpha t} y \frac{\partial}{\partial y} f_{Y(t)}(y)$$

(2) $$\frac{\partial^2}{\partial x^2} f_{X(t)}(x) = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left[\frac{\partial y}{\partial x}\frac{\partial}{\partial y} e^{\alpha t} f_{Y(t)}(x) \right] = e^{\alpha t}\frac{\partial}{\partial y}\left[e^{2\alpha t}\frac{\partial}{\partial y} f_{Y(t)}(y) \right] = e^{3\alpha t}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y)$$

(3) $$\frac{\partial}{\partial t} f_{X(t)}(x) = \frac{\partial}{\partial t}[e^{\alpha t} f_{Y(t)}(y)] = \alpha e^{\alpha t} f_{Y(t)}(y) + e^{\alpha t}\frac{\partial}{\partial t} f_{Y(t)}(y) + e^{\alpha t}\frac{\partial y}{\partial t}\frac{\partial}{\partial y} f_{Y(t)}(y)$$

$$= \alpha e^{\alpha t} f_{Y(t)}(y) + e^{\alpha t}\frac{\partial}{\partial t} f_{Y(t)}(y) + \alpha e^{\alpha t} y \frac{\partial}{\partial y} f_{Y(t)}(y)$$

giving

$$\alpha e^{\alpha t} f_{Y(t)}(y) + e^{\alpha t}\frac{\partial}{\partial t} f_{Y(t)}(y) + \alpha e^{\alpha t} y \frac{\partial}{\partial y} f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2 e^{3\alpha t}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y) + \alpha e^{\alpha t} f_{Y(t)}(y) + \alpha e^{\alpha t} y \frac{\partial}{\partial y} f_{Y(t)}(y)$$

or

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$$\frac{\partial}{\partial t} f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2 e^{2\alpha t} \frac{\partial^2}{\partial y^2} f_{Y(t)}(y)$$

which is a variant of the standard heat equation, the heat equation with an exponential diffusion coefficient. This is the equation that is now solved. Converting the partial differential equation in the transition density to an ordinary differential equation in the transition moment generating function,

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{Y(t)}(y)dy = \tfrac{1}{2}\sigma^2 e^{2\alpha t} \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial y^2} f_{Y(t)}(y)dy$$

or

$$\frac{\partial}{\partial t} M_{Y(t)}(\theta) = \tfrac{1}{2}\sigma^2 e^{2\alpha t}\theta^2 M_{Y(t)}(\theta)$$

or in the cumulant generating function

$$\frac{\partial}{\partial t} \log M_{Y(t)}(\theta) = \tfrac{1}{2}\sigma^2 e^{2\alpha t}\theta^2$$

which is an exact derivative. Integrating from time zero to time t over the dummy s,

$$\int_0^t \frac{\partial}{\partial s} \log M_{Y(s)}(\theta)ds = \tfrac{1}{2}\sigma^2\theta^2 \int_0^t e^{2\alpha s} ds$$

gives

$$\log M_{Y(t)}(\theta) - \log M_{Y(0)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 \left(\frac{e^{2\alpha t} - 1}{2\alpha} \right)$$

or

$$\log M_{Y(t)}(\theta) = \theta y_0 + \tfrac{1}{2}\sigma^2\theta^2 \left(\frac{e^{2\alpha t} - 1}{2\alpha} \right)$$

which is the cumulant generating function of a normal random variable with mean y_0 and variance $\sigma^2 (e^{2\alpha t} - 1) / 2\alpha$. The cumulant generating function of the original diffusion variable, $X(t) = e^{-\alpha t} Y(t)$, in terms of the transformed variable is

$$\log K_{X(t)}(\theta) = \log K_{e^{-\alpha t} Y(t)}(\theta) = \log K_{Y(t)}(e^{-\alpha t}\theta)$$

Noting that $X(0) = Y(0)$, and hence $y_0 = x_0$, the cumulant generating function of the original diffusion variable becomes

$$\log M_{Y(t)}(\theta) = (e^{-\alpha t}\theta)x_0 + \tfrac{1}{2}\sigma^2 (e^{-\alpha t}\theta)^2 \left(\frac{e^{2\alpha t} - 1}{2\alpha} \right)$$

that is,

$$\log M_{Y(t)}(\theta) = x_0 e^{-\alpha t}\theta + \tfrac{1}{2}\sigma^2\theta^2 \left(\frac{1 - e^{-2\alpha t}}{2\alpha} \right)$$

This is the cumulant generating function of a normal variable with mean $x_0 e^{-\alpha t}$ and variance $\sigma^2 (1 - e^{-2\alpha t}) / 2\alpha$

$$X(t) \sim N\left(x_0 e^{-\alpha t}, \sigma^2 \frac{1 - e^{-2\alpha t}}{2\alpha} \right)$$

for which the transition density function is

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$$f_{X(t)}(x) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{(1-e^{-2\alpha t})/2\alpha}} \exp\left(- \frac{[x - x_0 e^{-\alpha t}]^2}{\sigma^2 (1-e^{-2\alpha t})/\alpha} \right) \qquad \text{for } -\infty < x < \infty \text{ and } t \geq 0$$

which is the solution to the forward equation. Alternatively, the variant of the heat equation with an exponential diffusion coefficient can be converted to the standard heat equation with a change of time variable, $s = \sigma^2 (e^{2\alpha t} - 1)/2\alpha$, the derivative of the time change being

$$\frac{\partial s}{\partial t} = \sigma^2 e^{-2\alpha t}$$

giving

$$\tfrac{1}{2}\sigma^2 e^{2\alpha t} \frac{\partial^2}{\partial y^2} f_{Y(t)}(y) = \frac{\partial}{\partial t} f_{Y(t)}(y) = \frac{\partial s}{\partial t}\frac{\partial}{\partial s} f_{Y(s)}(y) = \sigma^2 e^{2\alpha t} \frac{\partial}{\partial s} f_{Y(s)}(y)$$

and canceling $\sigma^2 e^{2\alpha t}$ from both sides gives the standard heat equation

$$\tfrac{1}{2}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y) = \frac{\partial}{\partial s} f_{Y(s)}(y)$$

Solution can then follow by the ordinary methods by taking the transition distribution of a standard Brownian motion corresponding to the standard heat equation, $Y(s) \sim N(y_0, s)$, and retransforming the transition distribution back to that of the time variable of the original forward equation and a further transformation back to the original diffusion variable $X(t) = e^{-\alpha t} Y(t)$. Solution via the variant of the heat equation was chosen because it stresses the exponential integration of the variance of the diffusion basis.

Solution to the Forward Equation of Modulated Brownian Motion: (Solving the Equation Directly)

Converting the forward equation of the density function to that of the moment generating function by multiplying by the exponential factor in the diffusion variable and integrating over the range of the diffusion

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{X(t)}(x)dx = \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(t)}(x)dx + \alpha \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x}\left(x\, f_{X(t)}(x)\right)dx$$

and applying the results for moment generating functions of derivatives of density functions gives

$$\frac{\partial}{\partial t} M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 M_{X(t)}(\theta) + \alpha \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x}\left(x\, f_{X(t)}(x)\right)dx$$

Integrating the last term by parts with

$$u = e^{\theta x} \qquad \text{and} \quad dv = \frac{\partial}{\partial x}\left(x\, f_{X(t)}(x)\right)dx$$

and again, assuming that the density function, $x f_{X(t)}(x)$, is sub-exponential so that $[xe^{\theta x} f_{X(t)}(x)]_{-\infty}^{\infty}$ vanishes, gives

$$\frac{\partial}{\partial t} M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 M_{X(t)}(\theta) - \alpha\theta \int_{-\infty}^{\infty} xe^{\theta x} f_{X(t)}(x)dx$$

or

$$\frac{\partial}{\partial t} M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 M_{X(t)}(\theta) - \alpha\theta \frac{\partial}{\partial \theta} M_{X(t)}(\theta)$$

which is a partial differential equation of first order in both the diffusion variable and time. Dividing both sides by dependent variable $M_{X(t)}(\theta)$ simplifies the partial differential equation to an equation for the cumulant generating function

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$$\frac{\partial}{\partial t}\log M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta\frac{\partial}{\partial\theta}\log M_{X(t)}(\theta)$$

This is the equivalent forward equation in the cumulant generating function. This equation describes how the cumulants of the components of the diffusion sum with time. There are two alternative methods to solve this equation. The first is by transforming the moment generating function parameter to remove the exponentially modulated summation of the diffusion cumulants and subsequently solving the resulting unit-root diffusion in an exponential diffusion coefficient. The second method is to remove the time domain with a Laplace transform, solve the resulting ordinary differential equation, and then subsequently reintroduce the time domain with an inverse Laplace transform.

Transforming the MGF Parameter

$$\frac{\partial}{\partial t}\log M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta\frac{\partial}{\partial\theta}\log M_{X(t)}(\theta)$$

Transforming the second independent variable, the moment generating function parameter, in order to remove the exponential weighting in the first order summation of the diffusion basis[17]

$$\phi = \theta e^{-\alpha t}$$

with inverse $\theta = \phi e^{\alpha t}$. The derivatives of the transform are $\frac{\partial\phi}{\partial t} = -\alpha\theta e^{-\alpha t} = -\alpha\phi$ and $\frac{\partial\phi}{\partial\theta} = e^{-\alpha t}$ so that

$$\textbf{(1)}\quad \frac{\partial}{\partial\theta}\log M_{X(t)}(\theta) = \frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\phi}\log M_{X(t)}(\phi e^{\alpha t})$$
$$= e^{-\alpha t}\frac{\partial}{\partial\phi}\log M_{X(t)}(\phi e^{\alpha t})$$
$$\textbf{(2)}\quad \frac{\partial}{\partial t}\log M_{X(t)}(\theta) = \frac{\partial}{\partial t}\log M_{X(t)}(\phi e^{\alpha t}) + \frac{\partial\phi}{\partial t}\frac{\partial}{\partial\phi}\log M_{X(t)}(\phi e^{\alpha t})$$
$$= \frac{\partial}{\partial t}\log M_{X(t)}(\phi e^{\alpha t}) - \alpha\theta e^{-\alpha t}\frac{\partial}{\partial\phi}\log M_{X(t)}(\phi e^{\alpha t})$$

Substituting the derivatives into the partial differential equation in the cumulant generating function leaves an equation with a single independent variable of time *t*

$$\frac{\partial}{\partial t}\log M_{X(t)}(\phi e^{\alpha t}) = \tfrac{1}{2}\sigma^2(\phi e^{\alpha t})^2$$

The is the standard heat equation in the cumulant generating function whereby the generating function parameter is modulated by the characteristic root α. Integrating

$$\int_0^t \frac{\partial}{\partial y}\log M_{X(y)}(\phi e^{\alpha y})dy = \tfrac{1}{2}\sigma^2\phi^2\int_0^t e^{2\alpha y}dy$$

$$\log M_{X(t)}(\phi e^{\alpha t}) - \log M_{X(0)}(\phi) = \tfrac{1}{2}\sigma^2\phi^2\left(\frac{e^{2\alpha t}-1}{2\alpha}\right)$$

$$\log M_{X(t)}(\phi e^{\alpha t}) = x_0\phi + \tfrac{1}{2}\sigma^2\phi^2\left(\frac{e^{2\alpha t}-1}{2\alpha}\right)$$

In terms of the original generating function parameter $\phi = \theta e^{-\alpha t}$, the solution is

[17] which is the equivalent transformation for the moment generating function parameter as that for the diffusion variable of $X(t) = e^{-\alpha t}Y(t)$ $X(t) = e^{\alpha t}Y(t)$ to transform the forward equation to the heat equation with exponential diffusion coefficient, i.e., $M_{X(t)}(\theta) = M\,e^{-\alpha t}Y(t)(\theta) = M_{Y(t)}(e^{-\alpha t}\theta) = M_{Y(t)}(\phi)$.

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$$\log M_{X(t)}(\theta) = x_0 e^{-\alpha t}\theta + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\theta^2$$

or a moment generating function of

$$M_{X(t)}(\theta) = \exp\left(x_0 e^{-\alpha t}\theta + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\theta^2\right)$$

which is the moment generating function of a normal variable with mean $x_0 e^{-\alpha t}$ and variance $\sigma^2(1 - e^{-2\alpha t}) / 2\alpha$.

Laplace Transform

Taking the Laplace transform of both sides of the forward equation in the cumulant generating function by multiplying both sides by an exponential factor in the time variable and integrating gives

$$\int_0^\infty e^{-st}\frac{\partial}{\partial t}\log M_{X(t)}(\theta)dt = \tfrac{1}{2}\sigma^2\theta^2\int_0^\infty e^{-st}dt - \alpha\theta\frac{\partial}{\partial\theta}\int_0^\infty e^{-st}\log M_{X(t)}(\theta)dt$$

Denoting the Laplace transform by $\lambda(s;\theta) = L\{\log M_{X(t)}(\theta)\}$

$$sL\{\log M_{X(t)}(\theta)\} - \log M_{X(0)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2\frac{1}{s} - \alpha\theta\frac{\partial}{\partial\theta}L\{\log M_{X(t)}(\theta)\}$$

and noting that $\log M_{X(0)}(\theta) = x_0\theta$, the ordinary differential equation becomes

$$\frac{\partial}{\partial\theta}\lambda(s;\theta) + \frac{s}{\alpha\theta}\lambda(s;\theta) = \frac{x_0}{\alpha} + \tfrac{1}{2}\sigma^2\frac{\theta}{s\alpha}$$

which is linear in λ. The logarithm of the integrating factor is

$$\log IF(\theta) = \int \frac{s}{\alpha\theta}d\theta = \frac{s}{\alpha}\log\theta$$

so that $IF(\theta) = \theta^{s/\alpha}$. Multiplying both sides by the integrating factor gives the differential in the moment generating function parameter

$$\frac{\partial}{\partial\theta}\left[\theta^{\frac{s}{\alpha}}\lambda(s;\theta)\right] = \left(\frac{x_0}{\alpha} + \tfrac{1}{2}\sigma^2\frac{\theta}{s\alpha}\right)\theta^{\frac{s}{\alpha}}$$

Integrating from zero to θ over a dummy variable for the moment generating function parameter gives the solution of the Laplace transform of the cumulant generating function,

$$\lambda(s;\theta) = \frac{x_0\theta}{s+\alpha} + \tfrac{1}{2}\sigma^2\frac{\theta^2}{s(s+2\alpha)}$$

and expanding with partial fractions

$$\lambda(s;\theta) = \frac{x_0\theta}{s+\alpha} + \tfrac{1}{2}\sigma^2\frac{1}{2\alpha}\left[\frac{1}{s} - \frac{1}{s+2\alpha}\right]\theta^2$$

Taking the inverse Laplace Transform of both sides recovers the cumulant generating function. Noting the inverse Laplace transforms of $1/s = 1$ and $1/(s - a) = e^{at}$, the inversion results in

$$\log M_{X(t)}(\theta) = L^{-1}\{\lambda(s;\theta)\} = x_0\theta L^{-1}\left\{\frac{1}{s+\alpha}\right\} + \tfrac{1}{2}\sigma^2\frac{1}{2\alpha}\left[L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+2\alpha}\right\}\right]\theta^2$$

or

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$$\log M_{X(t)}(\theta) = x_0 e^{-\alpha t}\theta + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\theta^2$$

and the moment generating function is

$$M_{X(t)}(\theta) = \exp\left(x_0 e^{-\alpha t}\theta + \tfrac{1}{2}\sigma^2\frac{1-e^{-2\alpha t}}{2\alpha}\theta^2\right)$$

which is the moment generating function of a normal with mean $x_0\, e^{-\alpha t}$ and variance $\sigma^2(1-e^{-2\alpha t})/2\alpha$, and hence, the density of

$$X(t) \sim N\left(x_0 e^{-\alpha t},\ \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t})\right)$$

satisfies the forward equation. The density function is

$$f_{X(t)|X(0)=x_0}(x \mid x_0) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{(1-e^{-2\alpha t})/2\alpha}}\exp\left(-\frac{[x-x_0e^{-\alpha t}]^2}{\sigma(1-e^{-2\alpha t})/\alpha}\right) \qquad -\infty < x < \infty,\quad t \ge 0$$

Steady State (Ergodic), Long Run, Distribution of a Moderated Brownian Motion

Zeroing the time side of the forward equation leaves a form of Hermite's differential equation solutions for which relate to the normal density function

$$0 = \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}f_{X(t)}(x) + \alpha\frac{\partial}{\partial x}\left(x\, f_{X(T)}(x)\right)$$

In the differentially factorized form

$$0 = \tfrac{1}{2}\sigma^2\frac{\partial}{\partial x}\left(e^{-\frac{1}{2}kx^2}\frac{\partial}{\partial x}e^{\frac{1}{2}kx^2}f_{X(t)}(x)\right) \qquad \text{where } k = \frac{\alpha}{\frac{1}{2}\sigma^2}$$

Integrating once

$$e^{-\frac{1}{2}kx^2}\frac{\partial}{\partial x}e^{\frac{1}{2}kx^2}f_{X(t)}(x) = a_0$$

or

$$\frac{\partial}{\partial x}e^{\frac{1}{2}kx^2}f_{X(t)}(x) = a_0e^{\frac{1}{2}kx^2}$$

and integrating again

$$e^{\frac{1}{2}kx^2}f_{X(t)}(x) = a_1 + a_0\int e^{\frac{1}{2}ky^2}dy$$

or

$$f_{X(t)}(x) = a_1e^{-\frac{1}{2}kx^2} + a_0e^{-\frac{1}{2}kx^2}\int_{-\infty}^{x} e^{\frac{1}{2}ky^2}dy$$

The term containing the integral is divergent so $a_0 = 0$ giving

$$f_{X(t)}(x) = a_1e^{-\frac{1}{2}kx^2}$$

Integrating over the entire domain,

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$$\int_{-\infty}^{\infty} f_{X(t)}(x)dx = 1 = a_1\int_{-\infty}^{\infty} e^{-\frac{1}{2}kx^2}\,dx = \sqrt{2\pi / k} = \sqrt{2\pi\sigma^2 / 2\alpha}$$

gives the normalization factor $a_1 = \sqrt{k / 2\pi}$. The stationary steady state distribution is zero-mean normal with standard deviation $\sigma / \sqrt{2\alpha}$ or a variance of $1 / k = \sigma^2 / 2\alpha$

$$X(t) \sim N(0, \sigma^2 / 2\alpha)$$

which has the density function

$$f_{X(t)}(x) = \frac{1}{(\sigma / \sqrt{2\alpha})\sqrt{2\pi}} e^{-\frac{1}{2}x^2 /(\sigma^2 / 2\alpha)}$$

which is the density of a zero-mean Gaussian-Hilbert space. The ergodic density is only valid for recurrent Ornstein-Uhlenbeck diffusion, otherwise the transition distribution is unchanging at all time scales for a unit root or infinite for a divergent modulated root

MODULATED BROWNIAN MOTION

Introduction

The Ornstein-Uhlenbeck process is an exponential integration of a Gaussian diffusion basis with a non-zero infinitesimal mean. This process mean reverts to (if recurrent), or away from (if transient), the basis mean. From the perspective of the parameterization of the diffusion basis, the relationship between the two Ornstein-Uhlenbeck processes is similar to that between Brownian motion and standard Brownian motion.

In terms of the stochastic differential, the basis mean can be added through **(1)** a linear transformation with an addition of a time differential to the deterministic part of the process. Adding the deterministic differential term $\alpha\mu dt$ gives the stochastic differential equation that mean reverts to the value μ at rate α

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$$

where $dB_t \sim N(0,dt)$. This is a first order diffusion with infinitesimal parameters $\mu(x, t) = \alpha(\mu - x)$ and $\sigma(x, t) = \sigma$. This transformation is equivalent to **(2)** letting the stochastic differential Brownian term dB_t have non-zero mean or adding drift to the standard Brownian motion in the stochastic differential (giving the basis a non-zero mean), i.e., replacing dB_t with $mdt + dB_t$ in the expression $dX_t = -\alpha X_t dt + \sigma\, dB_t$ where $m = \mu/\sigma$. This is geometrically equivalent to a horizontal upward shift of the diffusion $dX_t = -\alpha X_t dt + \sigma\, dB_t$.

(1) adding differential drift to the differential Brownian term; Defining the drifting Brownian motion as U_t where $U_t = (\alpha\mu/\sigma)\, t + B_t$. The Ornstein-Uhlenbeck diffusion with the stochastic term U_t rather than B_t is $dX_t = -\alpha X_t dt + \sigma\, dU_t$. The differential of the stochastic term, the differential Brownian motion, is $dU_t = (\alpha\mu/\sigma)dt + \sigma\, dB_t$. Substituting the differential into the stochastic differential gives the equation in terms of the driftless stochastic term,

$$dX_t = -\alpha X_t dt + \sigma\left(\frac{\alpha\mu}{\sigma}\, dt + dB_t\right) = \alpha\,(\mu - X_t)dt + \sigma dB_t$$

(2) Vertical shift; Shifting the diffusion random variable upwards by μ units so that the reversion is towards the horizontal line μ rather than the horizontal axis; if $dX_t^* = -\alpha X_t^* dt + \sigma\, dB_t$ then the process $X_t = \mu + X_t^*$ has the stochastic differential equation $dX_t = \alpha(\mu - X_t)dt + \sigma\, dB_t$. This is seen by calculating the differential of $X_t = \mu + X_t^*$ which is $dX_t = dX_t^*$ and substituting in dX_t^*,

$dX_t = -\alpha X_t^{*} dt + \sigma dB_t$ and further substituting X_t^{*} giving

$dX_t = -\alpha(X_t - \mu)dt + \sigma dB_t$, that is,

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$$

Alternatively and more simply, substituting the vertical translation $X_t^* = \mu - X_t$ into the diffusion differential $dX_t^* = -\alpha X_t^* dt + \sigma\, dB_t$ gives $d(\mu - X_t) = -\alpha(\mu - X_t)\, dt + \sigma\, dB_t$ and since $-\sigma dB_t = \sigma dB_t$ and $d\mu = 0$

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$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$$

These methods for including an additional deterministic differential term are only applicable to processes that are additive, i.e., only for modulated Brownian motion

In anticipation of the results of obtaining the integral solution and solving the forward equation as well as to get an intuitive feel of how the Ornstein-Uhlenbeck process differs from the unit-root[18] diffusions and log-diffusions, the transition distribution of the Ornstein-Uhlenbeck process is normal with mean $x_0e^{-\alpha t} + \mu(1 - e^{-\alpha t})$ and variance $\sigma^2(1 - e^{-2\alpha t})/2\alpha$,

$$X_t \sim N\left(x_0e^{-\alpha t} + \mu(1 - e^{-\alpha t}),\ \sigma^2\frac{1 - e^{-2\alpha t}}{2\alpha}\right)$$

For large values of time, whenever the process mean reverts and is recurrent, the mean and variance reduce to values that are independent of time, μ and $\sigma^2/2\alpha$. The transition density becomes a normal bell that is stationary over time and a further forward outlook does not imply additional variation. This leaves a continuum of time invariant normal curves centered over the mean μ of the diffusion basis. This is a Gaussian Hilbert space and a diffusion basis, with an up-scaling of the stationary variance of the components by $1/2\alpha$, where the diffusion root α is proportional to a Newtonian differential *dt*, $\alpha \propto dt$.

For small values of time *t*, noting that α is very close to zero, the distribution approximates arithmetic Brownian motion. For very small values of time the exponential functions of the transition cumulants are dominated by the linear term of the power series expansion and are therefore approximately linear in time so that the variable resembles linear drift and Brownian (linear) variance i.e., for small *t*, $e^{-\alpha t} \approx 1 - \alpha t$ and $e^{-2\alpha t} \approx 1 - 2\alpha t$. The process then approximates a standard Brownian motion with mean $x_0(1 - \alpha t) + \mu\alpha t$ and variance $\sigma^2(1 - 1 + 2\alpha t)/2\alpha = \sigma^2 t$. That is, for small values of time, the transition distribution of the Ornstein-Uhlenbeck process is

$$X_t \sim N\left(x_0 + \alpha(\mu - x_0)t,\ \sigma^2 t\right)$$

which is the transition distribution of a Brownian motion. For large values of time, the transition distribution of the recurrent Ornstein-Uhlenbeck process is

$$X_t \sim N\left(\mu,\ \sigma^2/2\alpha\right)$$

which is the distribution of a zero order diffusion and a diffusion basis.

In the terminology of physicists[19], for small values of time, the process is Brown noise with an inverse square spectral density, $1/f^2$. For large values of time, the process is white with a flat (uniform) spectral density $1/f^0$. At intermittent values of time, anywhere between small values and large values, the process is between Brown and white noise which is where pink noise is found with an inverse spectral density, $1/f$. Therefore, a mean reverting Ornstein-Uhlenbeck is a pink noise process. Integrating with a infinitesimal moderated recurrent (non-unit) root applied to white noise generates pink noise, just as integrating with an infinitesimal unit-root applied to white noise generates Brown noise (Brownian motion).

In some regards, modulated Brownian motion is more important than Brownian motion as it is the general first order normal diffusion. Brownian motion is considered separately and in its own right for the fact that it is the algorithm of the Central Limit Theorem and is persistent and fractal, as well as much simpler mathematically. This becomes apparent in BOOK 3 which discusses the difference between unit and modulated roots in detail.

Integral Solution to Modulated Brownian Motion

The stochastic differential increment is

$$dX_t = \alpha(\mu - X_t)dt + \sigma dB_t$$

$$dX_t + \alpha X_t dt = \alpha\mu dt + \sigma dB_t$$

[18] actually a zero infinitesimal characteristic root.
[19] Inta, R., (2005/2007), personal communication.

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The integrating factor is the same as previously

$$IF(t) = \exp\left(\int \alpha dt\right) = e^{\alpha t}$$

Multiplying both sides by the integrating factor and differentially factorizing

$$e^{\alpha t}(dX_t + \alpha X_t dt) = e^{\alpha t}(\alpha\mu dt + \sigma dB_t)$$

$$d(e^{\alpha t}X_t) = e^{\alpha t}(\alpha\mu dt + \sigma dB_t)$$

$$\frac{d}{dt}(e^{\alpha t}X_t) = e^{\alpha t}\left(\alpha\mu + \sigma\frac{dB_t}{dt}\right)$$

which is identical to the equivalent process that mean reverts to zero apart from an additional constant term multiplied by the integrating factor. Integrating gives

$$\int_0^t \frac{d}{dy}(e^{\alpha y}X_y)dy = \sigma\int_0^t e^{\alpha y}\left(\alpha\mu + \sigma\frac{dB_y}{dy}\right)dy$$

$$e^{\alpha t}X_t = x_0 + \mu(e^{\alpha t} - 1) + \sigma\int_0^t e^{\alpha y}dB_y$$

or

$$X_t = x_0e^{-\alpha t} + \mu(1 - e^{-\alpha t}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha y}dB_y$$

which has a drift term added to the process that mean reverts to zero, the drift term increasing as the complement of exponential damping, rather than linearly as for drift of a Brownian motion. A further integration by parts gives

$$X_t = x_0e^{-\alpha t} + \mu(1 - e^{-\alpha t}) + \sigma B_t - \alpha\sigma e^{-\alpha t}\int_0^t e^{\alpha y}B_y dy$$

Alternatively, the integral solution of the process can proceed with a change of variable. The direct integration of differential Brownian motion can be achieved with the same change of time variable *t* to

$$s = \frac{e^{2\alpha t} - 1}{2\alpha}$$

which has Jacobian derivative of

$$\frac{ds}{dt} = e^{2\alpha t} = 1 + 2\alpha s$$

and the inverse function

$$t(s) = \frac{1}{2\alpha}\log(1 + 2\alpha s)$$

The integrating factor in the new variable is

$$e^{\alpha t} = \sqrt{1 + 2\alpha s}$$

The differential equation with the change of variable is then

$$\frac{ds}{dt} \times \frac{d}{ds}(X_{t(s)}\sqrt{1 + 2\alpha s}) = \alpha\mu\sqrt{1 + 2\alpha s} + \sigma\sqrt{1 + 2\alpha s}\frac{dB_t}{dt}$$

or

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$$\frac{d}{ds}(X_{t(s)}\sqrt{1+2\alpha s}) = \alpha\mu\sqrt{1+2\alpha s}\times\frac{dt}{ds} + \sigma\sqrt{1+2\alpha s}\,\frac{dt}{ds}\times\frac{dB_t}{dt}$$
$$= \frac{\alpha\mu}{\sqrt{1+2\alpha s}} + \sigma\sqrt{1+2\alpha s}\,\frac{dt}{ds}\times\frac{dB_t}{dt}$$

and by distribution theory, it was previously proved that

$$\sqrt{1+2\alpha s}\,\frac{dt}{ds}\times\frac{dB_t}{dt} \sim \frac{1}{ds}N(0,ds)$$

which implies that

$$\sqrt{1+2\alpha s}\,\frac{dt}{ds}\times\frac{dB_t}{dt} = \frac{dB_s}{ds}$$

giving

$$\frac{d}{ds}(X_{t(s)}\sqrt{1+2\alpha s}) = \frac{\alpha\mu}{\sqrt{1+2\alpha s}} + \sigma\frac{dB_s}{ds}$$

Integrating

$$\int_0^s \frac{d}{dy}(X_{t(y)}\sqrt{1+2\alpha y})dy = \int_0^s \frac{\alpha\mu}{\sqrt{1+2\alpha y}}\,dy + \sigma\int_0^s \frac{dB_y}{dy}\,dy$$

gives

$$X_{t(s)}\sqrt{1+2\alpha s} = X_{t(0)} + \mu\sqrt{1+2\alpha s} - \mu + \sigma B_s$$

or

$$X_{t(s)} = \frac{x_0}{\sqrt{1+2\alpha s}} + \mu - \frac{\mu}{\sqrt{1+2\alpha s}} + \frac{\sigma}{\sqrt{1+2\alpha s}}B_s$$

which has the distribution

$$X_{t(s)} \sim N\left(\frac{x_0}{\sqrt{1+2\alpha s}} + \mu - \frac{\mu}{\sqrt{1+2\alpha s}}, \frac{\sigma^2 s}{1+2\alpha s}\right)$$

changing back to linear time, noting that $1/\sqrt{1+2\alpha s} = e^{-\alpha t}$, gives

$$X_t = x_0 e^{-\alpha t} + \mu(1-e^{-\alpha t}) + \sigma e^{-\alpha t}B_{(e^{2\alpha t}-1)/2\alpha}$$

The transition distribution of a recurrent Ornstein-Uhlenbeck[20] process is then

$$X_t \sim N\left(x_0 e^{-\alpha t} + \mu(1-e^{-\alpha t}), \sigma^2\frac{1-e^{-2\alpha t}}{2\alpha}\right)$$

which has a transition mgf inclusive of the backward variables of

$$M_{X(T)|X(t_0)}(\theta) = \exp\left(x_0 e^{-\alpha(T-t_0)}\theta + \mu\left(\frac{1-e^{-\alpha(T-t_0)}}{\alpha}\right)\theta + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha(T-t_0)}}{2\alpha}\right)\theta^2\right)$$

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[20] The transition distribution of a transient Ornstein-Uhlenbeck process is

$$X_t \sim N\left(x_0 e^{\alpha t} + \mu(e^{\alpha t}-1), \sigma^2\frac{e^{2\alpha t}-1}{2\alpha}\right)$$

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Formation of the Forward and Backward Equations of Modulated Brownian Motion

A. Formation of the Forward Equation

Modulating the mgf parameter

$$M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) = \exp\left(x_0\theta + \mu\left(\frac{e^{\alpha(T-t_0)}-1}{\alpha}\right)\theta + \tfrac{1}{2}\sigma^2\left(\frac{e^{2\alpha(T-t_0)}-1}{2\alpha}\right)\theta^2 \right)$$

Differentiating the resulting mgf partially with respect to the forward time variable gives the forward equation in the mgf

$$\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) + \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) \frac{d}{dT}\theta e^{\alpha(T-t_0)}$$
$$= \left(\mu e^{\alpha(T-t_0)}\theta + \tfrac{1}{2} e^{2\alpha(T-t_0)}\sigma^2\theta^2\right) M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)})$$

i.e.,

$$\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) + \alpha e^{\alpha(T-t_0)}\theta \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)}) = \left(\mu e^{\alpha(T-t_0)}\theta + \tfrac{1}{2} e^{2\alpha(T-t_0)}\sigma^2\theta^2\right) M_{X(T)|X(t_0)}(\theta e^{\alpha(T-t_0)})$$

and in the unmodulated mgf parameter

$$\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta) + \alpha\theta \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta) = \left(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2\right) M_{X(T)|X(t_0)}(\theta)$$

Differentiating the mgf partially once and then twice with respect to forward diffusion variable, and then again with respect to the mgf parameter and the forward diffusion variable ,i.e.,

$$\frac{\partial}{\partial x} M_{X(T)|X(t_0)}(\theta) \quad \text{and} \quad \frac{\partial^2}{\partial x^2} M_{X(T)|X(t_0)}(\theta) \quad \text{and} \quad \frac{\partial^2}{\partial x \partial \theta} M_{X(T)|X(t_0)}(\theta)$$

for

(1) $\theta M_{X(T)|X(t_0)}(\theta) = -\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{X(T)|X(t_0)}(x \mid x_0) dx$

(2) $\theta \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta) = -\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x\, f_{X(T)|X(t_0)}(x \mid x_0) dx$

(3) $\theta^2 M_{X(T)|X(t_0)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) dx$

so that the time derivative can be written in terms of the transition density function as

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty} e^{\theta x} f_{X(T)|X(t_0)}(x \mid x_0) dx - \alpha \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x\, f_{X(T)|X(t_0)}(x \mid x_0) dx = \int_{-\infty}^{\infty} e^{\theta x}\left(\tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} - \mu \frac{\partial}{\partial x}\right) f_{X(T)|X(t_0)}(x \mid x_0) dx$$

and upon inversion of the mgf results in the forward equation of modulated Brownian motion, the modulated heat equation with convection, in the transition density

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{X(T)|X(t_0)}(x \mid x_0) - \frac{\partial}{\partial x}\left((\mu - \alpha x) f_{X(T)|X(t_0)}(x \mid x_0)\right)$$

In differentially factorized form, this is

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(e^{-\alpha(x^2-2\mu x)/\sigma^2} \frac{\partial}{\partial x} e^{\alpha(x^2-2\mu x)/\sigma^2} f_{X(T)|X(t_0)}(x \mid x_0) \right)$$

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B. Formation of the Backward Equation

Differentiating the mgf partially with respect to the backward time variable

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = \left(\alpha x_0 e^{-\alpha(T-t_0)}\theta - \mu e^{-\alpha(T-t_0)}\theta - \tfrac{1}{2}\sigma^2 e^{-2\alpha(T-t_0)}\theta^2 \right) M_{X(T)|X(t_0)}(\theta)$$

Differentiating partially with respect to the backward diffusion variable

$$\frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) = e^{-\alpha(T-t_0)}\theta M_{X(T)|X(t_0)}(\theta)$$

and again

$$\frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta) = e^{-2\alpha(T-t_0)}\theta^2 M_{X(T)|X(t_0)}(\theta)$$

Hence, the time derivative can be written as

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = \left(\alpha x_0 \frac{\partial}{\partial x_0} - \mu \frac{\partial}{\partial x_0} - \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} \right) M_{X(T)|X(t_0)}(\theta)$$

which results the backward equation of standard modulated Brownian motion (OU), the Hermite conjugate of the modulated heat equation with convection, in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} f_{X(T)|X(t_0)}(x \mid x_0) - (\mu - \alpha x_0) \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0)$$

In differentially factored form, this is

$$-\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 e^{\alpha(x_0^2 - 2\mu x_0)/\sigma^2} \frac{\partial}{\partial x_0} \left(e^{\alpha(x_0^2 - 2\mu x_0)/\sigma^2} \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0) \right)$$

Solution to the Forward Equation for Modulated Brownian Motion: (Transforming to the Heat Equation)

Taking the forward equation of an Ornstein-Uhlenbeck process that mean reverts to μ

$$\frac{\partial}{\partial t} f_{X(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{X(t)}(x) - \alpha \frac{\partial}{\partial x}\left((\mu - x) f_{X(t)}(x) \right)$$

By the same principle as the forward equation for the Ornstein-Uhlenbeck process mean reverting to zero, transforming the diffusion variable $X(t)$ with an additional additive translation to

$$Y(t) = \mu + e^{\alpha t}\left(X(t) - \mu \right)$$

and the transformed independent forward diffusion variable is $y = \mu + e^{\alpha t}(x - \mu)$. This transformation says that the corrected diffusion variables, by the mean μ of the diffusion basis components, $(y - \mu)$ and $(x - \mu)$, are proportional by the exponential time correlating factor. This transformation has the same Jacobian as the equivalent process that mean reverts to zero, which is

$$J = \frac{dx}{dy} = e^{-\alpha t}$$

The normalized density function of the transformed diffusion variable is

$$f_{Y(t)}(y) = f_{X(t)}(x) \mid J \mid = e^{-\alpha t} f_{X(t)}(x)$$

which implies $f_{X(t)}(x) = e^{\alpha t} f_{Y(t)}(y)$ and the relation $(x - \mu) f_{X(t)}(x) = (y - \mu) f_{Y(t)}(y)$. The relevant partial derivatives of the transform are

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$$\frac{\partial y}{\partial x} = e^{\alpha t} \qquad \text{and} \qquad \frac{\partial y}{\partial t} = \alpha e^{\alpha t}(x-\mu) = \alpha e^{\alpha t}[(y-\mu)e^{-\alpha t}] = \alpha(y-\mu)$$

The transformed partial derivatives of the forward equation are

$$\textbf{(1)}\ \frac{\partial}{\partial x}[(\mu - x) f_{X(t)}(x)] = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}[(\mu - y) f_{Y(t)}(y)] = e^{\alpha t}\frac{\partial}{\partial y}[(\mu - y) f_{Y(t)}(y)]$$

$$= e^{\alpha t}(\mu - y)\frac{\partial}{\partial y} f_{Y(t)}(y) - e^{\alpha t} f_{Y(t)}(y)$$

$$\textbf{(2)}\ \frac{\partial^2}{\partial x^2} f_{X(t)}(x) = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left[\frac{\partial y}{\partial x}\frac{\partial}{\partial y} e^{\alpha t} f_{Y(t)}(x)\right] = e^{\alpha t}\frac{\partial}{\partial y}\left[e^{2\alpha t}\frac{\partial}{\partial y} f_{Y(t)}(y)\right] = e^{3\alpha t}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y)$$

$$\textbf{(3)}\ \frac{\partial}{\partial t} f_{X(t)}(x) = \frac{\partial}{\partial t}[e^{\alpha t} f_{Y(t)}(y)] = \alpha e^{\alpha t} f_{Y(t)}(y) + e^{\alpha t}\frac{\partial}{\partial t} f_{Y(t)}(y) + e^{\alpha t}\frac{\partial y}{\partial t}\frac{\partial}{\partial y} f_{Y(t)}(y)$$

$$= \alpha e^{\alpha t} f_{Y(t)}(y) + e^{\alpha t}\frac{\partial}{\partial t} f_{Y(t)}(y) + \alpha e^{\alpha t}(y-\mu)\frac{\partial}{\partial y} f_{Y(t)}(y)$$

The transformation of the diffusion component being identical to that of the process that mean reverts to zero. The forward equation simplifies to the variant of the heat equation that has a diffusion coefficient that is a time dependent exponential function,

$$\alpha e^{\alpha t} f_{Y(t)}(y) + e^{\alpha t}\frac{\partial}{\partial t} f_{Y(t)}(y) + \alpha e^{\alpha t}(y-\mu)\frac{\partial}{\partial y} f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2 e^{3\alpha t}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y) - \alpha e^{\alpha t}(\mu - y)\frac{\partial}{\partial y} f_{Y(t)}(y) + \alpha e^{\alpha t} f_{Y(t)}(y)$$

and upon simplification, this is

$$\frac{\partial}{\partial t} f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2 e^{2\alpha t}\frac{\partial^2}{\partial y^2} f_{Y(t)}(y)$$

This equation was solved for the diffusion that mean reverts to zero by the technique of conversion to an ordinary differential equation in the moment generating function. This equation will be transformed to the standard heat equation from which the transition density is obtained from the two successive inverse transformations, one with respect to time and the other with respect to the diffusion variable. The transformation of the linear time variable to obtain the standard heat equation is $s = \sigma^2(e^{2\alpha t} - 1)/2\alpha$ which has derivative $\partial s/\partial t = \sigma^2 e^{2\alpha t}$. The heat equation with exponential diffusion coefficient becomes

$$\tfrac{1}{2}\sigma^2 e^{2\alpha t}\frac{\partial^2}{\partial y^2} f_{Y(s)}(y) = \frac{\partial}{\partial t} f_{Y(s)}(y) = \frac{\partial s}{\partial t}\frac{\partial}{\partial s} f_{Y(s)}(y) = \sigma^2 e^{2\alpha t} f_{Y(s)}(y)$$

or

$$\tfrac{1}{2}\frac{\partial^2}{\partial y^2} f_{Y(s)}(y) = \frac{\partial}{\partial s} f_{Y(s)}(y)$$

which is the standard heat equation and is the equation of standard Brownian motion with transition distribution

$$Y(s) \sim N(y_0, s)$$

Changing back to linear time by replacing s with $\sigma^2(e^{2\alpha t} - 1)/2\alpha$ gives

$$Y(t) \sim N\left(y_0, \sigma^2\frac{e^{2\alpha t} - 1}{2\alpha}\right)$$

Transforming back to the original diffusion variable $X(t) = \mu + e^{-\alpha t}[Y(t) - \mu]$

$$Y(t) - \mu \sim N\left(y_0 - \mu, \sigma^2\frac{e^{2\alpha t} - 1}{2\alpha}\right)$$

or

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$$\mu + e^{-\alpha t}\Big(Y(t)-\mu\Big) \sim N\Big(\mu + e^{-\alpha t}(y_0-\mu), \sigma^2\frac{1-e^{-2\alpha t}}{2\alpha}\Big)$$

and since $X(0) = \mu + [Y(0) - \mu]$ so that $y_0 = x_0$, the transition distribution of the original variable is

$$X(t) \sim N\Big(x_0 e^{-\alpha t} + \mu(1-e^{-\alpha t}), \sigma^2\frac{1-e^{-2\alpha t}}{2\alpha}\Big)$$

which is a normal variable with mean $x_0\, e^{-\alpha t} + \mu(1 - e^{-\alpha t})$ and variance $\sigma^2(1 - e^{-2\alpha t})/2\alpha$ which has the transition density function

$$f_{X(t)|X(t)}(x \mid x_0) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{(1-e^{-2\alpha t})/2\alpha}} \exp\Big(-\frac{[x - x_0 e^{-\alpha t} - \mu(1-e^{-\alpha t})]^2}{\sigma^2(1-e^{-2\alpha t})/\alpha}\Big) \quad \text{for } -\infty < x < \infty \text{ and } t \geq 0$$

which is the solution to the forward equation.

Solution to the Forward Equation for Modulated Brownian Motion: (Solving the Equation Directly)

Converting the forward equation in the transition density to the forward equation in the transition moment generating function by multiplying by the moment generating function exponential and integrating over the range of the diffusion

$$\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0)dx = \tfrac{1}{2}\sigma^2\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial^2}{\partial x^2} f_{X(t)|X(t_0)}(x \mid x_0)dx - \alpha\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial}{\partial x}\Big((\mu - x)\, f_{X(t)|X(t_0)}(x \mid x_0)\Big)dx$$

or

$$\frac{\partial}{\partial t} M_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 M_{X(t)}(\theta) + \alpha\mu\theta M_{X(t)}(\theta) - \alpha\theta\frac{\partial}{\partial\theta} M_{X(t)}(\theta)$$

Dividing through by $M_{X(t)}(\theta)$ gives the forward equation in the cumulant generating function

$$\frac{\partial}{\partial t} K_{X(t)}(\theta) = \tfrac{1}{2}\sigma^2\theta^2 + \alpha\mu\theta - \alpha\theta\frac{\partial}{\partial\theta} K_{X(t)}(\theta)$$

the solution for which can proceed by either **i.** transforming the moment generating function parameter to remove the exponential weighting in the integration of the diffusion basis or **ii.** by removing the time domain with a Laplace transform.

Transforming the MGF Parameter

With the same transformation as the Ornstein-Uhlenbeck process that mean reverts to zero

$$\phi = \theta e^{-\alpha t}$$

with inverse $\phi e^{\alpha t}$ and the same derivatives of the transform, $\frac{\partial\phi}{\partial t} = -\alpha\theta e^{-\alpha t} = -\alpha\phi$ and $\frac{\partial\phi}{\partial\theta} = e^{-\alpha t}$

(1) $$\frac{\partial}{\partial\theta}\log M_{X(t)}(\theta) = e^{-\alpha t}\frac{\partial}{\partial\phi}\log M_{X(t)}(\phi e^{\alpha t})$$

(2) $$\frac{\partial}{\partial t}\log M_{X(t)}(\theta) = \frac{\partial}{\partial t}\log M_{X(t)}(\phi e^{\alpha t}) - \alpha\theta e^{-\alpha t}\frac{\partial}{\partial\phi}\log M_{X(t)}(\phi e^{\alpha t})$$

the equation in the cumulant generating function reduces to

$$\frac{\partial}{\partial t}\log M_{X(t)}(\phi e^{\alpha t}) = \tfrac{1}{2}\sigma^2(\phi e^{\alpha t})^2 + \alpha\mu\phi e^{\alpha t}$$

Integrating

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$$\int_0^t \frac{\partial}{\partial y}\log M_{X(y)}(\phi e^{\alpha y})dy = \tfrac{1}{2}\sigma^2\phi^2\int_0^t e^{2\alpha y}dy + \alpha\mu\phi\int_0^t e^{\alpha t}dy$$

gives

$$\log M_{X(t)}(\phi e^{\alpha t}) - \log M_{X(0)}(\phi) = \tfrac{1}{2}\sigma^2\phi^2\left(\frac{e^{2\alpha t}-1}{2\alpha}\right) + \alpha\mu\phi\left(\frac{e^{\alpha t}-1}{\alpha}\right)$$

or

$$\log M_{X(t)}(\phi e^{\alpha t}) = x_0\phi + \tfrac{1}{2}\sigma^2\phi^2\left(\frac{e^{2\alpha t}-1}{2\alpha}\right) + \alpha\mu\phi\left(\frac{e^{\alpha t}-1}{\alpha}\right)$$

In terms of the original cumulant generating function, the solution is

$$\log M_{X(t)}(\theta) = x_0 e^{-\alpha t}\theta + \alpha\mu\left(\frac{1-e^{-\alpha t}}{\alpha}\right)\theta + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\theta^2$$

or a moment generating function of

$$M_{X(t)}(\theta) = \exp\left(x_0 e^{-\alpha t}\theta + \alpha\mu\left(\frac{1-e^{-\alpha t}}{\alpha}\right)\theta + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\theta^2\right)$$

which is the moment generating function of a normal variable with mean $x_0e^{-\alpha t} + \mu(1 - e^{-\alpha t})$ and variance $\sigma^2(1 - e^{-2\alpha t})/2\alpha$.

Laplace Transform

Taking the Laplace transform of both sides by multiplying by the Laplace exponential e^{st} and integrating over the time domain,

$$\int_0^\infty e^{st}\frac{\partial}{\partial t}K_{X(t)}(\theta)dt = \int_0^\infty e^{st}\left(\tfrac{1}{2}\sigma^2\theta^2 + \alpha\,\mu\theta\right)dt - \alpha\theta\frac{\partial}{\partial\theta}\int_0^\infty e^{st}K_{X(t)}(\theta)dt$$

or

$$sL\left\{K_{X(t)}(\theta)\right\} - K_{X(0)}(\theta) = \frac{1}{s}\left(\tfrac{1}{2}\sigma^2\theta^2 + \alpha\,\mu\theta\right)dt - \alpha\theta\frac{\partial}{\partial\theta}L\left\{K_{X(t)}(\theta)\right\}$$

where $L\{K_{X(t)}(\theta)\}$ is the Laplace transform of the cumulant generating function[21]. This is a linear first order ordinary differential equation. Noting that $K_{X(t)}(\theta) = \theta x_0$ and rearranging terms

$$\alpha\theta\frac{\partial}{\partial\theta}L\left\{K_{X(t)}(\theta)\right\} + sL\left\{K_{X(t)}(\theta)\right\} = \frac{1}{s}\left(\tfrac{1}{2}\sigma^2\theta^2 + \alpha\,\mu\theta\right)dt + \theta x_0$$

or

$$\frac{\partial}{\partial\theta}L\left\{K_{X(t)}(\theta)\right\} + \frac{s}{\alpha\theta}L\left\{K_{X(t)}(\theta)\right\} = \frac{1}{\alpha s}\left(\tfrac{1}{2}\sigma^2\theta + \alpha\,\mu\right)dt + \frac{x_0}{\alpha}$$

Differentially factorizing the equation with the integrating factor $\exp\left(\int \frac{s}{\alpha\theta}d\theta\right) = \exp\left(\frac{s}{\alpha}\log\theta\right) = \theta^{s/\alpha}$

[21] Noting that $L\{e^{-k\alpha}\} = 1/(s + k\alpha)$ and $L\{1/(s + k\alpha)\} = e^{-k\alpha}$

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$$\frac{\partial}{\partial\theta}\left[\theta^{s/\alpha}L\left\{K_{X(t)}(\theta)\right\}\right]=\frac{1}{\alpha s}\left(\tfrac{1}{2}\sigma^2\theta^{s/\alpha+1}+\alpha\,\mu\theta^{s/\alpha}\right)+\theta^{s/\alpha}\,\frac{x_0}{\alpha}$$

integration of which gives

$$\theta^{s/\alpha}L\left\{K_{X(t)}(\theta)\right\}=\frac{1}{\alpha s}\left(\frac{\tfrac{1}{2}\sigma^2\theta^{s/\alpha+2}}{s/\alpha+2}+\frac{\alpha\mu\theta^{s/\alpha+1}}{s/\alpha+1}\right)+\frac{\theta^{s/\alpha+1}}{s/\alpha+1}\,\frac{x_0}{\alpha}$$

or

$$L\left\{K_{X(t)}(\theta)\right\}=\frac{\tfrac{1}{2}\sigma^2\theta^2}{s(s+2\alpha)}+\frac{\alpha\mu\theta}{s(s+\alpha)}+\frac{x_0\theta}{s+\alpha}$$

and returning to the time domain by taking the inverse Laplace transform gives the solution to the forward equation in terms of the cumulant generating function

$$K_{X(t)}(\theta)=\tfrac{1}{2}\sigma^2\theta^2L^{-1}\left\{\frac{1}{s(s+2\alpha)}\right\}+\alpha\mu\theta\;L^{-1}\left\{\frac{1}{s(s+\alpha)}\right\}+x_0\theta\;L^{-1}\left\{\frac{1}{s+\alpha}\right\}$$

or by partial fractions

$$K_{X(t)}(\theta)=\tfrac{1}{2}\sigma^2\theta^2\;\frac{1}{2\alpha}L^{-1}\left\{\frac{1}{s}-\frac{1}{s+2\alpha}\right\}+\mu\theta\;L^{-1}\left\{\frac{1}{s}-\frac{1}{s+\alpha}\right\}+x_0\theta\;L^{-1}\left\{\frac{1}{s+\alpha}\right\}$$

giving

$$K_{X(t)}(\theta)=\tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\theta^2+\mu(1-e^{-\alpha t})\theta+x_0e^{-\alpha t}\theta$$

which is the cumulant generating function of a normal variable with mean $x_0e^{-\alpha t}+\mu(1-e^{-\alpha t})$ and variance $\sigma^2(1-e^{-2\alpha t})/2\alpha$ so that the transition distribution is

$$X(t)\sim N\left(x_0e^{-\alpha t}+\mu(1-e^{-\alpha t}),\,\sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\right)$$

for which the density function is the solution to the forward equation,

$$f_{X(t)}(x)=\frac{1}{\sigma\sqrt{2\pi}\sqrt{(1-e^{-2\alpha t})/2\alpha}}\exp\left(-\frac{[x-x_0e^{-\alpha t}-\mu(1-e^{-\alpha t})]^2}{\sigma^2(1-e^{-2\alpha t})/\alpha}\right)\qquad\text{for }-\infty<x<\infty\text{ and }t\geq0$$

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The Transition Distribution

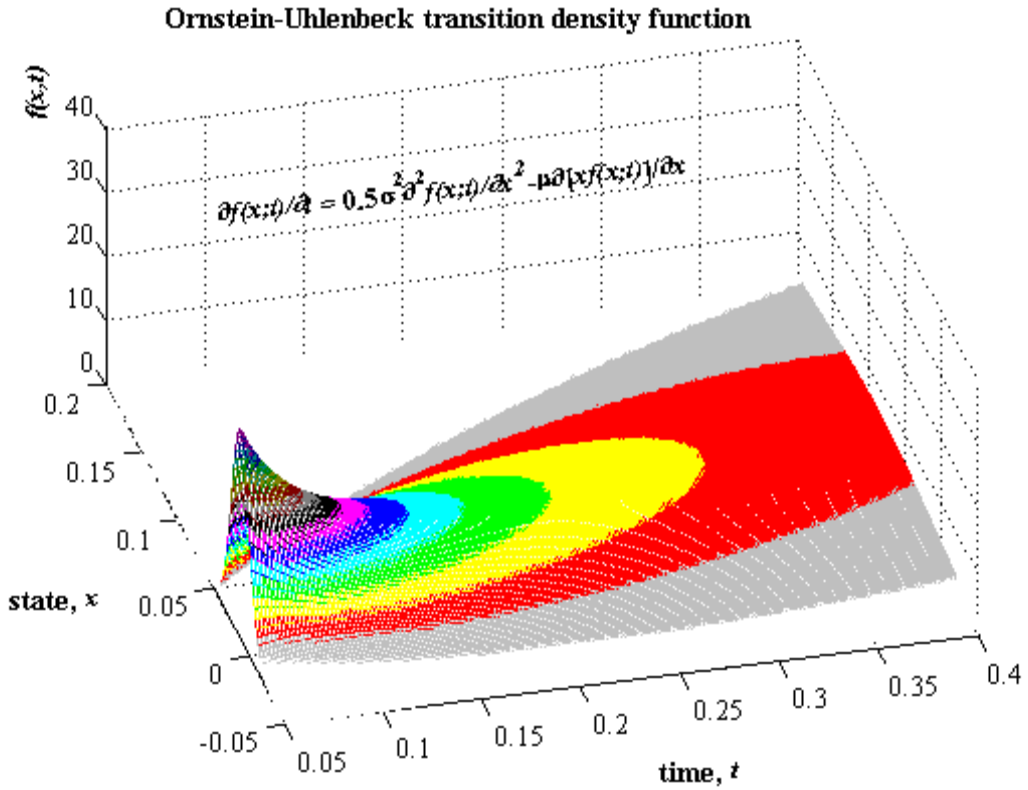


Figure 19: Ornstein-Uhlenbeck transition density surface (α,μ,σ) = (0.10,0.50,0.15)

When the basis variance zeroes $\sigma \to 0$, this surface becomes a Dirac singularity traced out over the exponentially decayed initial value and integrated moderated basis mean, as per mean reversion and independence of the intial conditions due to the moderated root. The diffusion equations becomes $X(t) \to x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t})$ for which the transition density is

$$f_{X(t)|X(0)}(x \mid x_0) = \delta_{x_0 e^{-\alpha t} + \mu(1-e^{-\alpha t})}(x)$$

Covariance of modulated Brownian motion

The autocorrelation between two successive points in time of the process, s and t , is given by

$$Cov[X(s), X(t)] = \frac{\sigma^2}{2\alpha}(e^{-\alpha(t-s)} - e^{-\alpha(t+s)}) \qquad \text{for } 0 < s < t$$

The second order properties specify multidimensional normals, marginals of which are normal so the auto-joint transition distribution is bivariate normal;

$$\begin{bmatrix} X(s) \\ X(t) \end{bmatrix} \sim BivN\left(\begin{bmatrix} x_0 e^{-\alpha s} + \mu(1 - e^{-\alpha s}) \\ x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}) \end{bmatrix}, \begin{bmatrix} \sigma^2(1 - e^{-2\alpha s})/2\alpha & \sigma^2(e^{-\alpha(t-s)} - e^{-\alpha(t+s)})/2\alpha \\ \sigma^2(e^{-\alpha(t-s)} - e^{-\alpha(t+s)})/2\alpha & \sigma^2(1 - e^{-2\alpha t})/2\alpha \end{bmatrix} \right)$$

Further Statistical Description of Modulated Brownian Motion

For the distributional statistics – quantiles, raw moments, central moments, standardized central moments, and entropy – of the Ornstein-Uhlenbeck process, substitute the mean and variance of the transition distribution of Brownian motion with the mean and variance of the Ornstein-Uhlenbeck process. These are both additive Gaussian processes for which differ only in the structure of the time functions of the transition mean and the transition variance, the only two transition cumulants that are non-zero. For diffusions of arbitrary order, see the relevant tables of transition means and variances in BOOK 3 of this dissertation.

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Estimating the Characteristic Root by the Method of Moments

Say one observes an Ornstein-Uhlenbeck process and that the mean and variance of the diffusion basis is given, and a preliminary estimate of the characteristic root α is required. Assume that the correlation of the process at some lag $t - s$, where neither s or t is zero, is given or calculated from incremental observations or recordings of the process. Letting this be $\rho_{t,s}$. Matching the lagged correlation gives the equation, whose solution is numeric and does not give an explicit estimator

$$\rho_{t,s} = \frac{e^{-\alpha(t-s)} - e^{-\alpha(t+s)}}{\sqrt{1-e^{-2\alpha t}}\sqrt{1-e^{-2\alpha s}}}$$

which can be solved easily for the characteristic root with a numerical root finding procedure such as Newton-Raphson. As an approximation, expanding the exponentials in a power series and truncating to degree one gives

$$\rho_{t,s} = \sqrt{\frac{1+2\alpha s}{1+2\alpha t}}$$

which is solved analytically for the approximate method of moments estimator of the characteristic root

$$\hat{\alpha} = \frac{1-\rho_{t,s}^2}{2(t\rho_{t,s}^2 - s)}$$

The parameter α is generally small so the truncated power series at o(α) should be relatively close to the exact value and the approximation to the method of moments estimator should consequently be relatively accurate. The approximation can also be used as the starting value for Newton-Raphson to solve for the exact method of moments estimate.

Steady State (Ergodic), Long Run, Distribution of Modulated Brownian Motion

Zeroing the time side of the forward equation leaves another form of Hermite's differential equation, solutions for which also relate to the normal density function,

$$0 = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{X(t)}(x) - \alpha \frac{\partial}{\partial x}\left((\mu - x)\, f_{X(t)}(x \mid x_0)\right)$$

In the differentially factorized forward form this is

$$0 = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(e^{-\frac{1}{2}kx^2+\mu k x} \frac{\partial}{\partial x} e^{\frac{1}{2}kx^2-\mu k x} f_{X(t)}(x)\right) \qquad \text{where } k = \frac{\alpha}{\frac{1}{2}\sigma^2}$$

Integrating once

$$e^{-\frac{1}{2}kx^2+\mu k x} \frac{\partial}{\partial x} e^{\frac{1}{2}kx^2-\mu k x} f_{X(t)}(x) = a_0$$

or

$$\frac{\partial}{\partial x} e^{\frac{1}{2}kx^2-\mu k x} f_{X(t)}(x) = a_0 e^{\frac{1}{2}kx^2-\mu k x}$$

Integrating again,

$$e^{\frac{1}{2}kx^2-\mu k x} f_{X(t)}(x) = a_1 + a_0 \int e^{\frac{1}{2}kx^2-\mu k x} dx$$

or

$$f_{X(t)}(x) = a_1 e^{-\frac{1}{2}kx^2+\mu k x} + a_0 e^{-\frac{1}{2}kx^2+\mu k x} \int e^{\frac{1}{2}kx^2-\mu k x} dx$$

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The right-hand-side must converge since the process is recurrent and since the term containing the integral is divergent, $a_0 = 0$. This leaves the expression

$$f_{X(t)}(x) = a_1 e^{-\frac{1}{2}kx^2 + \mu k x}$$

Factorizing the exponent gives

$$f_{X(t)}(x) = a_1 e^{\frac{1}{2}k\mu^2} e^{-\frac{1}{2}k(x^2-\mu)}$$

The infinitesimal characteristic root is positive since the diffusion is recurrent so the density is proportional to a normal density with mean μ and variance $1/k = \sigma^2/2\alpha$; the second constant of integration is

$$a_1 = \frac{e^{-\alpha\mu^2/\sigma^2}}{\sqrt{\sigma^2/2\alpha}\sqrt{2\pi}}$$

Hence the steady state distribution is

$$\begin{aligned} X(t) &\sim N(\mu, 1/k) \\ &\sim N(\mu, \sigma^2/2\alpha) \end{aligned}$$

for which the density and the solution to the steady state forward equation is

$$f_{X(t)}(x) = \frac{1}{\sqrt{\sigma^2/2\alpha}\sqrt{2\pi}} e^{-\frac{1}{2}(x^2-\mu)/(\sigma^2/2\alpha)} \qquad x \forall (-\infty, \infty)$$

This is the density of additive Gaussian white noise, a reparameterized normal diffusion basis and a Gaussian-Hilbert space.

Simulations

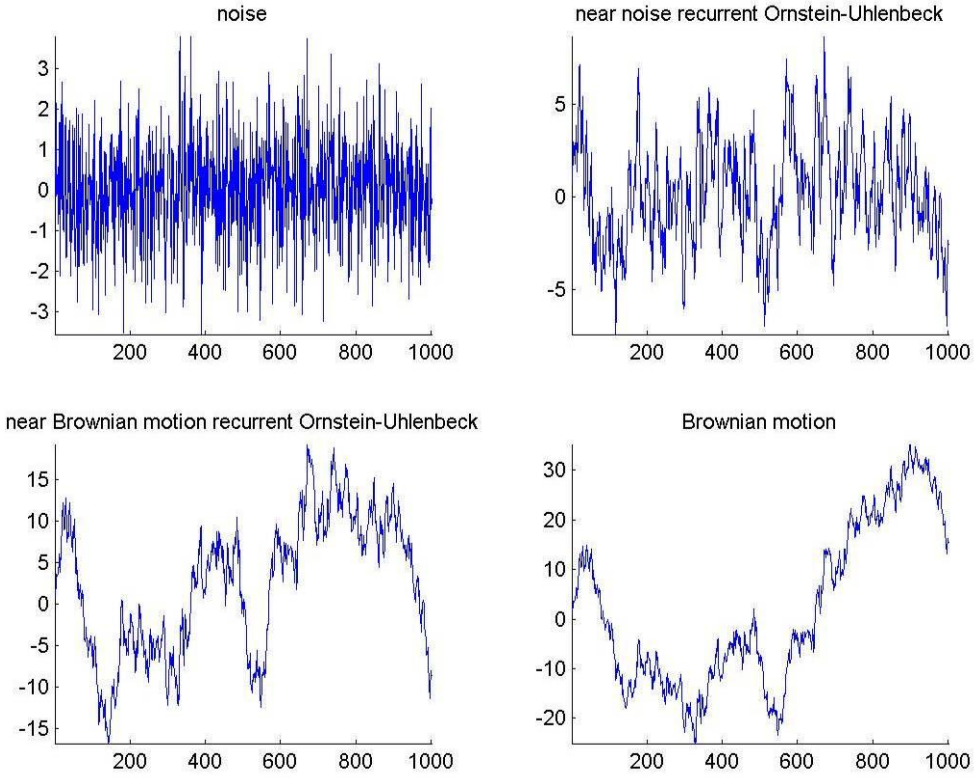


Figure 20: Paths of a spectrum of Ornstein-Uhlenbeck processes with the same diffusion basis

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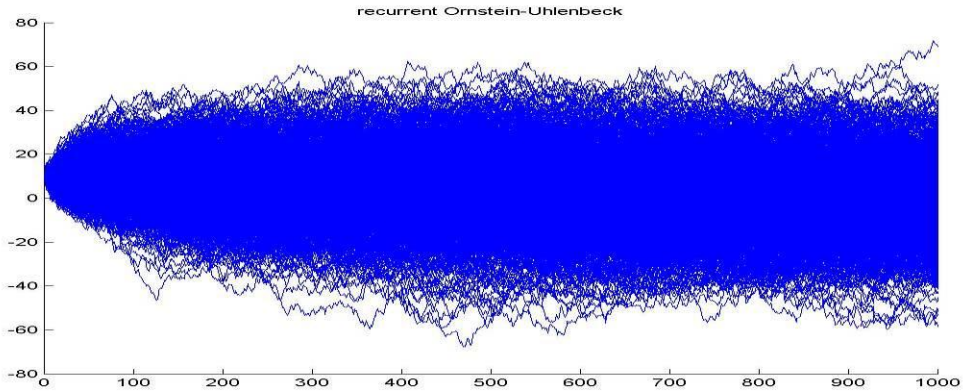


Figure 21: 1000 simulated paths of a recurrent Ornstein-Uhlenbeck process

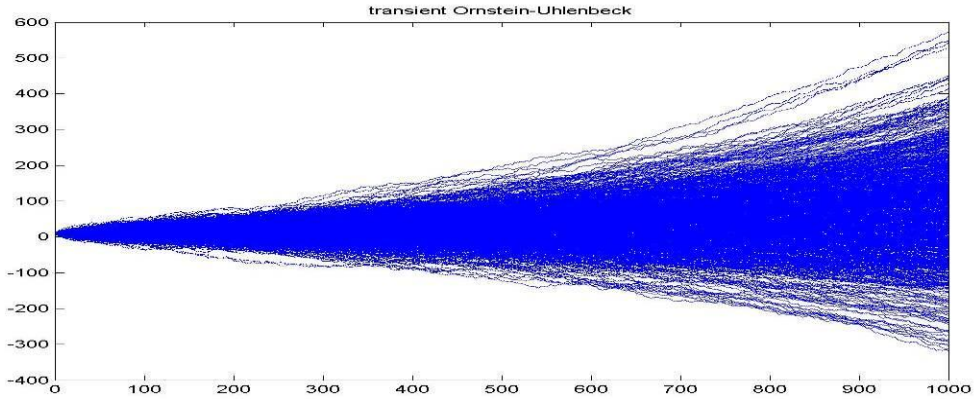


Figure 22: 1000 simulated paths of a transient Ornstein-Uhlenbeck process

The first four diagrams show a cross section of the spectrum of first order diffusion, in terms of trend propensity, that enclose the property of recurrence – from white noise ($1 / f^{0}$), to two recurrent Ornstein-Uhlenbeck processes, to Brownian motion ($1 / f^{2}$). One of the intermediate paths is near noise Ornstein-Uhlenbeck while the other is near Brownian motion Ornstein-Uhlenbeck, i.e., processes with, respectively, zero trend propensity, faint trend propensity, moderate trend propensity, and full trend propensity. Any trend propensity greater than a unit-root, i.e., that with excess trend propensity, results in a transient process for which each path is convex away from the origin that goes on an 'excursion' only never to return (diverge to infinity). One thousand paths of such a variable are shown in the last diagram.

MODULATED PROPORTIONAL BROWNIAN MOTION (GEOMETRIC ORSTEIN-UHLENBECK PROCESS)

Introduction

A geometric Ornstein-Uhlenbeck process is modulated proportional Brownian motion and is the exponential transformation of an ordinary Ornstein-Uhlenbeck process. This is in parallel to geometric Brownian motion as the exponential transformation of ordinary Brownian motion. When the time scaling parameter of the geometric Ornstein-Uhlenbeck process zeroes, the process reverts to a geometric Brownian motion.

Both the mean and variance are proportional and the diffusion mean reverts in logarithms towards the fixed value μ. The infinitesimal parameters are $\mu(x, t) = \alpha(\mu - \log x)x$ and $\sigma(x, t) = \sigma x$, showing that the diffusion is both modulated and proportional. The differential equation of the variable is then

$$dX_t = \alpha X_t(\mu - \log X_t)dt + \sigma X_t dB_t$$

For small values of time the exponential factors in the cumulants of the transition distribution approximate linear functions and the process approximates geometric Brownian motion. For large values of time, if the process recurrent rather than transient so that it is mean reverting and the reversionary force is a pull towards rather than away from the mean value, then the exponential factors vanish. Subsequently, for large values of time, the transi-

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tion density becomes stationary and time invariant and the process becomes a continuum of orthogonal multiplicative log-normal variables. Hence, for small values of time, the transition distribution of the geometric Ornstein-Uhlenbeck process is

$$X_t \sim \log N\left(\log x_0 + \alpha(\mu - \tfrac{1}{2}\sigma^2 / \alpha - \log x_0)t,\ \ \sigma^2 t\right)$$

which is the transition distribution of a geometric Brownian motion, while for large values of time, the transition distribution of the process is

$$X_t \sim \log N\left(\mu - \tfrac{1}{2}\sigma^2 / \alpha,\ \ \sigma^2 / 2\alpha\right)$$

which is the distribution of a multiplicative log-normal Gaussian-Hilbert space, i.e., a multiplicative basis.

Integral Solution to Modulated Proportional Brownian Motion

From the stochastic differential

$$dX_t = \alpha X_t(\mu - \log X_t)dt + \sigma X_t dB_t \qquad \text{where } dB_t \sim N(0, dt)$$

transforming of the diffusion variable to logarithms by the ito-calculus or transforming the forward equation to the equivalent equation in the additive Ornstein-Uhlenbeck process, gives the unit-root additive equation

$$d\log X_t = \alpha\left(\mu - \tfrac{1}{2}\sigma^2 / \alpha - \log X_t\right)dt + \sigma dB_t$$

As per the additive Ornstein-Uhlenbeck process, the integrating factor of this equation is

$$IF(t) = e^{\int \alpha dt} = e^{\alpha t}$$

Multiplying both sides by the integrating factor and factorizing the differential in the logarithm of the diffusion variable

$$d\log X_t + \alpha \log X_t dt = \alpha\left(\mu - \tfrac{1}{2}\sigma^2 / \alpha\right)dt + \sigma dB_t$$

$$d(e^{\alpha t}\log X_t) = \alpha e^{\alpha t}\left(\mu - \tfrac{1}{2}\sigma^2 / \alpha\right)dt + \sigma e^{\alpha t} dB_t$$

There are then again two possible ways to proceed; either by direct integration or making a change of variable in *t* and then proceeding with the integration. By the first method, integrating the differential in the logarithm of the diffusion

$$\int_0^t \frac{d}{dy}(e^{\alpha y}\log X_y)dy = \alpha\int_0^t e^{\alpha y}\left(\mu - \tfrac{1}{2}\sigma^2 / \alpha\right)dy + \sigma\int_0^t e^{\alpha y} dB_y$$

$$e^{\alpha t}\log X_t - \log x_0 = \left(\alpha\mu - \tfrac{1}{2}\sigma^2\right)\frac{e^{\alpha t} - 1}{\alpha} + \sigma\int_0^t e^{\alpha y} dB_y$$

or

$$X_t = \exp\left(e^{-\alpha t}\log x_0 + \left(\alpha\mu - \tfrac{1}{2}\sigma^2\right)\frac{1 - e^{-\alpha t}}{\alpha} + \sigma e^{-\alpha t}\int_0^t e^{\alpha y} dB_y\right)$$

an application of parts in the differential Brownian term gives

$$X_t = \exp\left(e^{-\alpha t}\log x_0 + \left(\alpha\mu - \tfrac{1}{2}\sigma^2\right)\frac{1 - e^{-\alpha t}}{\alpha} + \sigma B_t - \alpha\sigma e^{-\alpha t}\int_0^t e^{\alpha y} B_y dy\right)$$

which, like the Ornstein-Uhlenbeck process which is the exponential index, clearly shows that the process is geometric Brownian motion when the characteristic root zeroes[22]. Alternatively, with a change of variable from *t*

[22] and the equivalent discrete root becomes a unit root.

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to $s = (e^{2\alpha t} - 1)/2\alpha$ for which the derivative is $ds/dt = e^{2\alpha t} = 1 + 2\alpha$ and the integrating factor in new variable is $e^{\alpha t} = \sqrt{1 + 2\alpha}$, gives

$$\frac{ds}{dt}\frac{d}{ds}(\sqrt{1+2\alpha s}\log X_{t(s)}) = \sqrt{1+2\alpha s}\left(\alpha\mu - \tfrac{1}{2}\sigma^2\right) + \sigma\sqrt{1+2\alpha s}\,\frac{dB_t}{dt}$$

or

$$\frac{d}{ds}(\sqrt{1+2\alpha s}\log X_{t(s)}) = \sqrt{1+2\alpha s}\left(\alpha\mu - \tfrac{1}{2}\sigma^2\right)\frac{dt}{ds} + \sigma\sqrt{1+2\alpha s}\,\frac{dt}{ds}\frac{dB_t}{dt}$$

$$= \frac{\alpha\mu - \frac{1}{2}\sigma^2}{\sqrt{1+2\alpha s}} + \sigma\frac{1}{\sqrt{1+2\alpha s}}\frac{dB_t}{dt}$$

and by distribution theory, the mathematics of which was covered in the section regarding the additive Ornstein-Uhlenbeck process,

$$\frac{d}{ds}(\sqrt{1+2\alpha s}\log X_{t(s)}) = \frac{\alpha\mu - \frac{1}{2}\sigma^2}{\sqrt{1+2\alpha s}} + \sigma\frac{dB_s}{ds}$$

Integrating

$$\int_0^s \frac{d}{dy}(\sqrt{1+2\alpha y}\log X_{t(y)})dy = \int_0^s \frac{\alpha\mu - \frac{1}{2}\sigma^2}{\sqrt{1+2\alpha y}}dy + \sigma\int_0^s \frac{dB_y}{dy}dy$$

$$\sqrt{1+2\alpha s}\log X_{t(s)} - \log x_{t(0)} = \frac{\alpha\mu - \frac{1}{2}\sigma^2}{\alpha}\left(\sqrt{1+2\alpha s} - 1\right) + \sigma B_s$$

$$\log X_{t(s)} = \frac{\log x_0}{\sqrt{1+2\alpha s}} + \frac{\alpha\mu - \frac{1}{2}\sigma^2}{\alpha}\left(1 - \frac{1}{\sqrt{1+2\alpha s}}\right) + \frac{\sigma}{\sqrt{1+2\alpha s}}B_s$$

that is,

$$X_{t(s)} = \exp\left\{\frac{\log x_0}{\sqrt{1+2\alpha s}} + \frac{\alpha\mu - \frac{1}{2}\sigma^2}{\alpha}\left(1 - \frac{1}{\sqrt{1+2\alpha s}}\right) + \frac{\sigma}{\sqrt{1+2\alpha s}}B_s\right\}$$

which has the transition distribution in the time change

$$X_{t(s)} \sim \log N\left\{\frac{\log x_0}{\sqrt{1+2\alpha s}} + \frac{\alpha\mu - \frac{1}{2}\sigma^2}{\alpha}\left(1 - \frac{1}{\sqrt{1+2\alpha s}}\right), \sigma^2\left(\frac{s}{1+2\alpha s}\right)\right\}$$

Changing back to the original linear time variable gives

$$X_t = \exp\left\{e^{-\alpha t}\log x_0 + (\alpha\mu - \tfrac{1}{2}\sigma^2)\left(\frac{1-e^{-\alpha t}}{\alpha}\right) + \sigma e^{-\alpha t}B_{(e^{2\alpha t}-1)/2\alpha}\right\}$$

which is the exponential of an Ornstein-Uhlenbeck process and consequently has a log-normal transition distribution

$$X_t \sim \log N\left\{e^{-\alpha t}\log x_0 + (\alpha\mu - \tfrac{1}{2}\sigma^2)\left(\frac{1-e^{-\alpha t}}{\alpha}\right),\ \sigma^2\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\right\}$$

the transition mgf, inclusive of the backward variables is

$$M_{X(T)|X(t_0)}(\theta) = \sum_{j=0}^{\infty}\frac{\theta^j}{j!}\exp\left(x_0 e^{-\alpha(T-t_0)}j + \mu\left(\frac{1-e^{-\alpha(T-t_0)}}{\alpha}\right)j + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha(T-t_0)}}{2\alpha}\right)j^2\right)$$

and the logarithm of the diffusion variable is a Brownian motion with transition mgf

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$$M_{\log X(T)|\log X(t_0)}(\theta) = \exp\left(e^{-\alpha(T-t_0)}\theta \log x_0 + \mu\left(\frac{1-e^{-\alpha(T-t_0)}}{\alpha}\right)\theta + \tfrac{1}{2}\sigma^2\left(\frac{1-e^{-2\alpha(T-t_0)}}{2\alpha}\right)\theta^2 \right)$$

Formation of the Forward and Backward Equations for Modulated Proportional Brownian Motion

A. Formation of the Forward Equation

From the mathematics of additive moderated Brownian motion, the forward equation of the underlying additive process is

$$\frac{\partial}{\partial T} f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0) = \left(\tfrac{1}{2}\sigma^2 \frac{\partial^2}{(\partial \log x)^2} - \frac{\partial}{\partial \log x}\left(\mu - \alpha \log x\right) \right) f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0)$$

As for unit root proportion Brownian motion, transforming back to the original variable with

$$f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0) = x\, f_{X(T)|X(t_0)}(x \mid x_0)$$

and the transformed partial derivatives

$$\frac{\partial}{\partial \log x} = \frac{\partial x}{\partial \log x}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x}$$

and

$$\frac{\partial^2}{(\partial \log x)^2} = \frac{\partial x}{\partial \log x}\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial \log x}\frac{\partial}{\partial x}\right) = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) = x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} = x\frac{\partial^2}{\partial x^2}x - x\frac{\partial}{\partial x}$$

since

$$x\frac{\partial^2}{\partial x^2}x\,g(x) = x\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}x\,g(x)\right) = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}g(x) + g(x)\right) = x^2\frac{\partial^2}{\partial x^2}g(x) + 2x\frac{\partial}{\partial x}g(x)$$

The forward equation becomes

$$\begin{aligned} x\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) &= \left(\tfrac{1}{2}\sigma^2 x\frac{\partial^2}{\partial x^2}x - \tfrac{1}{2}\sigma^2 x\frac{\partial}{\partial x} - x\frac{\partial}{\partial x}\left(\mu - \alpha\log x\right) \right) x\, f_{X(T)|X(t_0)}(x \mid x_0) \\ &= x\left(\tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}x - \frac{\partial}{\partial x}\left(\mu + \tfrac{1}{2}\sigma^2 - \alpha\log x\right) \right) x\, f_{X(T)|X(t_0)}(x \mid x_0) \end{aligned}$$

which, upon cancelation of the Jacobian is the forward equation of a proportional Brownian motion

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}x^2 f_{X(T)|X(t_0)}(x \mid x_0) - \frac{\partial}{\partial x}\left(\mu + \tfrac{1}{2}\sigma^2 - \alpha\log x\right) x\, f_{X(T)|X(t_0)}(x \mid x_0)$$

In differentially factored form,

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2\frac{\partial}{\partial x}\left(e^{-\alpha\left(\log^2 x - 2(\mu + \frac{1}{2}\sigma^2)\log x\right)/\sigma^2}\frac{\partial}{\partial x}e^{\alpha\left(\log^2 x - 2(\mu+\frac{1}{2}\sigma^2)\log x\right)/\sigma^2}\left(x^2 f_{X(T)|X(t_0)}(x \mid x_0)\right)\right)$$

B. Formation of the Backward Equation

From the mathematics of additive modulated Brownian motion, the backward equations of the underlying additive process is

$$\frac{\partial}{\partial t_0} f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0) = -\left(\tfrac{1}{2}\sigma^2\frac{\partial^2}{(\partial \log x_0)^2} + \left(\mu - \alpha x_0\right)\frac{\partial}{\partial \log x_0} \right) f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0)$$

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Transforming back to the original variable with

$$f_{\log X(T)|\log X(t_0)}(\log x \mid \log x_0) = x\, f_{X(T)|X(t_0)}(x \mid x_0)$$

and with

(1) $$\frac{\partial}{\partial \log x_0} = \frac{\partial x_0}{\partial \log x_0}\frac{\partial}{\partial x_0} = x_0 \frac{\partial}{\partial x_0}$$

(2) $$\frac{\partial^2}{(\partial \log x_0)^2} = x_0 \frac{\partial}{\partial x_0}\left(x_0 \frac{\partial}{\partial x_0}\right) = x_0^2 \frac{\partial^2}{\partial x_0^2} + x_0 \frac{\partial}{\partial x_0}$$

so

$$x\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\left(\tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{\partial x_0^2} + \tfrac{1}{2}\sigma^2 x_0 \frac{\partial}{\partial x_0} + \left(\mu - \alpha x_0\right) x_0 \frac{\partial}{\partial x_0}\right) x\, f_{X(T)|X(t_0)}(x \mid x_0)$$

$$= -x\left(\tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{\partial x_0^2} + \left(\mu + \tfrac{1}{2}\sigma^2 x_0 - \alpha x_0\right) x_0 \frac{\partial}{\partial x_0}\right) f_{X(T)|X(t_0)}(x \mid x_0)$$

and cancelling the Jacobian results in the backward equation of proportional modulated Brownian motion

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\left(\tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{\partial x_0^2} + \left(\mu + \tfrac{1}{2}\sigma^2 - \alpha x_0\right) x_0 \frac{\partial}{\partial x_0}\right) f_{X(T)|X(t_0)}(x \mid x_0)$$

The differentially factorized backward equation is

$$-\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 x_0^2 e^{\alpha(\log^2 x_0 - 2(\mu + \frac{1}{2}\sigma^2)\log x_0)/\sigma^2} \frac{\partial}{\partial x_0}\left(e^{-\alpha(\log^2 x_0 - 2(\mu + \frac{1}{2}\sigma^2)\log x_0)/\sigma^2} \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0)\right)$$

Solution to the Forward Equation for Modulated Proportional Brownian Motion (by Transforming to the Standard Heat Equation)

From the forward equation

$$\frac{\partial}{\partial t} f_{X(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x^2 f_{X(t)}(x)\right) - \alpha \frac{\partial}{\partial x}\left((\mu - \log x)\, x\, f_{X(t)}(x)\right)$$

transforming the diffusion variable $X(t)$ to logarithms, $Y(t) = \log X(t)$. The transformed independent forward diffusion variable is $y = \log x$ giving the Jacobian of the transformation

$$J = \frac{dy}{dx} = \frac{1}{x} = e^{-y}$$

The normalized density function of the transformed diffusion variable is

$$f_{Y(t)}(y) = f_{X(t)}(x)\left|\frac{dx}{dy}\right| = x\, f_{X(t)}(x)$$

which also implies $f_{Xi(t)}(x) = e^{-y} f_{Y(t)}(y)$. The relevant partial derivatives of the transform are

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(1) $\frac{\partial}{\partial x}[(\mu - \log x)x\, f_{X(t)}(x)] = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}(\mu - y)\, f_{Y(t)}(y)] = e^{-y}\frac{\partial}{\partial y}[(\mu - y)\, f_{Y(t)}(y)]$

(2) $\frac{\partial^2}{\partial x^2}[x^2 f_{X(t)}(x)] = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left[\frac{\partial y}{\partial x}\frac{\partial}{\partial y}[e^{-y} f_{Y(t)}(x)]\right] = e^{-y}\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} f_{Y(t)}(x) + f_{Y(t)}(x)\right]$

$$= e^{-y}\frac{\partial^2}{\partial y^2} f_{Y(t)}(x) + e^{-y}\frac{\partial}{\partial y} f_{Y(t)}(x)$$

(3) $\frac{\partial}{\partial t} f_{X(t)}(x) = e^{-y}\frac{\partial}{\partial t} f_{Y(t)}(x)$

which, upon substitution, gives the equation

$$e^{-y}\frac{\partial}{\partial t} f_{Y(t)}(x) = \tfrac{1}{2}\sigma^2 e^{-y}\frac{\partial^2}{\partial y^2} f_{Y(t)}(x) + \tfrac{1}{2}\sigma^2 e^{-y}\frac{\partial}{\partial y} f_{Y(t)}(x) - \alpha e^{-y}\frac{\partial}{\partial y}[(\mu - y) f_{Y(t)}(y)]$$

or

$$\frac{\partial}{\partial t} f_{Y(t)}(x) = \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2} f_{Y(t)}(x) + \tfrac{1}{2}\sigma^2\frac{\partial}{\partial y} f_{Y(t)}(x) - \alpha\frac{\partial}{\partial y}[(\mu - \tfrac{1}{2}\sigma^2/\alpha - y) f_{Y(t)}(y)]$$

which is the forward equation of an Ornstein-Uhlenbeck process that mean reverts to $\mu - \frac{1}{2}\sigma^2/\alpha$. This is the parameter μ with an ito correction $\frac{1}{2}\sigma^2/\alpha$. A further transformation of the diffusion variable

$$U(t) = (\mu - \tfrac{1}{2}\sigma^2/\alpha) + e^{\alpha t}\left(Y(t) - (\mu - \tfrac{1}{2}\sigma^2/\alpha)\right)$$

transforms the equation to

$$\frac{\partial}{\partial t} f_{U(t)}(y) = \tfrac{1}{2}\sigma^2 e^{2\alpha t}\frac{\partial^2}{\partial u^2} f_{U(t)}(u)$$

and a further transformation of linear time $s = \sigma^2(e^{2\alpha t} - 1)/2\alpha$ gives the standard heat equation

$$\frac{\partial}{\partial s} f_{U(s)}(y) = \tfrac{1}{2}\frac{\partial^2}{\partial u^2} f_{U(s)}(u)$$

which has a standard Brownian motion transition distribution, $U(s) \sim N(u_0, s)$. Transforming back to the original linear time variable gives

$$U(t) \sim N\left(u_0, \sigma^2\frac{e^{2\alpha t} - 1}{2\alpha}\right)$$

and further back to the logarithm of the variable

$$Y(t) = (\mu - \tfrac{1}{2}\sigma^2/\alpha) + e^{-\alpha t}\left(U(t) - (\mu - \tfrac{1}{2}\sigma^2/\alpha)\right)$$

noting that $y_0 = u_0$ gives the transition distribution for the logarithm of the diffusion variable which has a normal transition distribution of an ito-corrected Ornstein-Uhlenbeck process

$$Y(t) = \log X(t) \sim N\left(y_0 e^{-\alpha t} + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1 - e^{-\alpha t}), \sigma^2\left(\frac{1 - e^{-2\alpha t}}{2\alpha}\right)\right)$$

and the original diffusion variable is therefore log-normal

$$X(t) \sim \log N\left(\log x_0 e^{-\alpha t} + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1 - e^{-\alpha t}), \sigma^2\left(\frac{1 - e^{-2\alpha t}}{2\alpha}\right)\right)$$

the transition density of which, the solution to the forward equation, is

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$$f_{X(t)}(x) = \frac{1}{\sigma\sqrt{2\pi}\sqrt{(1-e^{-2\alpha t})/2\alpha}} \exp\left(- \frac{[\log x - \log x_0 e^{-\alpha t} - (\mu - \frac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t})]^2}{\sigma^2(1-e^{-2\alpha t})/\alpha} \right)$$

for $0 \le x < \infty$ and $t \ge 0$

Solution to the Forward Equation for Modulated Proportional Brownian Motion: (Solving the Equation Directly)

This is omitted for rather clear reasons. If one is interested, refer to the results for geometric Brownian motion and combine the techniques with those for solving the additive Ornstein-Uhlenbeck diffusion.

Further Statistical Description of the Modulated Proportional Brownian Motion:

For the distributional statistics – quantiles, raw moments, central moments, standardized central moments, and entropy – of the geometric Ornstein-Uhlenbeck process, substitute the log-mean and log-variance of the transition distribution of geometric Brownian motion with the log-mean and log-variance of the geometric Ornstein-Uhlenbeck process. These are both multiplicative Gaussian processes for which differ only in the structure of the time functions of the transition log-mean and the transition log-variance. For diffusions of arbitrary order, see the relevant tables of transition means and variances in BOOK 3 of this dissertation.

Covariance of Modulated Proportional Brownian Motion

The general raw (auto) joint moment of a geometric Brownian motion at times *s* and *t* is

$$E(X_t^{\theta_1} X_s^{\theta_2}) = E(e^{\theta_1 \log X_t + \theta_2 \log X_s}) = M_{\log X_t, \log X_s}(\theta_1, \theta_2)$$

which is the (auto) joint moment generating function of the underlying arithmetic Brownian motion. This is, after the ito-correction,

$$M_{\log X_t, \log X_s}(\theta_1, \theta_2) = \exp\left(\theta_1 e^{-\alpha t} \log x_0 + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t})\theta_1 + \theta_2 e^{-\alpha s} \log x_0 + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha s})\theta_2 \right)$$
$$\times \exp\left(\tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha t}}{2\alpha}\theta_1^2 + \tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha s}}{2\alpha}\theta_2^2 + \sigma^2 \frac{e^{-\alpha(t-s)} - e^{-\alpha(t+s)}}{2\alpha}\theta_1\theta_2 \right)$$

where $(s \wedge t) = \min(s, t)$. The first raw (auto) joint moment is

$$E(X_t X_s) = M_{\log X_t, \log X_s}(1,1)$$
$$= \exp\left((e^{-\alpha t} + e^{-\alpha s}) \log x_0 + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t}) + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha s}) \right)$$
$$\times \exp\left(\tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha t}}{2\alpha} + \tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha s}}{2\alpha} + \sigma^2 \frac{e^{-\alpha(t-s)} - e^{-\alpha(t+s)}}{2\alpha} \right)$$

The individual raw moments are

$$E(X_t) = M_{\log X_t, \log X_s}(1,0)$$
$$= \exp\left(e^{-\alpha t} \log x_0 + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t}) + \tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha t}}{2\alpha} \right)$$

and

$$E(X_s) = M_{\log X_t, \log X_s}(0,1)$$
$$= \exp\left(e^{-\alpha s} \log x_0 + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha s}) + \tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha s}}{2\alpha} \right)$$

giving the (auto) covariance of the geometric Brownian motion

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$$
\begin{aligned}
Cov(X_t X_s) &= M_{\log X_t,\log X_s}(1,1) - M_{\log X_t,\log X_s}(1,0) M_{\log X_t,\log X_s}(0,1) \\
&= \exp\left((e^{-\alpha t} + e^{-\alpha s})\log x_0 + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t}) + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha s})\right) \\
&\quad \times \exp\left(\tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha t}}{2\alpha} + \tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha s}}{2\alpha} + \sigma^2 \frac{e^{-\alpha(t-s)} - e^{-\alpha(t+s)}}{2\alpha}\right) \\
&\quad - \exp\left((e^{-\alpha t} + e^{-\alpha s})\log x_0 + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t}) + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha s})\right) \\
&\quad \times \exp\left(\tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha t}}{2\alpha} + \tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha s}}{2\alpha}\right) \\
&= \exp\left((e^{-\alpha t} + e^{-\alpha s})\log x_0 + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t}) + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha s})\right) \\
&\quad \times \exp\left(\tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha t}}{2\alpha} + \tfrac{1}{2}\sigma^2 \frac{1-e^{-2\alpha s}}{2\alpha}\right)\left\{\exp\left(\sigma^2 \frac{e^{-\alpha(t-s)} - e^{-\alpha(t+s)}}{2\alpha}\right) - 1\right\}
\end{aligned}
$$

Joint Transition Distribution of Modulated Proportional Brownian Motion

Further, the correlation between the log process at two points in time which is that for the underlying additive process, is

$$Cov[\log X(s), \log X(t)] = \frac{\sigma^2}{2\alpha}(e^{-\alpha(t-s)} - e^{-\alpha(t+s)})$$

which means that the joint-auto transition density on logs is also the same,

$$\begin{bmatrix}\log X(s)\\ \log X(t)\end{bmatrix} \sim BivN\left(\begin{bmatrix}e^{-\alpha s}\log(x_0) + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha s})\\ e^{-\alpha t}\log(x_0) + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t})\end{bmatrix}, \begin{bmatrix}\sigma^2(1-e^{-2\alpha s})/2\alpha & \sigma^2(e^{-\alpha(t-s)} - e^{-\alpha(t+s)})/2\alpha\\ \sigma^2(e^{-\alpha(t-s)} - e^{-\alpha(t+s)})/2\alpha & \sigma^2(1-e^{-2\alpha t})/2\alpha\end{bmatrix}\right)$$

On the original scale, the joint-auto transition density is bivariate log-normal,

$$\begin{bmatrix}X(s)\\ X(t)\end{bmatrix} \sim Biv\log N\left(\begin{bmatrix}e^{-\alpha s}\log(x_0) + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha s})\\ e^{-\alpha t}\log(x_0) + (\mu - \tfrac{1}{2}\sigma^2/\alpha)(1-e^{-\alpha t})\end{bmatrix}, \begin{bmatrix}\sigma^2(1-e^{-2\alpha s})/2\alpha & \sigma^2(e^{-\alpha(t-s)} - e^{-\alpha(t+s)})/2\alpha\\ \sigma^2(e^{-\alpha(t-s)} - e^{-\alpha(t+s)})/2\alpha & \sigma^2(1-e^{-2\alpha t})/2\alpha\end{bmatrix}\right)$$

Steady State (Ergodic), Long Run, Distribution of Modulated Proportional Brownian Motion

Zeroing the time side of the differentially factorized[23] forward equation,

$$0 = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(e^{-\frac{1}{2}k\log^2 x + \mu k\log x} \frac{\partial}{\partial x} e^{\frac{1}{2}k\log^2 x - \mu k\log x}\left(x^2 f_{X(T)|X(t_0)}(x \mid x_0)\right)\right) \quad \text{where } k = \frac{\alpha}{\frac{1}{2}\sigma^2}$$

Integrating once

$$e^{-\frac{1}{2}k\log^2 x + \mu k\log x} \frac{\partial}{\partial x} e^{\frac{1}{2}k\log^2 x - \mu k\log x}\left(x^2 f_{X(T)|X(t_0)}(x \mid x_0)\right) = a_0$$

or

---

[23] The logarithm of the integrating factor is $\log IF(x) = -\int \left(\frac{\mu}{x} - \frac{\log x}{x}\right)dx = -\mu\log x + \frac{1}{2}\log^2 x$ from the integral $I = \int \frac{\log x}{x}dx = \log^2 x - \int \frac{\log x}{x}dx = \log^2 x - I$ giving $I = \frac{1}{2}\log^2 x$ included here for the sake of the mathematics, as an alternative to substitution.

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$$\frac{\partial}{\partial x} e^{\frac{1}{2}k\log^2 x-\mu k\log x}\left(x^2 f_{X(T)|X(t_0)}(x\mid x_0)\right)=a_0 e^{\frac{1}{2}k\log^2 x-\mu k\log x}$$

and integrating again

$$e^{\frac{1}{2}k\log^2 x-\mu k\log x}\left(x^2 f_{X(T)|X(t_0)}(x\mid x_0)\right)=a_1+a_0\int e^{\frac{1}{2}k\log^2 x-\mu k\log x}dx$$

or

$$f_{X(T)|X(t_0)}(x\mid x_0)=a_1 x^{-2}e^{-\frac{1}{2}k\log^2 x+\mu k\log x}+a_0 x^{-2}e^{-\frac{1}{2}k\log^2 x+\mu k\log x}\int e^{\frac{1}{2}k\log^2 x-\mu k\log x}dx$$

Since the term containing the integral is divergent, $a_0 = 0$, leaving the expression

$$f_{X(T)|X(t_0)}(x\mid x_0)=a_1 x^{-1}e^{-\frac{1}{2}k\log^2 x+(\mu k-1)\log x}$$

Factorizing the index of the exponential gives

$$f_{X(T)|X(t_0)}(x\mid x_0)=a_1 e^{-\frac{1}{2}k(\mu-1/k)^2}x^{-1}e^{-\frac{1}{2}k(\log x-\mu+1/k)^2}$$

which is proportional to a log-normal density function with parameters $\mu-1/k=\mu-\frac{1}{2}\sigma^2/\alpha$ and $\frac{1}{2}\sigma^2/\alpha$ which implies that

$$a_1=ke^{-\frac{1}{2}k(\mu-1/k)^2}/\sqrt{2\pi}$$

without having to do the integration for the normalization factor. The distribution is

$$X(t)\sim \log N\left(\mu-\tfrac{1}{2}\sigma^2/\alpha,\ \tfrac{1}{2}\sigma^2/\alpha\right)$$

and the density for which, the solution to the steady state equation, is a time invariant log-normal density

$$\begin{aligned} f_{X(T)|X(t_0)}(x\mid x_0) &= \frac{1}{x(1/\sqrt{k})\sqrt{2\pi}}e^{-\frac{1}{2}k(\log x-\mu+1/k)^2} \\ &= \frac{1}{x(\sigma/\sqrt{2\alpha})\sqrt{2\pi}}e^{-\frac{1}{2}(\log x-\mu+\sigma^2/2\alpha)^2/(\sigma^2/2\alpha)} \end{aligned}$$

This is the density of a multiplicative geometric Gaussian basis, i.e., a multiplicative Gaussian-Hilbert space.

ECCENTRIC SQUARE-ROOT PROCESS

Introduction

An eccentric square-root process is a peculiar form of quadratic diffusion. The value to which a square-root process mean reverts generates the central part and the degrees of freedom of a non-central chi-square transition distribution. Consequently, the transition distribution of a square-root diffusion that mean reverts to zero is a non-central chi-square that does not have a central part so that the transition distribution has zero degrees of freedom. This form of square-root diffusion is referred to as an eccentric[24] square-root diffusion due to the form of the transition distribution since it is the purely eccentric part of a (scaled) non-central chi-square distribution. The eccentric square-root process forms a convolution ito-transform pair with convolution index zero. This is the difference between two two-parameter square root processes (forming a deconvolution ito-transform pair). The forward equation is not transformable to the standard heat equation since the diffusion does not form an ordinary ito-transform pair. The eccentric square-root diffusion has infinitesimal parameters $\mu(x,t)=\alpha x$ and $\sigma(x,t)=\sigma\sqrt{x}$, and the stochastic differential equation is

[24] Johnson, N. L., Kotz, S. K. and Balakrishnan, N., (1994) *Continuous Univariate Distributions, Volume 2, 2nd Edition*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons.

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$$dX_t = -\alpha X_t dt + \sigma\sqrt{X_t}\, dB_t$$

Upon solution, the transition mgf is

$$M_{X(T)|X(t_0)}(\theta) = \exp\left(\frac{x_0 e^{-\alpha(T-t_0)}\theta}{1 - \frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} \right)$$

Integral Solution to the Eccentric Square-Root Process

It is not possible to transform to the standard heat equation since the eccentric square-root diffusion does not form a direct ito transform pair and therefore does not have an analytic integral solution.

Formation of the Forward and Backward Equations of the Eccentric Square-Root Process

A. Formation of the Forward Equation

Differentiating the mgf partially with respect to the forward time variable gives the forward equation in the mgf

$$\begin{aligned}
\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta) &= \left(-\frac{\alpha x_0 e^{-\alpha(T-t_0)}\theta}{1 - \frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} + \frac{\frac{1}{2}\theta^2\sigma^2 x_0 e^{-2\alpha(T-t_0)}}{\left(1 - \frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2} \right) M_{X(T)|X(t_0)}(\theta) \\
&= x_0 e^{-\alpha(T-t_0)} \left(-\frac{\alpha\theta}{1 - \frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} + \frac{\frac{1}{2}\theta^2\sigma^2 e^{-\alpha(T-t_0)}}{\left(1 - \frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2} \right) M_{X(T)|X(t_0)}(\theta) \\
&= \frac{x_0 e^{-\alpha(T-t_0)}}{\left(1 - \frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2} \left(-\alpha\theta + \frac{1}{2}\theta^2\sigma^2 \right) M_{X(T)|X(t_0)}(\theta)
\end{aligned}$$

Differentiating partially with respect to the mgf parameter

$$\begin{aligned}
\frac{\partial}{\partial\theta} M_{X(T)|X(t_0)}(\theta) &= \frac{1}{\theta^2} \frac{x_0 e^{-\alpha(T-t_0)}}{\left(1/\theta - \frac{1}{2}\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2} \exp\left(\frac{x_0 e^{-\alpha(T-t_0)}}{1/\theta - \frac{1}{2}\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} \right) \\
&= \frac{x_0 e^{-\alpha(T-t_0)}}{\left(1 - \frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2} M_{X(T)|X(t_0)}(\theta)
\end{aligned}$$

The forward equation in the mgf can then be written as

$$\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta) = \left(-\alpha\theta + \frac{1}{2}\theta^2\sigma^2 \right) \frac{\partial}{\partial\theta} M_{X(T)|X(t_0)}(\theta)$$

Differentiating the mgf partially once with respect to the forward diffusion variable and once with respect to the mgf parameter, and then twice with respect to the forward diffusion variable and once with respect to the mgf parameter, i.e.,

$$\frac{\partial^2}{\partial x \partial\theta} M_{X(T)|X(t_0)}(\theta) \qquad \text{and} \qquad \frac{\partial^3}{\partial x^2 \partial\theta} M_{X(T)|X(t_0)}(\theta)$$

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for

(1) $\theta \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta) = -\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x\, f_{X(T)|X(t_0)}(x \mid x_0) dx$

(2) $\theta^2 \frac{\partial}{\partial \theta} M_{X(T)|X(t_0)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} x\, f_{X(T)|X(t_0)}(x \mid x_0) dx$

The forward equation in the mgf can then be written in terms of the transition density as

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) dx = \alpha \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} x\, f_{X(T)|X(t_0)}(x \mid x_0) dx + \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} x\, f_{X(T)|X(t_0)}(x \mid x_0) dx$$

which results in the forward equation of the eccentric square-root process in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \alpha \frac{\partial}{\partial x}\left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right) + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right)$$

In differentially factorized form, this is

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(e^{-\alpha x/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x} e^{\alpha x/\frac{1}{2}\sigma^2} \left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right)\right)$$

B. Formation of the Backward Equation

Differentiating the mgf partially with respect to the backward time variable

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = \left(\frac{\alpha x_0 e^{-\alpha(T-t_0)}\theta}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} - \frac{\frac{1}{2}\theta^2\sigma^2 x_0 e^{-2\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}\right) M_{X(T)|X(t_0)}(\theta)$$

$$= \left(\alpha x_0 \frac{\theta e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} - \tfrac{1}{2}\sigma^2 x_0 \frac{\theta^2 e^{-2\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}\right) M_{X(T)|X(t_0)}(\theta)$$

Differentiating partially with respect to the backward diffusion variable

$$\frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) = \left(\frac{\theta e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right) M_{X(T)|X(t_0)}(\theta)$$

and again

$$\frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta) = \left(\frac{\theta e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right)^2 M_{X(T)|X(t_0)}(\theta)$$

Hence, the time derivative can be written as

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = \left(\alpha x_0 \frac{\partial}{\partial x_0} - \tfrac{1}{2}\sigma^2 x_0 \frac{\partial^2}{\partial x_0^2}\right) M_{X(T)|X(t_0)}(\theta)$$

which results the backward equation of the eccentric square-root process in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\tfrac{1}{2}\sigma^2 x_0 \frac{\partial^2}{\partial x_0^2} f_{X(T)|X(t_0)}(x \mid x_0) + \alpha x_0 \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0)$$

In differentially factorized form, this is

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$$-\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 x_0 e^{\alpha x_0 / \frac{1}{2}\sigma^2} \frac{\partial}{\partial x_0}\left(e^{-\alpha x_0 / \frac{1}{2}\sigma^2} \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0) \right)$$

Solution to the Forward Equation for the Square-Root Process: (by Transforming to the Standard Heat Equation)

It is not possible to transform the forward equation to the standard heat equation since the eccentric square-root diffusion does not form a direct ito transform pair.

Solution to the Forward Equation for the Square-Root Process: (Solving the Equation Directly)

From the eccentric square-root forward equation is

$$\frac{\partial}{\partial t} f_{X(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)}(x) \right) + \alpha \frac{\partial}{\partial x}\left(x\, f_{X(t)}(x) \right)$$

Multiplying by an exponential factor in the forward diffusion variable and integrating

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{X(t)}(x) dx = \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)}(x) \right) dx + \alpha \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x}\left(x\, f_{X(t)}(x) \right) dx$$

Integrating by parts gives the partial differential equation for the moment generating function

$$\frac{\partial}{\partial t} M_{X(t)}(\theta) = (\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta} M_{X(t)}(\theta)$$

subject to the initial condition of a the moment generating function of a Dirac singularity located at x_0

$$M_{X(0)}(\theta) = \int_0^{\infty} e^{x\theta} f_{X(0)|X(0)}(x \mid x_0) dx = \int_0^{\infty} e^{x\theta} \delta_{x_0}(x) dx = e^{x_0\theta}$$

Dividing both sides of the partial differential equation by the dependent variable gives the partial differential equation of the cumulant generating function

$$\frac{\partial}{\partial t} \log M_{X(t)}(\theta) = (\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta} \log M_{X(t)}(\theta)$$

Removing the derivative of the time dimension of the equation by defining the Laplace transform of the cumulant generating function

$$\lambda(\theta; s) = L\{\log M_{X(t)}(\theta)\}$$

gives

$$s\lambda(\theta; s) - \log M_{X(0)}(\theta) = (\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta}\lambda(\theta; s)$$

$$-x_0\theta = (\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta}\lambda(\theta; s) - s\lambda(\theta; s)$$

and rearranging gives a linear first order ordinary differential equation,

$$\frac{\partial}{\partial \theta}\lambda(\theta; s) - \frac{s}{\theta(\frac{1}{2}\sigma^2\theta - \alpha)}\lambda(\theta; s) = -\frac{x_0}{\frac{1}{2}\sigma^2\theta - \alpha}$$

The logarithm of the integrating factor, by partial fractions, is

$$\log IF(\theta) = \int_{-\infty}^{\infty} \frac{s}{\theta(\frac{1}{2}\sigma^2\theta - \alpha)} d\theta = \frac{s}{\alpha} \log\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta - \alpha} \right)$$

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Multiplying the partial differential equation through by the integrating factor gives the differential

$$\frac{\partial}{\partial\theta}\left[\lambda(\theta;s)\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)^{s/\alpha}\right]=-\frac{x_0}{\frac{1}{2}\sigma^2\theta-\alpha}\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)^{s/\alpha}$$

Integrating gives a solution in terms of a non-elementary integral,

$$\lambda(\theta;s)\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)^{s/\alpha}=-x_0\int\frac{1}{\frac{1}{2}\sigma^2\theta-\alpha}\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)^{s/\alpha}d\theta$$

Using the substitution $y=\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)$ transforms the integral to a beta form and gives a right hand side of

$$-\frac{x_0}{\frac{1}{2}\sigma^2}\int(1-y)^{-1}y^{s/\alpha}dy$$

This can be evaluated in series form

$$-\frac{x_0}{\frac{1}{2}\sigma^2}\int y^{s/\alpha}(1+y+y^2+y^3+\ldots)dy=-\frac{x_0}{\frac{1}{2}\sigma^2}\int(y^{s/\alpha}+y^{1+s/\alpha}+y^{2+s/\alpha}+y^{3+s/\alpha}+\ldots)dy$$

$$=-\frac{x_0}{\frac{1}{2}\sigma^2}\left(\frac{y^{s/\alpha+1}}{s/\alpha+1}+\frac{y^{s/\alpha+2}}{s/\alpha+2}+\frac{y^{s/\alpha+3}}{s/\alpha+3}+\frac{y^{s/\alpha+4}}{s/\alpha+4}+\ldots\right)$$

Giving the solution to the ordinary differential equation, in terms of *y*,

$$\lambda(\theta;s)y^{s/\alpha}=-\frac{x_0}{\frac{1}{2}\sigma^2}\left(\frac{y^{s/\alpha+1}}{s/\alpha+1}+\frac{y^{s/\alpha+2}}{s/\alpha+2}+\frac{y^{s/\alpha+3}}{s/\alpha+3}+\frac{y^{s/\alpha+4}}{s/\alpha+4}+\ldots\right)$$

or

$$\lambda(\theta;s)=-\frac{x_0\alpha}{\frac{1}{2}\sigma^2}\left(\frac{y}{s+\alpha}+\frac{y^2}{s+2\alpha}+\frac{y^3}{s+3\alpha}+\frac{y^4}{s+4\alpha}+\ldots\right)$$

Recovering the time dimension with a Laplace inversion gives a geometric progression so the series is summable

$$\log M_{X(t)}(\theta)=-\frac{x_0\alpha}{\frac{1}{2}\sigma^2}\left(L^{-1}\left\{\frac{y}{s+\alpha}\right\}+L^{-1}\left\{\frac{y^2}{s+2\alpha}\right\}+L^{-1}\left\{\frac{y^3}{s+3\alpha}\right\}+L^{-1}\left\{\frac{y^4}{s+4\alpha}\right\}+\ldots\right)$$

$$=-\frac{x_0\alpha}{\frac{1}{2}\sigma^2}(ye^{-\alpha t}+y^2e^{-2\alpha t}+y^3e^{-3\alpha t}+y^4e^{-4\alpha t}+\ldots)$$

$$=-\frac{x_0\alpha}{\frac{1}{2}\sigma^2}ye^{-\alpha t}(1-ye^{-\alpha t})^{-1}$$

$$=\frac{x_0e^{-\alpha t}\theta}{1-\frac{\sigma^2}{2\alpha}(1-e^{-\alpha t})\theta}$$

Exponentiating for the moment generating function,

$$M_{X(t)}(\theta)=\exp\left(\frac{x_0e^{-\alpha t}\theta}{1-\theta\sigma^2(1-e^{-\alpha t})/2\alpha}\right)$$

which is the moment generating function of a scaled non-central chi-square on zero degrees of freedom with non-centrality parameter

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$$\frac{x_0 e^{-\alpha t}}{\frac{\sigma^2}{4\alpha}(1-e^{-\alpha t})}$$

and variance scale factor of

$$\frac{\sigma^2}{4\alpha}(1-e^{-\alpha t})$$

This factor is the variance of an underlying Ornstein-Uhlenbeck process with the rate of mean reversion parameter, α, doubled since the Ornstein-Uhlenbeck diffusion has been squared. Therefore,

$$X(t) \sim k\,\chi_0^2(x_0 e^{-\alpha t}/k) \qquad \text{where} \quad k = \frac{\sigma^2}{4\alpha}(1-e^{-\alpha t})$$

The Transition Distribution of the Eccentric Square-Root Diffusion

The transition distribution of a scaled non-central chi-square has a real set of cumulants but it does not have an analytic transition density function. The transition distribution $X(t) \sim k\,\chi_0^2(x_0 e^{-\alpha t}/k)$ has the interpretation of a (scaled) central chi-square with n degrees of freedom $k\,\chi_n^2(0)$ deconvolved from a (scaled) central chi-square on n degrees of freedom $k\,\chi_n^2(x_0 e^{-\alpha t}/k)$, that is,

$$\begin{aligned} X(t) &\sim k\,\chi_0^2(x_0 e^{-\alpha t}/k) \\ &\sim k\,\chi_n^2(x_0 e^{-\alpha t}/k) - k\,\chi_n^2(0) \end{aligned}$$

both of which being well defined. The density can be obtained numerically by inverting the moment generating function along the Bromwhich contour or by solving the forward equation numerically. Software is yet to include functions related to this distribution in the relevant routines.

Steady State (Ergodic), Long Run, Distribution of the Eccentric Square-Root Diffusion

Zeroing the time side of the forward equation leaves a form of Laguerre's (associated) differential equation, solutions for which relate to the gamma density function, in this case, for a zero shape parameter;

$$0 = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)}(x)\right) + \alpha \frac{\partial}{\partial x}\left(x\, f_{X(t)}(x)\right)$$

In differentially factorized form

$$0 = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(e^{-kx}\frac{\partial}{\partial x}e^{kx}\left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right)\right) \qquad \text{where } k = \frac{\alpha}{\frac{1}{2}\sigma^2}$$

Integrating once

$$e^{-kx}\frac{\partial}{\partial x}e^{kx}[x\ f_{X(t)}(x)] = a_0$$

or

$$\frac{\partial}{\partial x}e^{kx}[x\ f_{X(t)}(x)] = a_0 e^{kx}$$

Integrating again

$$e^{kx}[x\ f_{X(t)}(x)] = a_1 + a_0\frac{e^{kx}}{k}$$

or

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$$f_{X(t)}(x) = a_1 \frac{e^{-kx}}{x} + a_0 \frac{1}{kx}$$

consistent with the solutions to the other ergodic distributions, $a_0 = 0$,

$$f_{X(t)}(x) = a_1 \frac{e^{-kx}}{x}$$

The integral of this function is the complementary exponential integral, the complement to the exponential integral which is defined as

$$Ei(u) = \int_u^\infty \frac{e^{-x}}{x} dx$$

which diverges at $u = 0$. It is also the value of the gamma function with zero argument, $\Gamma(0)$, which is infinite. The value of the constant of proportionality is then $a_1 = 1/\Gamma(0)$,

$$f_{X(t)}(x) = \frac{x^{-1}e^{-kx}}{\Gamma(0)}$$

which is a gamma density with zero shape parameter or a chi-square distribution on zero degrees of freedom. This function has unit area on an infinitesimal width and is therefore a Dirac singularity located at the origin,

$$\begin{aligned} X(t) &\sim \lim_{\varepsilon \to 0} U(0-\varepsilon, 0+\varepsilon) \\ &\sim \delta_0(x) \end{aligned}$$

One could immediately see this by observing that the non-centrality parameter in the non-central chi-square transition distribution of the eccentric square-root process zeroes leaving a non-central chi-square on zero degrees of freedom.

SQUARE-ROOT PROCESS

Introduction

A square-root process is a first order quadratic diffusion. Adding a differential term $\mu\alpha\, dt$ to the eccentric square root results in a first order diffusion process with infinitesimal parameters $\mu(x,t) = \alpha(\mu - x)$ and $\sigma(x,t) = \sigma\sqrt{x}$ that mean reverts towards the value μ. The stochastic differential equation is

$$dX_t = \alpha(\mu - X_t)dt + \sigma\sqrt{X_t}\, dB_t$$

The transition distribution has a non-zero number of degrees of freedom, possibly fractional, which demonstrates the fact that the component of the infinitesimal mean that is independent of the value of the diffusion variable adds the central part of the non-central chi-square distribution. This process has neither independent increments nor independent ratios.

Integral Solution to the Square-Root Process

The stochastic differential equation of the three-parameter square-root diffusion does not have an analytic integral solution in terms of standard Brownian motion since it is not possible to transform the forward equation to the standard heat equation. This is the result of the fact that the three-parameter square-root diffusion is the result of convolving individual two-parameter square-root diffusions. This can alternatively be inferred from the fact that the three-parameter square-root diffusion does not form a direct ito transform pair. The convolution index of the three-parameter square-root is $\nu = 4\alpha\mu/\sigma^2$. A special case of the three-parameter square-root process, then, the exception that does transform to the standard heat equation and form a direct ito-transform pair, is the two-parameter square-root which is parameterized in such a way that it has a unit convolution index $\nu = 4\alpha\mu/\sigma^2 = 1$. In this case, the diffusion is a single squared Ornstein-Uhlenbeck process (with zero basis mean) and has a chi-square transition distribution with a single degree of freedom. Since the two-parameter square-root diffusion is transformable to the standard heat equation, it also has an analytic integral solution. The

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integral solution for that which is composed of a two dimensional parameter space (rather than three) is obtained in this section since it is complementary to the material covered in BOOK 3.

Integral Solution of the Two-Parameter Square-Root Diffusion

The stochastic differential of the three-parameter square-root diffusion can be stated as

$$dX_t = 2\alpha\left(n\frac{\sigma^2}{2\alpha} - X_t \right)dt + 2\sigma\sqrt{X_t}\,dB_t$$

When $n = 1$, the process becomes the two-parameter square-root diffusion with differential increment

$$dX_t = 2\alpha\left(\frac{\sigma^2}{2\alpha} - X_t \right)dt + 2\sigma\sqrt{X_t}\,dB_t$$

By the ito-calculus, or by transforming the corresponding forward or backward equation, the transformation of the forward equation considered in the next section, the process can be written in terms of the square-root of the diffusion,

$$d\sqrt{X_t} = -\alpha\sqrt{X_t}\,dt + \sigma dB_t$$

or

$$\frac{d\sqrt{X_t}}{dt} + \alpha\sqrt{X_t} = \sigma\frac{dB_t}{dt}$$

which is an Ornstein-Uhlenbeck process in the variable $\sqrt{X_t}$. This is a first order linear differential equation with a differential Brownian motion input, that of a zero mean diffusion basis. The integrating factor is

$$IF(t) = e^{\int \alpha dt} = e^{\alpha t}$$

Multiplying by the integrating factor and differentially factorizing the equation gives

$$\frac{d}{dt}[e^{\alpha t}\sqrt{X_t}] = \sigma e^{\alpha t}\frac{dB_t}{dt}$$

Integrating gives

$$\sqrt{X_t} = e^{-\alpha t}\sqrt{x_0} + \sigma e^{-\alpha t}\int_0^t e^{\alpha t}\,dB_t$$

or

$$X_t = \left(e^{-\alpha t}\sqrt{x_0} + \sigma e^{-\alpha t}\int_0^t e^{\alpha t}\,dB_t \right)^2$$

Or alternatively, from the time change of the Ornstein-Uhlenbeck process $s = (e^{2\alpha t} - 1) / 2\alpha$, the differential becomes[25]

$$\frac{d}{ds}[\sqrt{1+2\alpha s}\sqrt{X_{(2\alpha)^{-1}\log(1+2\alpha s)}}] = \sigma\frac{dB_s}{ds}$$

Integrating

$$\int_0^s \frac{d}{dy}[\sqrt{1+2\alpha y}\sqrt{X_{(2\alpha)^{-1}\log(1+2\alpha s)}}]dy = \sigma\int_0^s \frac{dB_y}{dy}\,dy$$

gives

[25] See the mathematics for the Ornstein-Uhlenbeck process for the exact details.

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$$\sqrt{X_{(2\alpha)^{-1}\log(1+2\alpha s)}} = \frac{\sqrt{x_0}}{\sqrt{1+2\alpha s}} + \frac{\sigma}{\sqrt{1+2\alpha s}} B_s$$

or

$$X_{(2\alpha)^{-1}\log(1+2\alpha s)} = \left(\frac{\sqrt{x_0}}{\sqrt{1+2\alpha s}} + \frac{\sigma}{\sqrt{1+2\alpha s}} B_s \right)^2$$

and in the original linear time variable,

$$X_t = \left(e^{-\alpha t} \sqrt{x_0} + \alpha B_{(1-e^{-2\alpha t})/2\alpha} \right)^2$$

Noting that

$$\sqrt{X_t} \sim N\left(e^{-\alpha t} \sqrt{x_0}, \sigma^2 \frac{1-e^{-2\alpha t}}{2\alpha} \right)$$

and from the relation $Y^2 \sim s^2 \chi_1^2(m/s^2)$ if $Y \sim N(m, s^2)$, the distribution of the integral solution is non-central chi-square with a single degree of freedom,

$$X_t \sim \sigma^2 \left(\frac{1-e^{-2\alpha t}}{2\alpha} \right) \chi_1^2 \left(\frac{x_0 e^{-2\alpha t}}{\sigma^2 (1-e^{-2\alpha t})/2\alpha} \right)$$

With a parameterization that is consistent with the parameterization of the eccentric square-root $\alpha' = 2\alpha$ and $\sigma' = 2\alpha$, the transition distribution for the two parameter square-root stochastic differential

$$dX_t = \alpha' \left(\sigma'^2/\alpha' - X_t \right) dt + \sigma' \sqrt{X_t} dB_t$$

is

$$X_t \sim \frac{\sigma'^2}{4} \left(\frac{1-e^{-\alpha' t}}{\alpha'} \right) \chi_1^2 \left(\frac{x_0 e^{-\alpha' t}}{\sigma'^2 (1-e^{-\alpha' t})/4\alpha'} \right)$$

The MGF of a Non-Central Chi-Square Distribution

Using the fact that a non-central chi-square distribution

$$X \sim \chi_n^2(\lambda)$$

is a central chi-square distribution with n plus two times a Poisson distributed variable with mean $\lambda/2$, i.e., $n + 2N$ of degrees of freedom, where N is a Poisson variable with intensity of one half the non-centrality parameter $N \sim P(\lambda/2)$

$$X \sim \chi_{n+2N}^2(0)$$

With this in mind, the moment generating function of a non-central chi-square distribution is then

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$$
\begin{aligned}
M_{\chi_n^2(\lambda)}(\theta) &= E\left(e^{\theta\chi_n^2(\lambda)}\right) \\
&= \int_0^\infty e^{\theta y} f_{\chi_n^2(\lambda)}(y)dy \\
&= \int_0^\infty e^{\theta x} \sum_{j=0}^\infty p_{P(\lambda/2)}(j) f_{\chi^2_{n+2j}}(y)dy \\
&= \sum_{j=0}^\infty p_{P(\lambda/2)}(j) \int_0^\infty e^{\theta x} f_{\chi^2_{n+2j}}(y)dy \\
&= \sum_{j=0}^\infty p_{P(\lambda/2)}(j) M_{\chi^2_{n+2j}}(\theta) \\
&= \sum_{j=0}^\infty e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} \times (1-2\theta)^{-\frac{1}{2}n+j} \\
&= (1-2\theta)^{-\frac{1}{2}n} e^{-\lambda/2} \sum_{j=0}^\infty \frac{[\lambda(1-2\theta)^{-1}/2]^j}{j!} \\
&= (1-2\theta)^{-\frac{1}{2}n} e^{-\lambda/2+\lambda(1-2\theta)^{-1}/2} \\
&= (1-2\theta)^{-\frac{1}{2}n} e^{\lambda\theta(1-2\theta)^{-1}}
\end{aligned}
$$

i.e.,

$$M_{\chi_n^2(\lambda)}(\theta) = (1-2\theta)^{-\frac{1}{2}n} e^{\lambda\theta(1-2\theta)^{-1}}$$

The transition mgf of the three parameter square root process is then

$$M_{X(T)|X(t_0)}(\theta) = \left(1-\tfrac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^{-\frac{2\alpha\mu}{\sigma^2}} \exp\left(\frac{x_0 e^{-\alpha(T-t_0)}\theta}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right)$$

and that of the two parameter square root is obtained by setting/constraining $2\alpha\mu/\sigma^2 = 1/2$ so that the gamma part of the (scaled) non-central chi-square distribution has a single degree of freedom.

Formation of the Forward and Backward Equations of the Square-Root Process

A. Formation of the Forward Equation

Differentiating the mgf partially with respect to the forward time variable gives the forward equation in the mgf

$$
\begin{aligned}
\frac{\partial}{\partial T} M_{X(T)|X(t_0)}(\theta) &= \frac{2\alpha\mu}{\sigma^2} \tfrac{1}{2}\alpha\theta\sigma^2 e^{-\alpha(T-t_0)} \left(1-\tfrac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^{-\frac{2\alpha\mu}{\sigma^2}-1} \exp\left(\frac{x_0 e^{-\alpha(T-t_0)}\theta}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right) \\
&\quad - \left(\frac{\alpha x_0 e^{-\alpha(T-t_0)}\theta}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} - \frac{\frac{1}{2}\theta^2\sigma^2 x_0 e^{-2\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}\right) M_{X(T)|X(t_0)}(\theta) \\
&= \left((\alpha\mu-\alpha x_0)\frac{\theta e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} + \tfrac{1}{2}\sigma^2 x_0 \frac{\theta^2 e^{-2\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}\right) M_{X(T)|X(t_0)}(\theta) \\
&= \mu\frac{\alpha\theta e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} + \frac{x_0 e^{-\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}\left(-\alpha\theta+\tfrac{1}{2}\theta^2\sigma^2\right) M_{X(T)|X(t_0)}(\theta)
\end{aligned}
$$

Differentiating partially with respect to the mgf parameter

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$$\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)$$
$$=\mu(1-e^{-\alpha(T-t_0)})\left(1-\tfrac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^{-\frac{2\alpha\mu}{\sigma^2}-1}\exp\left(\frac{x_0e^{-\alpha(T-t_0)}}{1/\theta-\frac{1}{2}\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right)$$
$$+\frac{1}{\theta^2}\frac{x_0e^{-\alpha(T-t_0)}}{\left(1/\theta-\frac{1}{2}\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}\left(1-\tfrac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^{-\frac{2\alpha\mu}{\sigma^2}}\exp\left(\frac{x_0e^{-\alpha(T-t_0)}}{1/\theta-\frac{1}{2}\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right)$$
$$=\frac{\mu(1-e^{-\alpha(T-t_0)})}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)}M_{X(T)|X(t_0)}(\theta)+\frac{x_0e^{-\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}M_{X(T)|X(t_0)}(\theta)$$

so that

$$\frac{x_0e^{-\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}M_{X(T)|X(t_0)}(\theta)=\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)-\frac{(1-e^{-\alpha(T-t_0)})}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)}\mu M_{X(T)|X(t_0)}(\theta)$$

The time derivative can be written as

$$\frac{\partial}{\partial T}M_{X(T)|X(t_0)}(\theta)$$
$$=\mu\left(\frac{\alpha\theta e^{-\alpha(T-t_0)}-\left(-\alpha\theta+\frac{1}{2}\theta^2\sigma^2\right)(1-e^{-\alpha(T-t_0)})}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)}\right)M_{X(T)|X(t_0)}(\theta)+\left(-\alpha\theta+\tfrac{1}{2}\theta^2\sigma^2\right)\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)$$
$$=\mu\left(\frac{\alpha\theta e^{-\alpha(T-t_0)}+\alpha\theta(1-e^{-\alpha(T-t_0)})-\frac{1}{2}\theta^2\sigma^2(1-e^{-\alpha(T-t_0)})}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)}\right)M_{X(T)|X(t_0)}(\theta)+\left(-\alpha\theta+\tfrac{1}{2}\theta^2\sigma^2\right)\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)$$
$$=\alpha\mu\left(\frac{\theta-\frac{1}{2}\theta^2\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)}\right)M_{X(T)|X(t_0)}(\theta)+\left(-\alpha\theta+\tfrac{1}{2}\theta^2\sigma^2\right)\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)$$

i.e.,

$$\frac{\partial}{\partial T}M_{X(T)|X(t_0)}(\theta)=\alpha\mu\theta M_{X(T)|X(t_0)}(\theta)+\left(-\alpha\theta+\tfrac{1}{2}\theta^2\sigma^2\right)\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)$$

Differentiating the mgf partially once with respect to forward diffusion variable, then once with respect to forward diffusion variable and once with respect to the mgf parameter, and then twice with respect to forward diffusion variable and once with respect to the mgf parameter ,i.e.,

$$\frac{\partial}{\partial x}M_{X(T)|X(t_0)}(\theta)\quad\text{and}\quad\frac{\partial^2}{\partial x\partial\theta}M_{X(T)|X(t_0)}(\theta)\text{ and }\quad\frac{\partial^3}{\partial x^2\partial\theta}M_{X(T)|X(t_0)}(\theta)$$

for

(1) $\theta M_{X(T)|X(t_0)}(\theta)=-\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial}{\partial x}f_{X(T)|X(t_0)}(x\,|\,x_0)dx$

(2) $\theta\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)=-\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial}{\partial x}x\,f_{X(T)|X(t_0)}(x\,|\,x_0)dx$

(3) $\theta^2\frac{\partial}{\partial\theta}M_{X(T)|X(t_0)}(\theta)=\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial^2}{\partial x^2}x\,f_{X(T)|X(t_0)}(x\,|\,x_0)dx$

so that the forward equation in the mgf can then be written in terms of the transition density as

$$\frac{\partial}{\partial T}\int_{-\infty}^{\infty}e^{\theta x}f_{X(T)|X(t_0)}(x\,|\,x_0)dx=-\alpha\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial}{\partial x}(\mu-x)\,f_{X(T)|X(t_0)}(x\,|\,x_0)dx+\tfrac{1}{2}\sigma^2\int_{-\infty}^{\infty}e^{\theta x}\frac{\partial^2}{\partial x^2}x\,f_{X(T)|X(t_0)}(x\,|\,x_0)dx$$

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Inverting the mgf results in the forward equation of the three parameter square root process in the transition density function

$$\frac{\partial}{\partial T} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right) - \alpha \frac{\partial}{\partial x}\left((\mu - x)\, f_{X(T)|X(t_0)}(x \mid x_0)\right)$$

In differentially factorized form, this is expressible as

$$\frac{\partial}{\partial t} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(x^{\alpha\mu/\frac{1}{2}\sigma^2} e^{-\alpha x/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x} x^{-\alpha\mu/\frac{1}{2}\sigma^2} e^{\alpha x/\frac{1}{2}\sigma^2} \left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right)\right)$$

B. Formation of the Backward Equation

Differentiating the mgf partially with respect to the backward time variable

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = -\frac{2\alpha\mu}{\sigma^2} \tfrac{1}{2}\theta\sigma^2 e^{-\alpha(T-t_0)} \left(1 - \tfrac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^{-\frac{2\alpha\mu}{\sigma^2}-1} \exp\left(\frac{x_0 e^{-\alpha(T-t_0)}\theta}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right)$$

$$+\left(\frac{\alpha x_0 e^{-\alpha(T-t_0)}\theta}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} - \frac{\frac{1}{2}\theta^2\sigma^2 x_0 e^{-2\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}\right) M_{X(T)|X(t_0)}(\theta)$$

$$= \left((\alpha x_0 - \alpha\mu)\frac{\theta e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha} - \tfrac{1}{2}\sigma^2 x_0 \frac{\theta^2 e^{-2\alpha(T-t_0)}}{\left(1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha\right)^2}\right) M_{X(T)|X(t_0)}(\theta)$$

Differentiating partially with respect to the backward diffusion variable

$$\frac{\partial}{\partial x_0} M_{X(T)|X(t_0)}(\theta) = \left(\frac{e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right) M_{X(T)|X(t_0)}(\theta)$$

and again

$$\frac{\partial^2}{\partial x_0^2} M_{X(T)|X(t_0)}(\theta) = \left(\frac{e^{-\alpha(T-t_0)}}{1-\frac{1}{2}\theta\sigma^2(1-e^{-\alpha(T-t_0)})/\alpha}\right)^2 M_{X(T)|X(t_0)}(\theta)$$

Hence, the time derivative can be written as

$$\frac{\partial}{\partial t_0} M_{X(T)|X(t_0)}(\theta) = \left(-\alpha(\mu - x_0)\frac{\partial}{\partial x_0} - \tfrac{1}{2}\sigma^2 x_0 \frac{\partial}{\partial x_0^2}\right) M_{X(T)|X(t_0)}(\theta)$$

which results the backward equation of the square-root process in the transition density upon inversion of the mgf

$$\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = -\tfrac{1}{2}\sigma^2 x_0 \frac{\partial}{\partial x_0^2} f_{X(T)|X(t_0)}(x \mid x_0) - \alpha(\mu - x_0)\frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0)$$

or in differentially factorized form

$$-\frac{\partial}{\partial t_0} f_{X(T)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 x_0^{1-\alpha\mu/\frac{1}{2}\sigma^2} e^{\alpha x_0/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x_0}\left(x_0^{\alpha\mu/\frac{1}{2}\sigma^2} e^{-\alpha x_0/\frac{1}{2}\sigma^2} \frac{\partial}{\partial x_0} f_{X(T)|X(t_0)}(x \mid x_0)\right)$$

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Solution to the Forward Equation for the Square-Root Process: (by Transforming to the Standard Heat Equation)

Like the integral solution, this is not possible except for the special case of the two-parameter square-root whereby the convolution index is unitary, $v = 4\alpha\mu/\sigma^2 = 1$. This special case is again considered because of the relevance to the material in BOOK 3.

Solving the Forward Equation of the Two-Parameter Square-Root Diffusion: (by Transforming to the Standard Heat Equation)

Considering only the two-parameter square-root diffusion since this is the only square-root process that is transformable to the heat equation

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) = 2\sigma^2 \frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)|X(t_0)}(x \mid x_0)\right) - 2\alpha \frac{\partial}{\partial x}\left((\sigma^2/2\alpha - x)\, f_{X(t)|X(t_0)}(x \mid x_0)\right)$$

or with the shorthand

$$\frac{\partial}{\partial t} f_{X(t)}(x) = 2\sigma^2 \frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)}(x)\right) - 2\alpha \frac{\partial}{\partial x}\left((\sigma^2/2\alpha - x)\, f_{X(t)}(x)\right)$$

Transforming the diffusion variable to $Y(t) = \sqrt{X(t)}$. The Jacobian of the transformation is $dx/dy = 2$ and the transition density function of the transformed variable is

$$f_{X(t)}(y) = \frac{1}{2y} f_{Y(t)}(y)$$

The relevant partial derivatives of the forward equation are

(1) $$\frac{\partial}{\partial t} f_{X(t)}(x) = \frac{1}{2y}\frac{\partial}{\partial t} f_{Y(t)}(x)$$

(2) $$\frac{\partial}{\partial x} f_{X(t)}(x) = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left(\frac{1}{2y} f_{Y(t)}(x)\right)$$
$$= \frac{1}{4y}\frac{\partial}{\partial y}\left(\frac{1}{y} f_{Y(t)}(x)\right)$$

(3) $$\frac{\partial}{\partial x}\left[x\ f_{X(t)}(x)\right] = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left(y^2 \frac{1}{2y} f_{Y(t)}(x)\right)$$
$$= \frac{1}{4y}\frac{\partial}{\partial y}\left(y\, f_{Y(t)}(x)\right)$$

(4) $$\frac{\partial^2}{\partial x^2}\left[x\, f_{X(t)}(x)\right] = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left[\frac{1}{4y}\frac{\partial}{\partial y}\left(y\, f_{Y(t)}(y)\right)\right]$$
$$= \frac{1}{8y}\frac{\partial}{\partial y}\left[\frac{1}{y} f_{Y(t)}(y) + \frac{\partial}{\partial y} f_{Y(t)}(y)\right]$$
$$= \frac{1}{8y}\left[\frac{\partial^2}{\partial y^2} f_{Y(t)}(y) + \frac{\partial}{\partial y}\left(\frac{1}{y} f_{Y(t)}(y)\right)\right]$$

Inserting the partial derivatives into the forward equation gives

$$\frac{1}{2y}\frac{\partial}{\partial t} f_{Y(t)}(x) = \sigma^2 \frac{1}{4y}\left[\frac{\partial^2}{\partial y^2} f_{Y(t)}(y) + \frac{\partial}{\partial y}\left(\frac{1}{y} f_{Y(t)}(y)\right)\right] - \sigma^2 \frac{1}{4y}\frac{\partial}{\partial y}\left(\frac{1}{y} f_{Y(t)}(x)\right) + 2\alpha \frac{1}{4y}\frac{\partial}{\partial y}\left(y\, f_{Y(t)}(x)\right)$$

and canceling the Jacobian factor on both sides of the equation as well as the terms

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$$\tfrac{1}{2}\sigma^2 \frac{\partial}{\partial y}\left(\frac{1}{y} f_{Y(t)}(x)\right)$$

which only cancel for the two-parameter square-root diffusion (and not the three-parameter square-root), leaves the forward equation of a zero basis mean Ornstein-Uhlenbeck process

$$\frac{\partial}{\partial t} f_{Y(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} f_{Y(t)}(y) + \alpha \frac{\partial}{\partial y}\left(y\, f_{Y(t)}(x)\right)$$

A further change of diffusion variable $Y(t) = e^{\alpha t} X(t)$ reduces this equation to the heat equation with exponential diffusion coefficient and a subsequent time change $s = (e^{2\alpha t} - 1) / 2\alpha$ reduces this to the standard heat equation. The mathematics for these transformations is given in the preceding sections regarding Ornstein-Uhlenbeck diffusions. Hence, it is known that the transition distribution of this equation is normal with mean $y_0 e^{-\alpha t}$ and variance $\sigma^2 (1 - e^{-2\alpha t}) / 2\alpha$,

$$Y(t) \sim N\left(y_0 e^{-\alpha t}, \sigma^2 \frac{1 - e^{-2\alpha t}}{2\alpha}\right)$$

and that of the original diffusion variable, with initial condition $x_0 = y_0^2$, is scaled non-central chi-square with a single degree of freedom and non-centrality parameter $x_0 e^{-2\alpha t} / [\sigma^2 (1 - e^{-2\alpha t}) / 2\alpha]$,

$$X_t \sim \sigma^2\left(\frac{1 - e^{-2\alpha t}}{2\alpha}\right)\chi_1^2\left(\frac{x_0 e^{-2\alpha t}}{\sigma^2 (1 - e^{-2\alpha t}) / 2\alpha}\right)$$

Solution to the Forward Equation for the Three-Parameter Square-Root Process: (Solving the Equation Directly)

The forward equation of the three-parameter square-root diffusion is

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)|X(t_0)}(x \mid x_0)\right) - \alpha \frac{\partial}{\partial x}\left((\mu - x)\, f_{X(t)|X(t_0)}(x \mid x_0)\right)$$

Transforming the forward equation to a differential equation in the moment generating function

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) dx = \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)|X(t_0)}(x \mid x_0)\right) - \alpha \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x}\left((\mu - x)\, f_{X(t)|X(t_0)}(x \mid x_0)\right)$$

From the results regarding converting derivatives in the forward diffusion to functions of the moment generating function and the moment generating function parameter with integration by parts, the equation reduces to the partial differential equation

$$\begin{aligned}\frac{\partial}{\partial t} M_{X(t)}(\theta) &= \tfrac{1}{2}\sigma^2\theta^2 \frac{\partial}{\partial \theta} M_{X(t)}(\theta) - \alpha\mu\theta M_{X(t)}(\theta) + \alpha\theta \frac{\partial}{\partial \theta} M_{X(t)}(\theta) \\ &= (\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta} M_{X(t)}(\theta) - \alpha\mu\theta M_{X(t)}(\theta)\end{aligned}$$

subject to the initial condition of an mgf of a Dirac singularity

$$M_{X(0)}(\theta) = \int_{-\infty}^{\infty} e^{x\theta} f_{X(0)}(x) dx = \int_{-\infty}^{\infty} e^{x\theta} \delta_{x_0}(x) dx = e^{x_0\theta}$$

the moment generating function of the Dirac singularity at the start of the diffusion. Dividing through by the dependent variable, the moment generating function, gives the forward equation of the cumulant generating function

$$\frac{\partial}{\partial t} \log M_{X(t)}(\theta) = (\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta} \log M_{X(t)}(\theta) - \alpha\mu\theta$$

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which is a partial differential equation and is identical to that of the simpler process that mean reverts to zero with the addition of a linear term in the independent variable of the transition distribution moment generating function parameter. Utilizing the Laplace transform of the cgf $\lambda(\theta;s) = L\{\log M_{X(t)}(\theta)\}$ to remove the time derivative

$$\frac{\partial}{\partial t} L\{\log M_{X(t)}(\theta)\} = (\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta} L\{\log M_{X(t)}(\theta)\} + L\{\alpha\mu\theta\}$$

or

$$s\lambda(\theta;s) - \log M_{X(0)}(\theta) = (\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta}\lambda(\theta;s) + \frac{\alpha\mu\theta}{s}$$

that is,

$$(\tfrac{1}{2}\sigma^2\theta^2 - \alpha\theta)\frac{\partial}{\partial \theta}\lambda(\theta;s) - s\lambda(\theta;s) = -x_0\theta - \frac{\alpha\mu\theta}{s}$$

which is a linear first order ordinary differential equation. The integrating factor is identical to that of the eccentric square-root although the equation is now both linear and harmonic in the Laplace parameter. The differential becomes, after applying the integrating factor,

$$\frac{\partial}{\partial\theta}\left[\lambda(\theta;s)\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)^{s/\mu}\right] = -\frac{x_0 + \alpha\mu/s}{\frac{1}{2}\sigma^2\theta-\alpha}\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)^{s/\mu}$$

integrating

$$\lambda(\theta;s)\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)^{s/\mu} = -\left(x_0 + \frac{\alpha\mu}{s}\right)\int \frac{1}{\frac{1}{2}\sigma^2\theta-\alpha}\left(\frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}\right)^{s/\mu} d\theta$$

and again introducing the substitution $y = \dfrac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta-\alpha}$ simplifies the integral equation to

$$\lambda(\theta;s)y^{s/\mu} = -\frac{1}{\frac{1}{2}\sigma^2}\left(x_0 + \frac{\alpha\mu}{s}\right)\int y^{s/\mu}(1-y)^{-1}d\theta$$

Evaluating the integral as a power series, from the previous section,

$$\lambda(\theta;s)y^{s/\mu} = -\frac{1}{\frac{1}{2}\sigma^2}\left(x_0 + \frac{\alpha\mu}{s}\right)\left(\frac{y^{s/\alpha+1}}{s/\alpha+1} + \frac{y^{s/\alpha+2}}{s/\alpha+2} + \frac{y^{s/\alpha+3}}{s/\alpha+3} + \frac{y^{s/\alpha+4}}{s/\alpha+4} + \ldots\right)$$

and simplifying

$$\lambda(\theta;s) = -\frac{\alpha}{\frac{1}{2}\sigma^2}\left(x_0 + \frac{\alpha\mu}{s}\right)\left(\frac{y}{s+\alpha} + \frac{y^2}{s+2\alpha} + \frac{y^3}{s+3\alpha} + \frac{y^4}{s+4\alpha} + \ldots\right)$$

Splitting the infinite series into two, and applying term by term partial fractions to the second infinite series

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$$\lambda(\theta;s) = -\frac{\alpha}{\frac{1}{2}\sigma^2}\left(x_0 + \frac{\alpha\mu}{s}\right)\left(\frac{y}{s+\alpha} + \frac{y^2}{s+2\alpha} + \frac{y^3}{s+3\alpha} + \frac{y^4}{s+4\alpha} + \ldots\right)$$

$$= -\frac{\alpha x_0}{\frac{1}{2}\sigma^2}\left(\frac{y}{s+\alpha} + \frac{y^2}{s+2\alpha} + \frac{y^3}{s+3\alpha} + \frac{y^4}{s+4\alpha} + \ldots\right)$$

$$-\frac{\alpha^2\mu}{\frac{1}{2}\sigma^2 s}\left(\frac{y}{s+\alpha} + \frac{y^2}{s+2\alpha} + \frac{y^3}{s+3\alpha} + \frac{y^4}{s+4\alpha} + \ldots\right)$$

$$= -\frac{\alpha x_0}{\frac{1}{2}\sigma^2}\left(\frac{y}{s+\alpha} + \frac{y^2}{s+2\alpha} + \frac{y^3}{s+3\alpha} + \frac{y^4}{s+4\alpha} + \ldots\right)$$

$$-\frac{\alpha\mu}{\frac{1}{2}\sigma^2}\left(\frac{y}{1}\left[\frac{1}{s} - \frac{1}{s+\alpha}\right] + \frac{y^2}{2}\left[\frac{1}{s} - \frac{1}{s+2\alpha}\right] + \frac{y^3}{3}\left[\frac{1}{s} - \frac{1}{s+3\alpha}\right] + \frac{y^4}{4}\left[\frac{1}{s} - \frac{1}{s+4\alpha}\right] + \ldots\right)$$

Applying the inverse Laplace transform and recovering the time dimension and the log of the moment generating function

$$\log M_{X(t)}(\theta) = L^{-1}\{\lambda(\theta;s)\}$$

$$= -\frac{\alpha x_0}{\frac{1}{2}\sigma^2}\left(ye^{-\alpha t} + y^2e^{-2\alpha t} + y^3e^{-3\alpha t} + y^4e^{-4\alpha t} + \ldots -\right)$$

$$-\frac{\alpha\mu}{\frac{1}{2}\sigma^2}\left((y - ye^{-\alpha t}) + \frac{1}{2}(y^2 - y^2e^{-2\alpha t}) + \frac{1}{3}(y^3 - y^3e^{-3\alpha t}) + \frac{1}{4}(y^4 - y^4e^{-4\alpha t}) + \ldots -\right)$$

The first series is in geometric progression while the second is the difference of two logarithmic power series

$$-\log(1-x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \ldots$$

and hence the combined series is summable. The series simplifies, upon summation, to

$$\log M_{X(t)}(\theta) = -\frac{\alpha x_0}{\frac{1}{2}\sigma^2}ye^{-\alpha t}(1 - ye^{-\alpha t})^{-1} + \frac{\alpha\mu}{\frac{1}{2}\sigma^2}\left(-\log(1-y) + \log(1 - ye^{-\alpha t})\right)$$

Reintroducing the moment generating function parameter from the substitution

$$y = \frac{\frac{1}{2}\sigma^2\theta}{\frac{1}{2}\sigma^2\theta - \alpha}$$

gives a cumulant generating function of

$$\log M_{X(t)}(\theta) = \frac{x_0e^{-\alpha t}\theta}{1 - \frac{\sigma^2}{2\alpha}(1 - e^{-\alpha t})\theta} - \frac{2\alpha\mu}{\sigma^2}\log\left[1 - \frac{\sigma^2}{2\alpha}(1 - e^{-\alpha t})\theta\right]$$

and the moment generating function is then

$$M_{X(t)}(\theta) = \left[1 - \frac{\sigma^2}{2\alpha}(1 - e^{-\alpha t})\theta\right]^{-\frac{2\alpha\mu}{\sigma^2}}\exp\left(x_0e^{-\alpha t}\theta\left[1 - \theta\sigma^2(1 - e^{-\alpha t})/2\alpha\right]^{-1}\right)$$

which is that of a scaled non-central chi-square distribution with $4\alpha\mu/\sigma^2$ degrees of freedom and non-centrality parameter $x_0e^{-\alpha t}$. The scaling factor is again a variance scaling of $\sigma^2(1 - e^{-\alpha t})/4\alpha$. The transition distribution is

$$X(t) \sim k\,\chi^2_{4\alpha\mu/\sigma^2}(x_0e^{-\alpha t}/k) \qquad \text{where} \qquad k = \frac{\sigma^2}{4\alpha}(1 - e^{-\alpha t})$$

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The mean of this distribution is

$$k\left(\frac{4am}{\sigma^2}+\frac{x_0e^{-\alpha t}}{k}\right)=x_0e^{-\alpha t}+\mu(1-e^{-\alpha t})$$

which is the mean of an Ornstein-Uhlenbeck process. No moments or cumulants of higher order of the square-root process coincide with those of an Ornstein-Uhlenbeck process. The non-central chi-square distribution can be interpreted in a number of ways. Here, the distribution will firstly be interpreted as a standard chi-square distribution with a Poisson distributed parameter. Letting v be the number of degrees of freedom, $v = 4\alpha\mu / \sigma^2$ and λ be the non- centrality parameter, $\lambda = x_0e^{-\alpha t} / [\sigma^2 (1 - e^{-\alpha t}) / 4\alpha]$ and $k = \sigma^2 (1 - e^{-\alpha t}) / 4\alpha$ be the scale factor,

$$\begin{aligned}X(t) &\sim k\,\chi_v^2(\lambda)\\ &\sim k\,\chi_{v+2Y}^2\\ &\sim \Gamma(v/2+Y,2k)\\ &\sim \Gamma(v/2,2k)+\Gamma(Y,2k)\\ &\sim k\,\chi_v^2+k\,\chi_0^2(\lambda)\end{aligned}$$

Alternatively

$$\begin{aligned}X(t) &\sim \Gamma(v/2,2k)+\sum_{j=1}^{N(t)}X_j\\ &\sim \Gamma(v/2,2k)+\sum_{j=1}^{P(\lambda/2)}\Gamma(1,2k)\\ &\sim \Gamma(v/2,2k)+\Gamma\left(P(\lambda/2),2k\right)\\ &\sim k\,\chi_v^2+k\,\chi_0^2(\lambda)\end{aligned}$$

where and $\chi_{(\cdot)}$ is a central chi-square and Y is Poisson distributed with mean rate of one half the non-centrality parameter, $Y \sim P(\lambda / 2)$. The conditional distribution of $X(t)$ given $Y = y$ is

$$f_{X(t)|Y=y}(x\,|\,y)=\frac{x^{v/2+y-1}\exp(-x/2k)}{\Gamma(v/2+y)(2k)^{v/2+y}}$$

$X(t)$ has the density function[26], unconditionally after summing out the Poisson distributed parameter for the degrees of freedom,

$$\begin{aligned}f_{X(t)}(x) &= \sum_{y=0}^{\infty}\Pr(Y=y)\times f_{X(t)|Y=y}(x\,|\,y)\\ &= \sum_{y=0}^{\infty}\exp(-\lambda/2)\frac{(\lambda/2)^y}{y!}\times\frac{x^{v/2+y-1}\exp(-x/2k)}{\Gamma(v/2+y)(2k)^{v/2+y}}\end{aligned}$$

The infinite sum can be expressed in terms of an ordinary modified Bessel function of the first kind, noting that

[26] Johnson, N. L., Kotz, S. K. and Balakrishnan, N., (1994) *Continuous Univariate Distributions, Volume 2, 2nd Edition*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons.

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$$
\begin{aligned}
f_{\chi_n^2(\lambda)}(y) &= \sum_{j=0}^{\infty} \; p_{P(\lambda/2)}(j)\, f_{\chi^2_{n+sj}}(y) \\
&= \sum_{j=0}^{\infty} \; e^{-\lambda/2}\frac{(\lambda/2)^j}{j!} \times \frac{y^{n/2+j-1}e^{-y/2}}{2^{n/2+j}\Gamma(n/2+j)} \\
&= \tfrac{1}{2}e^{-\frac{1}{2}(\lambda+y)}\sum_{j=0}^{\infty} \; \frac{(\lambda/2)^j}{j!} \times \frac{(y/2)^{n/2+j-1}}{(n/2+j-1)!} \\
&= \tfrac{1}{2}e^{-\frac{1}{2}(\lambda+y)}(\sqrt{y/\lambda})^{\frac{1}{2}n-1}\sum_{j=0}^{\infty} \; \frac{(\sqrt{\lambda y}/2)^j}{j!} \times \frac{(\sqrt{\lambda y}/2)^{j+\frac{1}{2}n-1}}{(n/2+j-1)!} \\
&= \tfrac{1}{2}e^{-\frac{1}{2}(\lambda+y)}(\sqrt{y/\lambda})^{\frac{1}{2}n-1} I_{\frac{1}{2}n-1}(\sqrt{\lambda y})
\end{aligned}
$$

From the scaling relation of density functions, $f_Y(y) = k^{-1}DF_X(y/k) = k^{-1}f_X(y/k)$ for the standard scaling transform $Y = kX$, or for that of a scaled non-central chi-square distribution, $X(t) \sim k\chi_n^2(\lambda)$, the transition distribution can be written as

$$
\begin{aligned}
f_{X(t)}(x) &= \sum_{y=0}^{\infty} \Pr(Y=y) \times f_{X(t)|Y=y}(x \mid y) \\
&= \frac{1}{2k}\left(\frac{\sqrt{x}}{\sqrt{\lambda k}}\right)^{\nu/2-1} \exp\left(-\frac{\lambda + x/k}{2}\right) I_{\nu/2-1}\left(\frac{\sqrt{x}}{\sqrt{k/\lambda}}\right)
\end{aligned}
$$

where $\nu = 4\alpha\mu/\sigma^2$ and $\lambda = \dfrac{x_0 e^{-\alpha t}}{\sigma^2(1-e^{-\alpha t})/4\alpha}$ and $k = \dfrac{\sigma^2}{4\alpha}(1-e^{-\alpha t})$ and

$$
I_v(x) = \sum_{y=0}^{\infty} \; \frac{(x/2)^y}{y!} \times \frac{(x/2)^{y+v}}{(y+v)!}
$$

is the modified Bessel function of the first kind with the parameterization.

Steady State (Ergodic), Long Run, Distribution of a Recurrent Square-Root Diffusion

Zeroing the time side of the forward equation leaves another form of Laguerre's (associated) differential equation, solutions for which relate to the gamma density function, now with non-zero shape parameter,

$$
0 = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x\, f_{X(t)|X(t_0)}(x \mid x_0)\right) - \alpha \frac{\partial}{\partial x}\left((\mu - x)\, f_{X(t)|X(t_0)}(x \mid x_0)\right)
$$

In differentially factorized form

$$
\tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(x^{\,k\mu}e^{-kx}\frac{\partial}{\partial x}x^{-k\mu}e^{kx}\left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right)\right) = 0 \qquad \text{where } k = \frac{\alpha}{\frac{1}{2}\sigma^2}
$$

Integrating

$$
x^{\,k\mu}e^{-kx}\frac{\partial}{\partial x}x^{-k\mu}e^{kx}\left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right) = a_0
$$

or

$$
\frac{\partial}{\partial x}x^{-k\mu}e^{kx}\left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right) = a_0 x^{-k\mu}e^{kx}
$$

and integrating again

$$
x^{-k\mu}e^{kx}\left(x\, f_{X(T)|X(t_0)}(x \mid x_0)\right) = a_1 + a_0 \int \; x^{-k\mu}e^{kx}dx
$$

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or

$$f_{X(T)|X(t_0)}(x \mid x_0) = a_1 x^{k\mu-1} e^{-kx} + a_0 x^{k\mu-1} e^{-kx} \int x^{-k\mu} e^{kx} dx$$

The second term is divergent so the coefficient must be zero, $a_0 = 0$. This leaves

$$f_{X(T)|X(t_0)}(x \mid x_0) = a_1 x^{k\mu-1} e^{-kx}$$

which is proportional to a gamma density with shape parameter $k\mu = \alpha\mu / \frac{1}{2}\sigma^2$ and scale $1/k = \frac{1}{2}\sigma^2 / \alpha$

$$\begin{aligned} X(t) &\sim \Gamma\left(\frac{\alpha\mu}{\frac{1}{2}\sigma^2}, \frac{\frac{1}{2}\sigma^2}{\alpha} \right) \\ &\sim \frac{\sigma^2}{4\alpha} \Gamma\left(\frac{4\alpha\mu}{\sigma^2} / 2, 2 \right) \\ &\sim \frac{\sigma^2}{4\alpha} \chi^2_{4\alpha\mu/\sigma^2} \end{aligned}$$

The integral of the kernel is

$$a_1 \int_0^\infty x^{k\mu-1} e^{-kx} dx = k^{-k\mu} \Gamma(k\mu)$$

Hence $a_1 = k^{k\mu} / \Gamma(k\mu)$ and the density that solves the steady state equation is

$$f_{X(T)|X(t_0)}(x \mid x_0) = \frac{k^{k\mu} x^{k\mu-1} e^{-kx}}{\Gamma(k\mu)}$$

which is the density of additive gamma, or (scaled) central chi-square, Hilbert space (gamma white noise). Alternatively, referring to the moment generating function of the transition distribution and observing that as time t becomes large the non-centrality parameter vanishes and the mgf converges to that of a scaled central chi-square, i.e., the mgf of a gamma distribution

$$\lim_{t\to\infty} M_{X(t)}(\theta) = \left(1 - \theta \frac{\sigma^2}{2\alpha} \right)^{-\frac{2\alpha\mu}{\sigma^2}}$$

This inverts to the same density as that obtained by solving the ordinary differential equation of the steady state forward equation. This is an example of inferring results from applied mathematics, such as solving ordinary differential equations and partial differential equations, by reasoning from mathematical statistics.

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The Transition Distribution

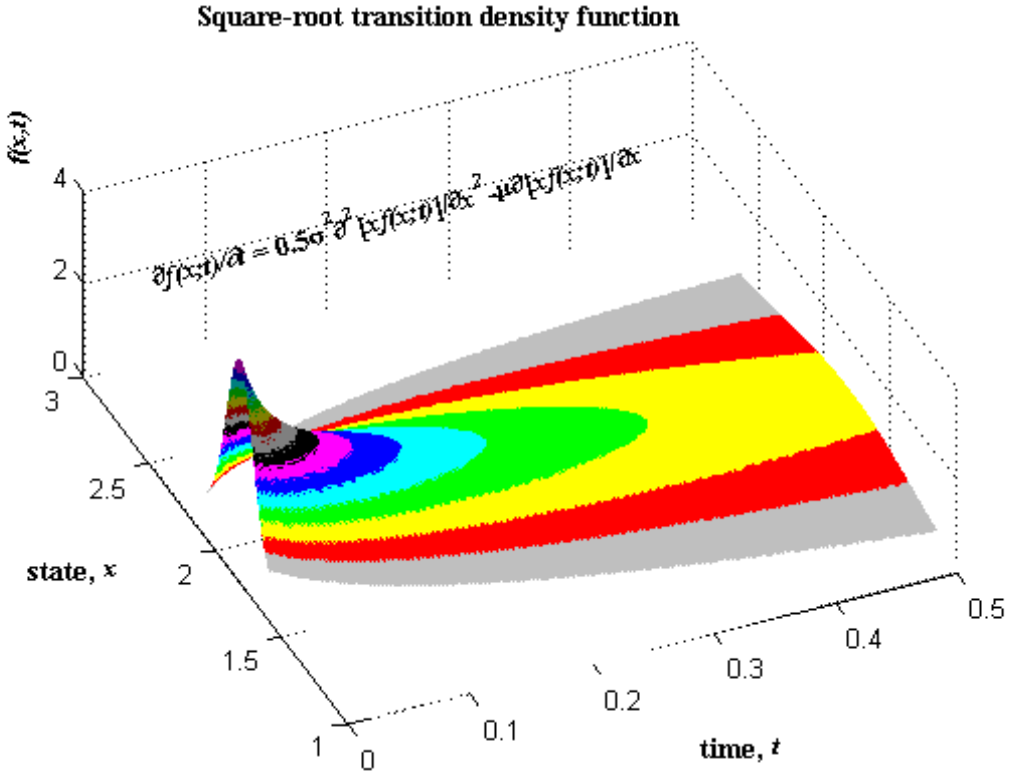


Figure 23: Square-root process transition density surface with parameters (α,μ,σ) = (0.10,0.50,0.50) and initial value of 2.

When the basis variance zeroes $\sigma \to 0$, this surface becomes a Dirac singularity traced out over the exponentially decayed initial value and integrated moderated basis mean $X(t) \to x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t})$ for which the transition density is

$$f_{X(t)|X(0)}(x \mid x_0) = \delta_{x_0 e^{-\alpha t} + \mu(1-e^{-\alpha t})}(x)$$

which coincides exactly with that of proportional Brownian motion (the Ornstein-Uhlenbeck process) since the transition means are the same.

BESSEL PROCESS – RADIUS OF A MULTIVARIATE ORNSTEIN-UHLENBECK DIFFUSION

Introduction

The Bessel diffusion is the positive square-root of the sum of squares of *n* recurrent independent Ornstein-Uhlenbeck processes that mean revert to zero. This is the distance from the origin of an *n* dimensional multivariate Ornstein-Uhlenbeck process. As long as the marginal Brownian motions are recurrent so too is their radius. The Bessel process is therefore recurrent given this condition. The infinitesimal parameters of the diffusion are $\mu(x,t) = \alpha(\mu/x - x)$ and $\sigma(x,t) = \sigma$. The differential increment of the process is

$$dX_t = \alpha\left(\frac{\mu}{X_t} - X_t\right)dt + \sigma dB_t$$

which has a single differential Brownian term which forms the Gaussian diffusion basis. The Bessel process has the property of **(1)** mean aversion from, and **(2)** mean reversion to, zero.

- As the variable X_t becomes large, the term $\alpha\mu / X_t$ becomes small while the reversionary term X_t becomes large pulling the variable back to $\alpha\mu / X_t$ (which is near zero), more strongly.

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- As the variable becomes small, the term $\alpha\mu / X_t$ becomes large and the reversionary term X_t becomes small (weak reversion) resulting in the variable diverging away from zero more strongly.

Therefore, there is an interplay between aversion from, and reversion to, zero. Alternatively, there is a reversionary force towards a hyperbolic function of the variable, which itself creates a reversionary force away from the linear reversionary function of the variable. This is what could reasonably be expected of a radius of a recurrent multidimensional diffusion.

The diffusion basis term is independent of the size of the diffusion which suggests that the process is Gaussian. As the transition distribution is the square-root of a non-central chi-square variable, it is not Gaussian and this is saying that the reciprocal term in the deterministic part of the stochastic differential, and how the forward process depends on the current value (which is the backward variable), changes the nature of the transition probability. A single standard Brownian motion differential term added to some deterministic function of a variable does not necessarily remain a Gaussian process when the deterministic function is not linear.

This process is not transformable to standard Brownian motion and does not form an ito-transform pair. Consequently, the stochastic differential equation does not have an explicit integral solution and the forward equation cannot be transformed to the standard heat equation. It is transformable, instead, to the forward equation of the three-parameter square-root diffusion.

Bessel process Forward and Backward Equations

The forward equation is

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{X(t)|X(t_0)}(x \mid x_0) - \alpha \frac{\partial}{\partial x}\left[\left(\frac{\mu}{x} - x\right) f_{X(t)|X(t_0)}(x \mid x_0)\right]$$

and in differentially factorized form

$$\frac{\partial}{\partial t} f_{X(t)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x}\left(x^{2\alpha\mu/\sigma^2} e^{-\alpha x^2/\sigma^2} \frac{\partial}{\partial x} x^{-2\alpha\mu/\sigma^2} e^{\alpha x^2/\sigma^2} f_{X(t)|X(t_0)}(x \mid x_0)\right)$$

The backward equation is

$$-\frac{\partial}{\partial t_0} f_{X(t)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} f_{X(t)|X(t_0)}(x \mid x_0) + \alpha\left(\frac{\mu}{x_0} - x_0\right)\frac{\partial}{\partial x_0} f_{X(t)|X(t_0)}(x \mid x_0)$$

and in differentially factorized form

$$-\frac{\partial}{\partial t_0} f_{X(t)|X(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 x_0^{-\alpha\mu/\frac{1}{2}\sigma^2} e^{\alpha x_0^2/\sigma^2} \frac{\partial}{\partial x_0}\left(x_0^{\alpha\mu/\frac{1}{2}\sigma^2} e^{-\alpha x_0^2/\sigma^2} \frac{\partial}{\partial x_0} f_{X(t)|X(t_0)}(x \mid x_0)\right)$$

Solution to the Forward Equation for the Bessel Process: (Transforming to the Forward Equation of the Square-Root Process)

Transforming the diffusion variable $X(t)$ to the square, $Y(t) = X(t)^2$; The transformed independent variable is $y = x^2$ giving the Jacobian of the transformation

$$J = \frac{dx}{dy} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$$

The normalized density function of the transformed diffusion variable is

$$f_{Y(t)}(y) = f_{X(t)}(x) \mid J \mid = \frac{1}{2x} f_{X(t)}(x)$$

which implies

$$f_{X(t)}(x) = 2\sqrt{y}\, f_{Y(t)}(y)$$

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as well as the relation

$$(x-\mu)\,f_{X(t)}(x) = (y-\mu)\,f_{Y(t)}(y)$$

The relevant partial derivatives of the transform are

$$\frac{\partial y}{\partial x} = \frac{1}{2\sqrt{y}} \qquad \text{and} \qquad \frac{\partial y}{\partial t} = 0$$

The transformed partial derivatives of the forward equation are

(1) $$\frac{\partial}{\partial x}\left[\left(\frac{\mu}{x}-x\right)f_{X(t)}(x)\right] = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left[\left(\frac{\mu}{\sqrt{y}}-\sqrt{y}\right)2\sqrt{y}\,f_{Y(t)}(y)\right] = 2\sqrt{y}\,\frac{\partial}{\partial y}\left[\left(2\mu-2y\right)f_{Y(t)}(y)\right]$$

$$= 4\sqrt{y}\,\frac{\partial}{\partial y}\left[\left(\mu-y\right)f_{Y(t)}(y)\right]$$

(2) $$\frac{\partial}{\partial x}\,f_{X(t)}(x) = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}2\sqrt{y}\,f_{Y(t)}(y) = 2\sqrt{y}\left(\frac{1}{\sqrt{y}}\,f_{Y(t)}(y) + 2\sqrt{y}\,\frac{\partial}{\partial y}\,f_{Y(t)}(y)\right) = 2\left(\,f_{Y(t)}(y) + 2y\,\frac{\partial}{\partial y}\,f_{Y(t)}(y)\right)$$

$$= 2\left(2\,\frac{\partial}{\partial y}[y\,f_{Y(t)}(y)] - f_{Y(t)}(y)\right)$$

(3) $$\frac{\partial^2}{\partial x^2}\,f_{X(t)}(x) = \frac{\partial y}{\partial x}\frac{\partial}{\partial y}\left[2\left(2\,\frac{\partial}{\partial y}[y\,f_{Y(t)}(y)] - f_{Y(t)}(y)\right)\right] = 4\sqrt{y}\,\frac{\partial}{\partial y}\left(2\,\frac{\partial}{\partial y}[y\,f_{Y(t)}(y)] - f_{Y(t)}(y)\right)$$

$$= 8\sqrt{y}\,\frac{\partial^2}{\partial y^2}[y\,f_{Y(t)}(y)] - 4\sqrt{y}\,\frac{\partial}{\partial y}\,f_{Y(t)}(y)$$

(4) $$\frac{\partial}{\partial t}\,f_{X(t)}(x) = 2\sqrt{y}\,\frac{\partial}{\partial t}\,f_{Y(t)}(y)$$

The forward equation of the transformed diffusion variable becomes

$$2\sqrt{y}\,\frac{\partial}{\partial t}\,f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2 \times 8\sqrt{y}\,\frac{\partial^2}{\partial y^2}[y\,f_{Y(t)}(y)] - \tfrac{1}{2}\sigma^2 \times 4\sqrt{y}\,\frac{\partial}{\partial y}\,f_{Y(t)}(y) - 4\alpha\sqrt{y}\,\frac{\partial}{\partial y}\left[\left(\mu-y\right)f_{Y(t)}(y)\right]$$

canceling the Jacobian factor

$$\frac{\partial}{\partial t}\,f_{Y(t)}(y) = \tfrac{1}{2}\sigma^2\,\frac{\partial^2}{\partial y^2}[4y\,f_{Y(t)}(y)] - \frac{\partial}{\partial y}\left[2\alpha\left(\mu+\tfrac{1}{2}\frac{\sigma^2}{\alpha}-y\right)f_{Y(t)}(y)\right]$$

which is the forward equation of a three-parameter square-root diffusion with infinitesimal parameters $2\alpha(\mu + \frac{1}{2}\sigma^2 / \alpha - y)$ and $4\sigma^2 y$, which has a stochastic differential increment of

$$dY = 2\alpha(\mu + \tfrac{1}{2}\sigma^2/\alpha - y)dt + 2\sigma\sqrt{y}dB$$

The transition distribution of this process was shown to be a scaled non-central chi-square

$$Y(t) = X(t)^2 \sim k\,\chi^2_{4am/s^2}(\sqrt{y_0}e^{-at}/k) \qquad \text{where } k = \frac{s^2}{4a}(1-e^{-at})$$

and $y_0 = x_0^2$, $s = 2\sigma$, $a = 2\alpha$, and $m = \mu + \sigma^2/2\alpha$, the latter of which is the ito corrected mean level of the underlying square-root process. The transition distribution of the original diffusion variable is therefore the positive square-root (since it is a radius of a multivariate Ornstein-Uhlenbeck processes) of a scaled non-central chi-square distribution, which is a statistical distribution with a linear argument in a modified Bessel function of the first kind referred to as a (scaled) non-central chi distribution

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$$\sqrt{Y(t)} = X(t) \sim \sqrt{k}\,\chi_{4am/s^2}(x_0 e^{-at}/k) \qquad \text{where } k = \frac{s^2}{4a}(1-e^{-at})$$

From the positive square-root transformation of the transition cumulative density function of the three-parameter square-root process

$$F_{X(t)}(x) = \Pr(X(t) < x) = \Pr(\sqrt{Y(t)} < x) = \Pr(Y(t) < x^2) = F_{Y(t)}(x^2)$$

and by differentiation,

$$f_{X(t)}(x) = 2x f_{Y(t)}(x^2)$$

that is,

$$f_{\sqrt{k}\sqrt{\chi^2_{4am/s^2}(\lambda)}}(x) = 2x f_{k\chi^2_{4am/s^2}(\lambda)}(x^2)$$

The transition density function is then, like the square-root process, expressible in terms of a Poisson mixture or a Bessel function. The transition density in terms of the Poisson mixture is

$$f_{X(t)}(x) = 2\sum_{y=0}^{\infty} \exp(-\lambda/2)\frac{(\lambda/2)^y}{y!} \times \frac{x^{\nu+2y-1}\exp(-x^2/2k)}{\Gamma(\nu/2+y)(2k)^{\nu/2+y}}$$

and in terms of the modified Bessel function of the first kind,

$$f_{X(t)}(x) = \frac{x}{k}\left(\frac{x}{\sqrt{\lambda k}}\right)^{\nu/2-1} \exp\left(-\frac{\lambda + x^2/k}{2}\right) I_{\nu/2-1}\left(\frac{x}{\sqrt{k/\lambda}}\right)$$

where $\nu = 4am/s^2$, $k = \frac{s^2}{4a}(1-e^{-at})$, and $\lambda = x_0 e^{-at}/k = \frac{4x_0 a e^{-at}}{s^2(1-e^{-at})}$ and

$$I_\nu(x) = \sum_{y=0}^{\infty} \frac{(x/2)^y}{y!} \times \frac{(x/2)^{y+\nu}}{(y+\nu)!}$$

is the modified Bessel function of the first kind.

Steady State (Ergodic), Long Run, Distribution of a Recurrent Bessel Diffusion

Zeroing the time side of the forward equation leaves the square-root transformation of a form of Laguerre's (associated) differential equation, solutions for which relate to a square-root of a gamma density function

$$0 = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{X(t)}(x) - \alpha \frac{\partial}{\partial x}\left[\left(\frac{\mu}{x} - x\right) f_{X(t)}(x)\right]$$

Integrating,

$$a_0 = \tfrac{1}{2}\sigma^2 \frac{\partial}{\partial x} f_{X(t)}(x) - \alpha\left(\frac{\mu}{x} - x\right) f_{X(t)}(x)$$

which is a first order ordinary differential equation. The integrating factor is

$$IF(x) = \exp\left(-\int \frac{\alpha\mu}{\frac{1}{2}\sigma^2 x} + \frac{\alpha x}{\frac{1}{2}\sigma^2} dx\right) = x^{-2\alpha\mu/\sigma^2} e^{\alpha x^2/\sigma^2}$$

Multiplying by the integrating factor and differentially factorizing the equation

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$$\frac{a_0}{\frac{1}{2}\sigma^2} x^{-2\alpha\mu/\sigma^2} e^{\alpha x^2/\sigma^2} = \frac{\partial}{\partial x}\left(x^{-2\alpha\mu/\sigma^2} e^{\alpha x^2/\sigma^2} \quad f_{X(t)}(x) \right)$$

and integrating again

$$x^{-2\alpha\mu/\sigma^2} e^{\alpha x^2/\sigma^2} \; f_{X(t)}(x) = a_1 + a_0 \int \quad \frac{x^{-2\alpha\mu/\sigma^2} e^{\alpha x^2/\sigma^2}}{\frac{1}{2}\sigma^2} dx$$

or

$$f_{X(t)}(x) = a_1 x^{2\alpha\mu/\sigma^2} e^{-\alpha x^2/\sigma^2} + a_0 x^{2\alpha\mu/\sigma^2} e^{-\alpha x^2/\sigma^2} \int \quad \frac{x^{-2\alpha\mu/\sigma^2} e^{\alpha x^2/\sigma^2}}{\frac{1}{2}\sigma^2} dx$$

The second term is divergent so the coefficient must be zero, $a_0 = 0$. This leaves the expression

$$f_{X(t)}(x) = a_1 x^{2\alpha\mu/\sigma^2} e^{-\alpha x^2/\sigma^2}$$

which is proportional to a chi-density[27], the density of the square-root of a chi-square distribution with an arbitrary number of degrees of freedom. Integrating over the support, which is $[0,\infty)$,

$$\int_0^\infty f_{X(t)}(x)dx = a_1 \int_0^\infty x^{2\alpha\mu/\sigma^2} e^{-\alpha x^2/\sigma^2} dx = a_1 \frac{1}{2}\left(\frac{\sigma}{\sqrt{\alpha}}\right)^{\frac{2\alpha\mu}{\sigma^2}} \Gamma\left(\frac{\alpha\mu}{\sigma^2} + \frac{1}{2}\right) = 1$$

so the second constant of integration and the constant of proportionality is

$$a_1 = \frac{2}{\left(\frac{\sigma}{\sqrt{\alpha}}\right)^{\frac{2\alpha\mu}{\sigma^2}} \Gamma\left(\frac{\alpha\mu}{\sigma^2} + \frac{1}{2}\right)}$$

giving the steady state density

$$f_{X(t)}(x) = \frac{2x^{2\alpha\mu/\sigma^2} \exp(-\alpha x^2/\sigma^2)}{\left(\frac{\sigma}{\sqrt{\alpha}}\right)^{\frac{2\alpha\mu}{\sigma^2}} \Gamma\left(\frac{\alpha\mu}{\sigma^2} + \frac{1}{2}\right)}$$

This steady state transition distribution includes the scaled Rayleigh and scaled Maxwell distributions, when $2\alpha\mu/\sigma^2$ takes the values one and two respectively.

[27] Parzen, E., (1960) Modern Probability Theory and its Applications. John Wiley & Sons.

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ITO'S LEMMA FOR TRANSFORMATIONS OF FIRST ORDER DIFFUSION

Introduction

This part of BOOK 1 formally introduces the theory of the Ito calculus which is a particular aspect of the calculus of the random variables of first order normal diffusion. This aspect pertains to the concepts of the Ito series expansion, Ito transformations, and the Ito-transform pair and the convolution Ito-transform pair. Each of these concepts are introduced and discussed. There is an emphasis on distribution theory for the quadratic variation of differential Brownian motion and transformations of differential Brownian motion, and the transition distributions of the various transformations.

Ito Calculus

The Ito calculus is a technique for transforming first order normal diffusion. It is based upon a Taylor-Maclauren series expansion for functions of first order normal diffusion. In particular, the series differential expansion for a transformation Y of a univariate first order normal diffusion X and time t. This series expansion is $dY = Y_X\,dX + \frac{1}{2}Y_{XX}\,(dX)^2 + Y_t\,dt$. At any stage in this discussion it is well worth referring to this equation relating dY to dX and dt. This can also be written as a total derivative peculiar to the calculus of first order normal diffusion[28],

$$\frac{dY}{dt} = \frac{\partial Y}{\partial t}\frac{dt}{dt} + \frac{\partial Y}{\partial X}\frac{dX}{dt} + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}\frac{dX^2}{dt}$$

The method of Ito transforms is a convenient shorthand for avoiding transforming the transition density function in the related forward and backward equations and having to deal with the technicalities of Jacobians and the like, and thus provides the third method of solving the relevant forward equation, as a method of induction from the results of the Ito-calculus. Following transforming the stochastic differential equation, the transition distribution is inferred from that of the inverse transformation of the diffusion being transformed. Since the method relies on induction, it is felt that it is necessary to catalogue in a systematic and standardized fashion each of the processes that form an Ito-transform pair or convolution Ito-transform pair. Ideally, a table should be compiled, as a reference, so that the theory of any one pair can be identified, looked-up and implemented in an applied setting. It is the intention of this section to compile such a table. The table is included at the end of this section, after each is obtained from one or more transformations of standard Brownian motion by the Ito calculus and includes all of the important pairs and convolution pairs. The main processes of this table are those for which have had their forward equations solved for the transition density function, and the transition distribution, in the preceding work of this thesis in FOUNDATIONS I and that regarding the Ito calculus which now follows. I have chosen the terminology Ito-transform pair and convolution Ito-transform pair to keep it consistent with what is termed Ito calculus as described in this chapter regarding Taylor series expansions of first order diffusion variables and with the notion of tables of Fourier and Laplace transform pairs. Such a catalogue of pairs has the potential be as convenient for applied work as these traditional and well established tables have been to mathematicians throughout the course of the last few centuries.

Transformation of the Diffusion Variable

The statistical differential for a drifting Brownian motion is

$$dX = \mu dt + \sigma dB \qquad \text{subject to} \qquad X(t) \sim \delta(x)$$

where $dB \sim N(0,dt)$ and therefore $dX \sim N(\mu dt, \sigma^2 dt)$. Defining a new process Y as a transformation h of Brownian motion X. The transformation of the process Y is then $Y = h(X)$. Expanding the transformation in a Taylor series about the forward increment

$$h(X + dX) = h(X) + \frac{\partial h(X)}{\partial X}dX + \frac{1}{2}\frac{d^2\partial(X)}{\partial X^2}dX^2 + o(dX^2)$$

[28] Which is instrumental in explaining the bias in the Black & Scholes option formulae, as per BOOK 2.

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Subtracting $h(X)$ from both sides gives the total differential of the transformation in a series expansion

$$dh(X) = \frac{\partial h(X)}{\partial X} dX + \frac{1}{2} \frac{\partial^2 h(X)}{\partial X^2} dX^2 + o(dX^2)$$

with $Y = h(X)$ and subscripts denoting partial derivatives,

$$Y_X = \frac{\partial h(X)}{\partial X} \text{ and } Y_{XX} = \frac{\partial^2 h(X)}{\partial^2 X}$$

the shorthand for this expansion is $dY = Y_X dX + \frac{1}{2} Y_{XX} (dX)^2 + o[(dX)^2]$. The second term of the expansion contains the square of the statistical differential of the Brownian motion. Expanding the differential in the square in terms of standard Brownian motion dB gives

$$(dX)^2 = \mu^2 dt^2 + 2\mu\sigma dB dt + \sigma^2 dB^2$$

In the standard calculus of deterministic variables the terms of dt at higher order than one are $o(dt)$ leaving only linear changes on the dt scale, so that curvature and higher order functional behavior is negligible for infinitesimal differentials. Hence, the first $\mu^2 dt^2$ term vanishes for $dt \to 0$. The squares of differential standard Brownian motion dB(t) do not vanish but rather approach dt . Remembering that

$$\begin{aligned} dB(t) &= B(dt) \\ &\sim N(0, dt) \\ &\sim N(0,1)\sqrt{dt} \end{aligned}$$

so that

$$\begin{aligned} \left(dB(t)\right)^2 &= B(dt)^2 \\ &\sim N(0, dt)^2 \\ &\sim N(0,1)^2 dt \\ &\sim \chi_1^2 dt \end{aligned}$$

The expected value of the chi-square distributed stochastic term is dt and the variance is $2dt^2$. The variance is therefore $o(dt)$ and vanishes for the Newtonian differential leaving a deterministic constant equal to the expected value dt . Therefore, the square of standard Brownian motion is equal to the Newtonian time differential

$$dB(t)^2 = dt$$

Further, the second order differential between the stochastic and the Newtonian differential is

$$\begin{aligned} dB(t)dt &= B(dt)dt \\ &\sim N(0, dt)dt \\ &\sim N(0, dt^3) \\ &\sim \delta(0) + o(dt) \end{aligned}$$

which is normally distributed with zero mean and variance equal to the cube of the Newtonian time differential. The product of the differentials is therefore negligible. The distribution of this term is therefore a Dirac delta function located at zero. Hence

$$dB(t)dt = 0$$

with $o(dt)$ variation. The square of the drifting Brownian motion $X(t)$ is consequently proportional to the Newtonian time differential

$$dX^2 = \sigma^2 dt$$

with $o(dt)$ variation. All higher order powers of the Brownian motion are o(dt) in the Newtonian differential and are therefore negligible. Hence the key concept of the stochastic calculus of normal first order diffusion is that

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only the expected value of the squared Brownian term, the quadratic variation, of the higher order terms that is non-negligible. The drift-Brownian interaction and chi-square variation at this order vanishes on expected values with $dt \to 0$ since it is o(dt). This is summarized in the table[29] of distributions of products of Newtonian and stochastic differentials

Table 12: Products of Newtonian differentials and zero mean unit scale stochastic differentials

×	Newtonian differential dt	Stochastic differential dB
Newtonian differential dt	$o(dt)$	$N(0, dt^3) \to 0$
Stochastic differential dB	$N(0, dt^3) \to 0$	$\chi_1 dt \to dt$

The stochastic differential of the transformation of the Brownian motion becomes

$$dh(X) = \frac{\partial h(X)}{\partial X}(\mu dt + \sigma dB) + \tfrac{1}{2}\sigma^2 \frac{\partial^2 h(X)}{\partial X^2} dt + o(dt)$$

Collecting like terms in the differentials and dividing both sides by dt

$$\frac{dh(X)}{dt} = \left(\mu \frac{\partial h(X)}{\partial X} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 h(X)}{\partial X^2} \right) + \sigma \frac{\partial h(X)}{\partial X}\frac{dB}{dt} + \frac{o(dt)}{dt}$$

Letting the dt approach zero as per the standard calculus so that

$$\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$$

gives the total derivative of the transformation of the diffusion variable $h(X)$ as a function of differential Brownian motion

$$\frac{dh(X)}{dt} = \left(\mu \frac{\partial h(X)}{\partial X} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 h(X)}{\partial X^2} \right) + \sigma \frac{\partial h(X)}{\partial X}\frac{dB}{dt}$$

In differential form, this is Ito's formula

$$dh(X) = \left(\mu \frac{\partial h(X)}{\partial X} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 h(X)}{\partial X^2} \right) dt + \sigma \frac{\partial h(X)}{\partial X} dB$$

or

$$dY = \left(\mu Y_X + \tfrac{1}{2}\sigma^2 Y_{XX} \right) dt + \sigma Y_X dB$$

Transformation of the Diffusion Variable and Time

Defining a new process Y as a the transformation h now dependent on time t as well as the diffusion variable X. The variable of the transformation is then $Y = h(X,t)$. Applying the same technique and expanding the function in a Taylor series around the total forward increment

$$h(X + dX, t + dt)$$
$$= h(X,t) + \frac{\partial h(X,t)}{\partial X} dX + \frac{\partial h(X,t)}{\partial t} dt + \tfrac{1}{2}\frac{\partial^2 h(X,t)}{\partial X^2} dX^2 + \frac{\partial^2 h(X,t)}{\partial X \partial t} dX dt + o(dX^2) + o(dt) + o(dX dt)$$

[29] these tables of products, not including distributional theory, are considered in
(i) Baxter, M. and Rennie, A., (1996) *Financial Calculus: An Introduction to Derivative Pricing.* Cambridge University Press;
(ii) Shimko, D. C., (1992) *Finance in Continuous Time: A Primer.* Kolb Publishing, Miami, FL.
The distributional theory including the exact distributions of the products is from independent mathematical research of the author (and hence making possible Aaron' Lemma in FOUNDATIONS II).

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and subtracting $h(X,t)$ from both sides gives the total differential

$$dh(X,t) = \frac{\partial h(X,t)}{\partial X} dX + \frac{\partial h(X,t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 h(X,t)}{\partial X^2} dX^2 + \frac{\partial^2 h(X,t)}{\partial X \partial t} dXdt + o(dX^2) + o(dt) + o(dXdt)$$

Remembering that $dX \sim N(\mu dt, \sigma^2 dt)$ so that

$$\begin{aligned} dXdt &\sim N(\mu dt, \sigma^2 dt)dt \\ &\sim N(\mu dt^2, \sigma^2 dt^3) \\ &\sim \delta(\mu dt^2) \end{aligned}$$

so that the cross differential between the Brownian motion and the time differential has mean proportional to dt^2 and variance proportional to dt^3 and is hence negligible o(dt) both in value and variability. Therefore, the interaction of the time differential and Gaussian variation at this order vanish together leaving that of the squared differential Brownian motion. In particular, the variation in both cases is o(dt), and the squared Brownian chi-square differential is quadratic variation which is deterministic with a value equal to the infinitesimal variance σ^2 and the time differential dt , i.e., $\sigma^2 dt$,

$$\begin{aligned} dX &\sim N(\mu dt, \sigma^2 dt) \\ &\sim \sigma\sqrt{dt}\, N\left(\frac{\mu}{\sigma}\sqrt{dt}, 1\right) \end{aligned}$$

$$(dX)^2 \sim \chi_1^2\left(\frac{\mu^2}{\sigma^2} dt\right) \times \sigma^2 dt$$

and

$$\begin{aligned} E\left((dX)^2\right) &= \left(1 + \frac{\mu^2}{\sigma^2} dt\right)\sigma^2 dt \\ &= \sigma^2 dt + (\mu\, dt)^2 \\ &= \sigma^2 dt + o(dt) \end{aligned}$$

as well as

$$Var\left((dX)^2\right) = \sigma^4 (dt)^2 \left(2 + 4\frac{\mu^2}{\sigma^2} dt\right) \to o(dt)$$

The summary table of products of Newtonian and drifting stochastic differentials is

Table 13: Products of Newtonian differentials and stochastic differentials with drift and scale

×	Newtonian differential dt	Stochastic differential dX
Newtonian differential dt	$o(dt)$	$N(\mu dt^2, \sigma^2 dt^3) \to o(dt)$
Stochastic differential dX	$N(\mu dt^2, \sigma^2 dt^3) \to o(dt)$	$\chi_1^2\left(\frac{\mu^2}{\sigma^2} dt\right)\sigma^2 dt \to \sigma^2 dt + o(dt)$

The statistical differential of the transformation h becomes

$$\begin{aligned} dh(X,t) &= \frac{\partial h(X,t)}{\partial X}(\mu dt + \sigma dB) + \frac{\partial h(X,t)}{\partial t} dt + \tfrac{1}{2}\sigma^2 \frac{\partial^2 h(X,t)}{\partial X^2} dt + o(dt) \\ &= \left(\mu \frac{\partial h(X,t)}{\partial X} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 h(X,t)}{\partial X^2} + \frac{\partial h(X,t)}{\partial t}\right) dt + \sigma \frac{\partial h(X,t)}{\partial X} dB + o(dt) \end{aligned}$$

Dividing both sides by dt results in

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$$\frac{dh(X,t)}{dt}=\left(\mu\frac{\partial h(X,t)}{\partial X}+\tfrac{1}{2}\sigma^2\frac{\partial^2 h(X,t)}{\partial X^2}+\frac{\partial h(X,t)}{\partial t}\right)+\sigma\frac{\partial h(X,t)}{\partial X}\frac{dB}{dt}+\frac{o(dt)}{dt}$$

and letting the dt approach zero so that

$$\lim_{dt\to 0}\frac{o(dt)}{dt}=0$$

gives the total derivative of the transformation h of time t and the diffusion variable X as a function of differential Brownian motion

$$\frac{dh(X,t)}{dt}=\left(\mu\frac{\partial h(X,t)}{\partial X}+\tfrac{1}{2}\sigma^2\frac{\partial^2 h(X,t)}{\partial X^2}+\frac{\partial h(X,t)}{\partial t}\right)+\sigma\frac{\partial h(X,t)}{\partial X}\frac{dB}{dt}$$

This is Ito's formula when in differential form

$$dh(X,t)=\left(\mu\frac{\partial h(X,t)}{\partial X}+\tfrac{1}{2}\sigma^2\frac{\partial^2 h(X,t)}{\partial X^2}+\frac{\partial h(X,t)}{\partial t}\right)dt+\sigma\frac{\partial h(X,t)}{\partial X}dB$$

or

$$dY=\left(\mu Y_X+\tfrac{1}{2}\sigma^2 Y_{XX}+Y_t\right)dt+\sigma Y_X\,dB$$

Ito's Lemma for Transformations of General Diffusion Processes

The transformations of an arithmetic Brownian motion is a special case of transformations of the general first order diffusion process. This is a motivation for such transformations, and pertinently and clearly shows that the infinitesimal increments of quadratic variation (the chi-square variation) is on the dt scale while all higher order variation is $o(dt)$ and is therefore negligible

$$\lim_{dt\to 0} o(dt)/dt=0$$

The general diffusion equation is

$$dX=\mu(X,t)dt+\sigma(X,t)dB \quad \text{subject to } X(t)\sim\delta(X)$$

where $dB\sim N(0,dt)$ and therefore the differential increment is distributed normally conditional on the backward variable

$$dX\mid X\sim N\left(\mu(X,t)dt,\sigma^2(X,t)dt\right)$$

The derivation of the Ito formulae for general diffusion runs exactly parallel to that of arithmetic Brownian motion starting with a Taylor series expansion. The difference being that the products of the Newtonian and stochastic differentials, which are proportional to the infinitesimal parameters, are then additionally dependent on the diffusion variable, the time variable, or both. The result is that the squared diffusion differential is proportional to the infinitesimal variance $\sigma^2(X,t)$ and a function of X or t or both. The table of second order differentials is

Table 14: Products of Newtonian differentials and stochastic differentials with arbitrary infinitesimal parameters

×	Newtonian differential dt	Stochastic differential dX
Newtonian differential dt	$o(dt)$	$N\left(\mu(X,t)dt^2,\sigma^2(X,t)dt^3\right)\to o(dt)$
Stochastic differential dX	$N\left(\mu(X,t)dt^2,\sigma^2(X,t)dt^3\right)\to o(dt)$	$\chi_1\sigma^2(X,t)dt\to\sigma^2(X,t)dt$

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Since the infinitesimal parameters are functions of the backward variables of the differential increment, they are constant with respect to the increment. As such, the derivations of the Ito formulae from here follow exactly the same lines and are simply quoted to avoid repetition of reasoning and duplication of results;

Transformation of the Diffusion Variable

$$dh(X) = \left(\mu(X,t)\frac{\partial h(X)}{\partial X} + \tfrac{1}{2}\sigma^2(X,t)\frac{\partial^2 h(X)}{\partial X^2} \right)dt + \sigma(X,t)\frac{\partial h(X)}{\partial X}\,dB$$

or

$$dY = \left(\mu(X,t)Y_X + \tfrac{1}{2}\sigma^2(X,t)Y_{XX} \right)dt + \sigma(X,t)Y_X\,dB$$

Transformation of the Diffusion Variable and Time

$$dh(X,t) = \left(\mu(X,t)\frac{\partial h(X,t)}{\partial X} + \tfrac{1}{2}\sigma^2(X,t)\frac{\partial^2 h(X,t)}{\partial X^2} + \frac{\partial h(X,t)}{\partial t} \right)dt + \sigma(X,t)\frac{\partial h(X,t)}{\partial X}\,dB$$

or

$$dY = \left(\mu(X,t)Y_X + \tfrac{1}{2}\sigma^2(X,t)Y_{XX} + Y_t \right)dt + \sigma(X,t)Y_X\,dB$$

Transformation of the Diffusion Variable, Chronological Time, and Cross-Sectional Time

It is sometimes required that a measure of time is two dimensional - one dimension of time measuring chronological or rolling time while the other measuring a cross section of times, for example, the age spectrum of a population or time-to-maturity for a fixed date. Some specific examples for which it is an absolutely essential to include both concepts of time in a model is cross-sectional lifetables over successive years, modeling IBNR[30] insurance claims, and a yield curve measured on successive days. The transformation is then

$$h(X,t,s)$$

where $X(t,s)$ is a two dimensional correlated Brownian motion. With relation to the stochastic calculus, the most rudimentary example is a Brownian mountain. The derivation of the relevant Ito formula involves the addition of a single Newtonian differential term and the derivation is therefore similar to the earlier derivations. This will not be developed further in this dissertation.

ITO-TRANSFORM AND CONVOLUTION ITO-TRANSFORM PAIRS

Proposition:

Any transformation h of a stochastic process X that has a Gaussian transition distribution, either Brownian motion or the Ornstein-Uhlenbeck process,

(1) $dX = \mu dt + \sigma dB$

(2) $dX = -\mu X dt + \sigma dB$

(3) $dX = \alpha(\mu - X)dt + \sigma dB$

defines a process Y that has a transition distribution that can be expressed in terms of a transformation h^{-1} of a Gaussian random variable.

[30] Incurred But Not yet Reported.

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Converse Proposition:

If the process *Y* of a particular stochastic differential can be transformed by Ito's lemma to a Brownian motion or an Ornstein-Uhlenbeck process, then the transition distribution of the process can be stated as the inverse transformation of a transition distribution of a Brownian motion or an Ornstein-Uhlenbeck process, both of which are Gaussian.

Either proposition may be of value in applied statistical work, such as modeling security dynamics for calculating an option premium. The utility is obtained with the quick and efficient transformations between the relevant processes and the underlying transition distributions, all of which have analytic formulae for option premia. This efficiency is improved by being able to pick out the pairs or convolution pairs by inspection or from a pre-complied table such as that at the end of this topic.

Mathematics of the Proposition: Implications for Statistical Work

Given a diffusion variable $X(t)$, if a transformation $h(X)$ can be found such that it results in a Brownian motion

$$dh(X_t) = \mu dt + \sigma dB_t$$

then by direct integration since the transformed process has a unit-root,

$$h(X_t) = h(x_0) + \mu t + \sigma B_t$$

and is normally distributed

$$\begin{aligned} h(X_t) &= h(x_0) + \mu t + \sigma B_t \\ &\sim N\left(h(x_0) + \mu t, \sigma^2 t\right) \end{aligned}$$

Inverting the transform *h* leaves the original variable *X*

$$X_t = h^{-1}[h(x_0) + \mu t + B_t]$$

so that

$$X_t \sim h^{-1}\left[N\left(h(x_0) + \mu t, \sigma^2 t\right)\right]$$

which is a statistical distribution in the parameters alone from which statistical work can follow with a set of standard normal or non-central chi-square tables. Additionally, given a process $X(t)$, if a transformation $h(X)$ can be found such that it results in an Ornstein-Uhlenbeck process, that introduces a moderator and correlation between the increments[31]

$$dh(X_t) = \alpha\left(\mu - h(X_t)\right)dt + \sigma dB_t$$

then by direct integration with a moderating integrating factor $IF(t) = e^{\alpha t}$

$$\int_0^t d[e^{\alpha y} h(X_t)] = \int_0^t e^{\alpha t}\left(\alpha\mu + \sigma\frac{dB_t}{dt}\right)dt$$

$$h(X_t) = h(x_0)e^{-\alpha t} + \mu(1 - e^{-\alpha t}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha t} dB(t)$$

which is normal

$$\begin{aligned} h(X_t) &= h(x_0)e^{-\alpha t} + \mu(1 - e^{-\alpha t}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha t} dB_t \\ &\sim N\left(h(x_0)e^{-\alpha t} + \mu(1 - e^{-\alpha t}),\ \sigma^2\frac{1 - e^{-2\alpha t}}{2\alpha}\right) \end{aligned}$$

Hence the inverse of *h* leaves the original variable

[31] a moderator for the integration of the diffusion basis.

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$$X_t = h^{-1}\left(h(x_0)e^{-\alpha t} + \mu(1-e^{-\alpha t}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha t} dB(t) \right)$$

and this implies the transition distribution

$$X_t \sim h^{-1}\left[N\left(x_0 e^{-\alpha t} + \mu(1-e^{-\alpha t}),\ \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t}) \right) \right]$$

This is a statistical distribution of the parameters alone from which statistical work can follow with a set of standard normal or non-central chi-square tables.

First Passage Times

Further results include implications for first passage time distributions, the distribution of time over state for a given diffusion. For instance, unit root diffusion defined by $dh(X_t) = \mu dt + \sigma dB_t$ has solution $h(X_t) = h(x_0) + \mu t + \sigma B_t$ which is normal for state over time and inverse Gaussian for time over state. If $y = h(x)$ and $y_0 = h(x_0)$ so that $x = h^{-1}(y)$ and $x_0 = h^{-1}(y_0)$, then

$$T_{y-y_0} \sim IG\left(\frac{y-y_0}{\mu}, \frac{(y-y_0)^2}{\sigma^2} \right) \text{ and } T_{h^{-1}(x)-h^{-1}(x_0)} \sim IG\left(\frac{h^{-1}(x)-h^{-1}(x_0)}{\mu}, \frac{[h^{-1}(x)-h^{-1}(x_0)]^2}{\sigma^2} \right)$$

First passage times will not be pursued further here.

Convolution Ito-Transform Pairs

Since normal variables are closed under the operation of convolution, the idea of an Ito-transform pair can be extended to include more complicated processes. A homogenous convolved sum of a transform of a process that has a Gaussian transition distribution, where the transform retains additivity, forms a convolution Ito-transform pair with the underlying Gaussian transition distribution. A process resulting from an *n*-fold convolution is said to be a convolution pair with index *n* . The benefits that accrue are the same as those for standard Ito-transform pairs, most importantly, a transition distribution that is some elementary transformation of a normal random variable. The transform considered here is always a square – the square of a Brownian motion or the square of an Ornstein-Uhlenbeck process. This terminology, convolution Ito-transform pairs, also includes the square-root of convolved Gaussian processes. Together, Ito-transform pairs and convolution Ito-transform pairs exhausts all of the processes of first order diffusion that have transition distributions of a known and analytic form up to a table lookup. Since the transition distributions are either normal, log-normal, central and non-central scaled chi-square, or the square-root of central and non-central scaled chi-square, the table lookup involves either standard normal tables or non-central chi-square (or Bessel) tables (or function calls).

These propositions follow from mathematical statistics and distribution theory. It is assumed knowledge of a technically competent person who is knowledgeable in the art of mathematical statistics[32].

Limitation to Ito-Transform and Convolution Ito-Transform Pairs

Although convenient, the set of Ito-transform and convolution Ito-transform pairs do not exhaust the set of diffusion processes defined by the general stochastic differential

$$dX = \mu(x,t)dt + \sigma(x,t)dz$$

This proposition is similar to the fact that not all differential equations have general analytic solutions, although many have, albeit only a small portion of the infinite number. Important examples of first order diffusion that do not form an Ito-transform pair or convolution Ito-transform pair are

[32] Fisher, R. A., (1956) *Statistical Methods and Scientific Inference*, Collier Macmillan.
cite K. Pearson.

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1) $dX = (\mu + \alpha / X - \beta X)dt + \sigma dz$

2) $dX = (\mu + \alpha\sqrt{X} + \beta X)dt + \sigma\sqrt{X}\,dz$

3) $dX = \alpha(\mu - X)dt + \sigma X^{\beta/2}dz$ (for arbitrary β)

4) $dX = \alpha(\mu - X)dt + \sigma X dz$

Note that this last equation $dX = \alpha(\mu - X)dt + \sigma X dz$ is not a process with a closed form transition distribution since the process without the single time differential is multiplicative not additive. If this process with the additional time differential had a closed form transition density then all of compound interest theory would generalize to log-normal interest rates. As it stands, only accumulation and discount factors do; additions of these do not.

There are also many processes, made up from combining several simpler processes by convolution, that are not expressible by a convenient stochastic differential like, for example, that of the general three-parameter square-root process. These may be, for whatever reason, just as important as those that do have a convenient expression for the associated stochastic differential increment. Like, for example, convolved squared Brownian motions and convolved squared Ornstein-Uhlenbeck processes that mean revert to μ rather than zero; these processes are on either side of convolved squared Ornstein-Uhlenbeck processes that mean revert to zero in terms of practical application and utility, one more general and one less general, as well as the statistical convenience of having closed form transition distributions of the same structure as the three-parameter square-root and closed form transition density functions. Yet neither of these have a convenient expression for the stochastic differential. The point being is that stochastic differentials may not be the most suitable way of expressing a convolution of homogenous first order diffusion. For example, generalizing the modeling of the three-parameter square-root process may best be achieved by concentrating on the physical interpretation of the basic underlying processes rather than trying to achieve a more fancy stochastic differential or trying to model and generalize the differential dynamics directly. It is my belief, with the mathematics of this thesis, that the differential dynamics of diffusion is best expressed in terms of differential equations with proper derivatives rather than differentials. This ensures that such equations remain differentially balanced when the order of the diffusion is increased above first, a topic for which is elaborated on fully in the later parts of this dissertation. In this regard, stochastic calculus fits in with the conventional calculus perfectly as well as extending it. Each of the important first order diffusions of FOUNDATIONS I are included under the umbrella of diffusion of arbitrary order in BOOK 3 of this dissertation. This book deals specifically with diffusion, and inverse diffusion, of arbitrary order for which analytic formulae for option premia are derived for all of the processes covered by the methodological umbrella defined by generalizing the Newtonian calculus to stochastic variables.

THE ITO-TRANSFORM PAIRS

Basic Transformations – Scale and Drift of Brownian Motion

Adding Drift and Scale to a Standard Brownian Motion

$$dY = dz$$
$$X = x_0 + \mu t + \sigma Y$$
$$dX = X_Y dY + \tfrac{1}{2} X_{YY}(dY)^2 + X_t dt$$
$$X_Y = \sigma \qquad X_{YY} = 0 \qquad X_t = \mu$$
$$dX = \sigma dz + \tfrac{1}{2}(0)dt + \mu dt$$
$$dX = \mu dt + \sigma dz$$

Removing Drift and Scale from a Brownian Motion (Standardizing to Standard Brownian Motion)

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$$dX = \mu dt + \sigma dz$$
$$Y = \frac{X - x_0 - \mu t}{\sigma}$$
$$dY = Y_X dX + \tfrac{1}{2} Y_{XX} (dX)^2 + Y_t dt$$
$$Y_X = \frac{1}{\sigma} \qquad Y_{XX} = 0 \qquad Y_t = -\frac{\mu}{\sigma}$$
$$dY = \frac{1}{\sigma}(\mu dt + \sigma dz) + \tfrac{1}{2}(0)\sigma^2 dt - \frac{\mu}{\sigma} dt$$
$$dY = dz$$

Ito-Transform from Standard Brownian Motion

$$dY = dz \sim N(0, dt)$$
$$Y = z \sim N(0, t)$$

inverse transformation $X = x_0 + \mu t + \sigma Y$

$$X \sim x_0 + \mu t + \sigma N(0,t)$$
$$\sim N\left(x_0 + \mu t, \sigma^2 t\right)$$

Ornstein-Uhlenbeck to Standard Brownian Motion

From the stochastic differential of the standard Ornstein-Uhlenbeck process

$$dX = -\alpha X dt + \sigma dB$$

which is now transformed to standard Brownian motion with the transformation

$$Y(s) = \frac{1}{\sigma}\sqrt{1+2\alpha s}\, X\left(\frac{1}{2\alpha}\log(1+2\alpha s)\right)$$

This has the partial derivatives

$$Y_X = \frac{1}{\sigma}\sqrt{1+2\alpha s} \qquad Y_{XX} = 0 \qquad \text{and} \qquad Y_S = \frac{\alpha}{\sigma\sqrt{1+2\alpha s}} X$$

from the bivariate expansion of the Ito calculus or total differential from standard calculus

$$dY = Y_X dX + \tfrac{1}{2} Y_{XX} (dX)^2 + Y_S ds$$

the increment becomes

$$dY(s) = \frac{1}{\sigma\sqrt{1+2\alpha s}}\left(-\alpha X(t) dt + \sigma dB(t)\right) + \frac{\alpha}{\sigma\sqrt{1+2\alpha s}} X(t) ds$$

and noting that $t = \frac{1}{2\alpha}\log(1+2\alpha s)$ so that $\frac{dt}{ds} = \frac{1}{1+2\alpha s}$, that is, $dt = \frac{ds}{1+2\alpha s}$ giving

$$dY(s) = \frac{1}{\sigma}\sqrt{1+2\alpha s}\left(-\alpha X(t)\frac{ds}{1+2\alpha s} + \sigma dB(t)\right) + \frac{\alpha}{\sigma\sqrt{1+2\alpha s}} X(t) ds$$

or

$$dY(s) = \sqrt{1+2\alpha s}\, dB(t)$$

From distribution theory

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$$\begin{aligned}\sqrt{1+2\alpha s}\, dB(t) &\sim \sqrt{1+2\alpha s}\, N(0, dt) \\ &\sim \sqrt{\frac{ds}{dt}} N(0, dt) \\ &\sim N\left(0, \frac{ds}{dt} dt\right) \\ &\sim N(0, ds)\end{aligned}$$

which is a Brownian motion in the time variable s, i.e., $dB(s) \sim N(0,ds)$ and hence

$$\sqrt{1+2\alpha s}\, dB(t) = dB(s)$$

which gives

$$dY(s) = dB(s)$$

which is the stochastic differential increment of a standard Brownian motion $dY = dB$ as required. Since it is known that the transition distribution of standard Brownian motion is normal with variance that is linear in time, the transition distribution in the time change is

$$Y(s) \sim N(y_0, s)$$

The original variable, that of the Ornstein-Uhlenbeck process, is then the inverse transformation of the original transformation $Y(s) = \sqrt{1 + 2\alpha s}\, X[\log(1 + 2\alpha s)/ 2\alpha] / \sigma$ applied to the Gaussian variable, i.e., the transformation

$$X(t) = \sigma e^{-\alpha t} Y\left(\frac{e^{2\alpha t} - 1}{2\alpha}\right)$$

giving

$$\begin{aligned}\sigma e^{-\alpha t} Y(s) &\sim \sigma e^{-\alpha t} N(y_0, s) \\ &\sim N(y_0 \sigma e^{-\alpha t}, \sigma^2 e^{-2\alpha t} s)\end{aligned}$$

Inclusive of the time change and noting that $Y(0) = X(0) / \sigma$, i.e., $x_0 = \sigma y_0$, the transition variable is,

$$X(t) = \sigma e^{-\alpha t} Y\left(\frac{e^{2\alpha t} - 1}{2\alpha}\right) \sim N\left(x_0 e^{-\alpha t}, \sigma^2 \frac{1 - e^{-2\alpha t}}{2\alpha}\right)$$

Standard Brownian Motion to Ornstein-Uhlenbeck

From the differential increment of standard Brownian motion

$$dX = dB$$

or

$$dX(t) = dB(t)$$

Transforming to

$$Y(s) = \sigma e^{-\alpha s} X\left(\frac{e^{2\alpha s} - 1}{2\alpha}\right)$$

which includes the time change $t = (e^{2\alpha t} - 1) / 2\alpha$. The transformation has the partial derivatives

$$Y_X = \sigma e^{-\alpha s} \qquad Y_{XX} = 0 \qquad \text{and} \quad Y_S = -\alpha \sigma e^{-\alpha s} X$$

In terms of the series expansion of the stochastic differential

$$dY(s) = Y_X dX(t) + \tfrac{1}{2} Y_{XX} [dX(t)]^2 + Y_S ds$$

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the differential of the transformation is

$$dY(s) = \sigma e^{-\alpha s} dB(t) - \alpha\sigma e^{-\theta s} X(t) ds$$

or

$$dY(s) = -\alpha Y(s) ds + \sigma e^{-\alpha s} dB(t)$$

The rate of change in the time change is

$\frac{dt}{ds} = e^{2\alpha s}$ so noting that $e^{-\alpha s} dB(t) = \sqrt{\frac{ds}{dt}} dB(t) \sim \sqrt{\frac{ds}{dt}} N(0, dt)$

that is,

$$e^{-\alpha s} dB(t) \sim N(0, ds)$$

which implies that $e^{-\alpha s} dB(t)$ is a Brownian motion in s

$$e^{-\alpha s} dB(t) = dB(s)$$

The stochastic differential can be written as

$$dY(s) = -\alpha Y(s) ds + \sigma dB(s)$$

or simply

$$dY = -\alpha Y ds + \sigma dB$$

which is the stochastic differential increment of an Ornstein-Uhlenbeck process as required. Since it is known that the transition distribution of standard Brownian motion is normal with variance that is linear in time

$$X(t) \sim N(x_0, t)$$

from which it can be inferred that the transformed variable is zero mean normal with variance $\sigma^2(1 - e^{-2\alpha t}) / 2\alpha$, i.e.,

$$Y(s) = \sigma e^{-\alpha s} X\left(\frac{e^{2\alpha s} - 1}{2\alpha}\right) \sim N\left(\sigma x_0 e^{-\alpha s}, \sigma^2 \frac{1 - e^{-2\alpha s}}{2\alpha}\right)$$

and noting that $Y(0) = \sigma X(0)$, i.e., $y_0 = \sigma x_0$,

$$Y(s) = \sigma e^{-\alpha s} X\left(\frac{e^{2\alpha s} - 1}{2\alpha}\right) \sim N\left(y_0 e^{-\alpha s}, \sigma^2 \frac{1 - e^{-2\alpha s}}{2\alpha}\right)$$

The Two Main Diffusion Transformations

- The **exponential** transformation $Y = \exp(X)$ which has derivatives $Y = Y_X = Y_{XX} = \exp(X)$ and inverse function $X = \log Y$
- The **square** transformation $Y = X^2$ which has derivatives $Y_X = 2X$ and $Y_{XX} = 2$ and inverse function $X = \sqrt{Y}$

These transformations of Brownian motion and the Ornstein-Uhlenbeck process, which are the two main forms of first order Gaussian diffusion, relate to transformations between normals, log-normals, and chi-squares (central and non-central).

The exponential transformation is the transformation that turns an additive process that sums (convolves) and subtracts (deconvolves), into a multiplicative process that multiplies (log-convolves) and divides (log-deconvolves). The converse holds for the inverse transformation, the logarithmic transformation. Multiplicative processes are transformed to additive processes. Squaring an additive process results in a process that is intermediate between additive and multiplicative. These are two forms of the power transformation, which is sometimes referred to as the Box-Cox power transformation. Including the null transformation, these provide the two ex-

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tremities and one midpoint on the spectrum of power transformation – null, square-root, and logarithmic. The nomenclature refers to the function applied to observed data, rather than the functional transform applied to the random variable, for which the inverse function applies and visa-versa.

Geometric Brownian Motion

Geometric Brownian motion Y is defined as the exponential transformation $h(\cdot) = \exp(\cdot)$ of Brownian motion X, i.e., the transformation $Y = h(X) = \exp(X)$ obtained from $dX = \mu dt + \sigma dz$. From the Ito formula, the stochastic differential increment is then

$$dY = Y_X(dX) + \tfrac{1}{2}Y_{XX}(dX)^2$$

$$\begin{aligned} dY &= Y(\mu dt + \sigma dz) + \tfrac{1}{2}Y(\sigma^2 dt) \\ &= (\mu + \tfrac{1}{2}\sigma^2)Ydt + \sigma Ydz \end{aligned}$$

Since $X = h^{-1}(Y) = \log(Y)$ is normal

$$X = \log Y \sim N\left(x_0 + \mu t, \sigma^2 t\right)$$

it is then known that because an exponentiated normal is a log-normal that Y is log-normal with $x_0 = \log y_0$

$$Y \sim \log N\left(\log y_0 + \mu t, \sigma^2 t\right)$$

Reparameterizing to the required form

$$dY = mYdt + sYdz$$

where $m = \mu + \tfrac{1}{2}\sigma^2$ and $s = \sigma$ are the parameters of the required process and $\mu = m - \tfrac{1}{2}s^2$ and $\sigma = s$ are the parameters of the underlying process. The transition distribution of Y is then

$$Y \sim \log N\left(\log y_0 + (m - \tfrac{1}{2}s^2)t,\ s^2 t\right)$$

Geometric Ornstein-Uhlenbeck

The geometric Ornstein-Uhlenbeck process Y is defined as the exponential transformation $Y = h(X) = \exp(X)$ of the Ornstein-Uhlenbeck process X that has the stochastic differential $dX = \alpha(\mu - X)dt + \sigma dz$. From the Ito formula, the stochastic differential increment is then

$$dY = Y_X(dX) + \tfrac{1}{2}Y_{XX}(dX)^2$$

$$\begin{aligned} dY &= Y(\alpha[\mu - X]dt + \sigma dz) + \tfrac{1}{2}Y(\sigma^2 dt) \\ &= \alpha(\mu + \tfrac{1}{2}\sigma^2 - \log Y)Ydt + \sigma Ydz \end{aligned}$$

Since $X = h^{-1}(Y) = \log(Y)$ is normal

$$X = \log Y \sim N\left(x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}),\ \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\right)$$

it is then known that Y is log-normal

$$Y \sim \log N\left(x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}),\ \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\right)$$

Reparameterizing to the required form

$$dY = a(m - \log Y)Ydt + sYdz$$

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where $a = \alpha$, $m = \mu + \frac{1}{2}\sigma^2$ and $s = \sigma$ are the parameters of the required process and $\alpha = a$, $\mu = m - \frac{1}{2}s^2$, and $\sigma = s$ are the parameters of the underlying process. With $x_0 = \log y_0$, the transition distribution of Y is

$$Y \sim \log N\left(e^{-a\,t} \log y_0 + (m - \tfrac{1}{2}s^2)(1 - e^{-a\,t}),\ \ \frac{s^2}{2a}(1 - e^{-2a\,t}) \right)$$

Squared Brownian Motion

Squared Brownian motion Y is defined as the square $h(\cdot) = (\cdot)^2$ of Brownian motion X, the transformation $Y = h(X) = X^2$ obtained from $dX = -\mu dt + \sigma dz$, noting that the Brownian drift has to be negative if the squared process is to be recurrent and mean revert. From the Ito formula, the stochastic differential is

$$dY = Y_X (dX) + \tfrac{1}{2} Y_{XX} (dX)^2$$

$$\begin{aligned} dY &= 2X(-\mu dt + \sigma dz) + \tfrac{1}{2}(2)(\sigma^2 dt) \\ &= (\sigma^2 - 2\mu X)dt + 2\sigma X dz \\ &= 2\mu\left(\frac{\sigma^2}{2\mu} - \sqrt{Y} \right) dt + 2\sigma\sqrt{Y}\, dz \end{aligned}$$

noting that it is not possible to reparameterize the differential increment to the form

$$dY = a(m - \sqrt{Y})dt + s\sqrt{Y}\, dB$$

because there are only two degrees of freedom in the parameterization of the process, the mean and variance of the Brownian motion, not three that this expression requires. The inverse transformation $X = h^{-1}(Y) = \sqrt{Y}$ is Brownian motion and is hence normal

$$X = \sqrt{Y} \sim N\left(x_0 + \mu t, \sigma^2 t \right)$$

Since a square of a normal is a scaled non-central chi-square (or central chi-square if the normal is zero mean) on one degree of freedom, it is then known that $Y = X^2$ is a scaled non-central chi-square with a single degree of freedom

$$Y \sim k\chi_1^2(\lambda)$$

where **(1)** the scale factor is the variance of the underlying Brownian motion $k = \sigma^2 t$ and **(2)** the non-centrality parameter is the square of the mean of the underlying Brownian motion divided by the scale factor

$$\lambda = (x_0 + \mu t)^2 / \sigma^2 t$$

When the drift of the underlying Brownian motion, μ, is set to zero the stochastic differential becomes $dY = \sigma^2 dt + 2\sigma\sqrt{Y}\, dz$. If the initial value of the diffusion is zero, the Ito-transform pair corresponding to the transition distribution of $X = \sqrt{Y} \sim N(0, \sigma^2 t)$ is a scaled central chi-square on a single degree of freedom, $Y \sim \sigma^2 t \times \chi_1^2$. This can alternatively be expressed distributionally as $Y \sim \Gamma(\frac{1}{2}, 2\sigma^2 t)$. This is squared zero drift Brownian motion starting at the origin and has only one parameter, the infinitesimal variance of the underlying Brownian motion.

Squared Ornstein-Uhlenbeck (2 Parameter Square-Root)

The two parameter square root process Y is defined as the square $h(\cdot) = (\cdot)^2$ of an Ornstein-Uhlenbeck process X that mean reverts to zero; the transformation $Y = h(X) = X^2$ obtained from the process with stochastic differential equation $dX = -\alpha X dt + \sigma dz$. From the Ito formula, the stochastic differential increment is then

$$dY = Y_X (dX) + \tfrac{1}{2} Y_{XX} (dX)^2$$

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$$
\begin{aligned}
dY &= 2X(-\alpha X dt + \sigma dz) + \tfrac{1}{2}(2)(\sigma^2 dt) \\
&= (\sigma^2 - 2\alpha X^2)dt + 2\sigma X dz \\
&= 2\alpha\left(\frac{\sigma^2}{2\alpha} - Y\right)dt + 2\sigma\sqrt{Y}\,dz
\end{aligned}
$$

Since the inverse transformation $X = h^{-1}(Y) = \sqrt{Y}$ is normal

$$X = \sqrt{Y} \sim N\left(x_0 e^{-\alpha t}, \sigma^2 \frac{1 - e^{-2\alpha t}}{2\alpha}\right)$$

it is then known that Y is a scaled non-central chi-square on a single degree of freedom

$$Y \sim k\chi_1^2(\lambda)$$

where the scale factor is the variance of the underlying Ornstein-Uhlenbeck diffusion

$$k = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

and the non-centrality parameter is the ratio of the square of the mean to that of the variance of the underlying Ornstein-Uhlenbeck diffusion

$$\lambda = \frac{x_0 e^{-\alpha t}}{\sigma^2(1 - e^{-2\alpha t})/2\alpha}$$

It is has been shown in earlier work by solving the forward equation, that the distribution without the additional time differential in the stochastic increment,

$$dY = -\alpha Y dt + \sigma\sqrt{Y}\,dz$$

is a scaled chi-square distribution on zero degrees of freedom and that an additional term of $\alpha\mu dt$, which is a deterministic constant, adds degrees of freedom to the non-central chi-square variable. The transition distribution can be obtained from the convolution Ito-transform pair for the three parameter square-root process for which the convolution index can be zeroed. A little ironically, this process is a valid diffusion resulting from the convolution of zero two parameter square root diffusions. It is more theoretically correct to think of this process in terms of deconvolution.

Alternative Specification of Two Parameter Square-Root (With an Additional Parameter, Non-Zero Basis Mean)

Alternatively, the slightly more flexible specification, obtained from squaring with the transformation $Y = h(X) = X^2$ an Ornstein-Uhlenbeck process that mean reverts to μ rather than zero, $dX = \alpha(\mu - X)dt + \sigma dz$,

$$
\begin{aligned}
dY &= 2X(\alpha[\mu - X]dt + \sigma dz) + \tfrac{1}{2}(2)(\sigma^2 dt) \\
&= (\sigma^2 + 2\alpha\mu X - 2\alpha X^2)dt + 2\sigma X dz \\
&= (\sigma^2 + 2\alpha\mu\sqrt{Y} - 2\alpha Y)dt + 2\sigma\sqrt{Y}\,dz \\
&= 2\alpha\left(\frac{\sigma^2}{2\alpha} + \mu\sqrt{Y} - Y\right)dt + 2\sigma\sqrt{Y}\,dz
\end{aligned}
$$

since $X = h^{-1}(Y) = \sqrt{Y}$ is normal

$$X = \sqrt{Y} \sim N\left(x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}),\ \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\right)$$

it is then known that Y is a scaled non-central chi-square on a single degree of freedom

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$$Y \sim k\chi_1^2(\lambda)$$

where the scale factor is the variance of the underlying Ornstein-Uhlenbeck process

$$k = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

and the non-centrality parameter is the ratio of the square of the mean to that of the variance of the underlying Ornstein-Uhlenbeck diffusion

$$\lambda = [x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t})]^2 \ / k$$

Note that this specification, with an underlying Ornstein-Uhlenbeck diffusion with non-zero basis mean, cannot be convolved into a closed form stochastic differential with some general convolution index. In this case, the processes do not algebraically aggregate because of the $\sqrt{Y}dt$ terms and the differential components can only be expressed as separate processes. The convolution can nevertheless still be defined and implemented with a non-central chi-square transition distribution but the stochastic differential increment does not simplify into a neat expression.

Discussion

All of the previous formulations can be estimated by the principle of likelihood based on the discrete[33] implementation of Brownian motion and Ornstein-Uhlenbeck processes. Analytic maximum likelihood estimators exist for each process forming an Ito-transform pair. This is not so for convolution Ito-transform pairs, considered next, since it is not possible to 'deconvolve' the component processes before writing down the likelihood. For the three-parameter square-root process, for example, it is not possible to separate out the $n = 4am / s^2$ independent squared homogenous Ornstein-Uhlenbeck components. Therefore, maximum likelihood estimation in this case, for general convolution index not equal to one, must be implemented numerically.

THE CONVOLUTION ITO-TRANSFORM PAIRS

This section discusses convolutions of Brownian motions and Ornstein-Uhlenbeck processes. Let X be the diffusion transition variable denoting the convolution, or cumulative linear supposition, of n first order diffusions each of the same diffusion type but not necessarily with identical parameterization, i.e., not necessarily homogenous,

$$Y = \sum_{i=1}^{n} h(X_i) = h(X_1) + h(X_2) + ... + h(X_n)$$

By the linearity property of differentials, the stochastic differential of the convolved processes is then

$$dY = \sum_{i=1}^{n} dh(X_i) = dh(X_1) + dh(X_2) + ... + dh(X_n)$$

This convolved differential is the basis for the next several transformations; those that form convolution Ito-transform pairs.

Introductory Example (Sums of Additive Processes): Convolved Brownian Motions and Ornstein-Uhlenbeck Processes

With $h(\cdot) = (\cdot)$ the convolution is a simple linear supposition

$$X = \sum_{i=1}^{n} X_i = X_1 + X_2 + ... + X_n$$

which has differential

$$dX = \sum_{i=1}^{n} dX_i = dX_1 + dX_2 + ... + dX_n$$

[33] Refer to BOOK 3.

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Let the j^{th} Brownian motion have stochastic differential increment $dX_j = \mu_j\,dt + \sigma_j\,dz$ and be **(i)** independent and then **(ii)** correlated with each of the other Brownian motions.

(i) Convolved Independent Brownian Motions

The convolution, or cumulative linear supposition, of the n independent diffusions is

$$\begin{aligned} dX &= \sum_{j=1}^{n} (\mu_j dt + \sigma_j dz) = \left(\sum_{j=1}^{n} \mu_j\right) dt + \sum_{j=1}^{n} \sigma_j dz \\ &\sim \left(\sum_{j=1}^{n} \mu_j\right) dt + \sum_{j=1}^{n} \sigma_j N(0, dt) \\ &\sim \left(\sum_{j=1}^{n} \mu_j\right) dt + N\left(0, \sum_{j=1}^{n} \sigma_j^2 dt\right) \\ &\sim \left(\sum_{j=1}^{n} \mu_j\right) dt + \sqrt{\sum_{j=1}^{n} \sigma_j^2}\, N(0, dt) \\ &\sim \mu dt + \sigma N(0, dt) \end{aligned}$$

so that the convolved process is again another Brownian motion

$$dX = \mu dt + \sigma dz$$

where the infinitesimal cumulants of the Brownian motion, also the cumulants of the transition distribution, are the sums of the infinitesimal cumulants of the component Brownian motions

$$\mu = \sum_{j=1}^{n} \mu_j \qquad \text{and} \qquad \sigma^2 = \sum_{j=1}^{n} \sigma_j^2$$

(ii) Convolved Correlated Brownian Motions

Say now each Brownian motion is correlated with each of the other Brownian motions, the correlation between the i^{th} and j^{th} Brownian motions being ρ_{ij}, i.e., the correlation between each stochastic term dz_i and dz_j is ρ_{ij} .The convolved processes is then

$$\begin{aligned} dX &= \sum_{j=1}^{n} (\mu_j dt + \sigma_j dz) = \left(\sum_{j=1}^{n} \mu_j\right) dt + \sum_{j=1}^{n} \sigma_j dz \\ &\sim \left(\sum_{j=1}^{n} \mu_j\right) dt + N\left(0, \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij} dt\right) \\ &\sim \left(\sum_{j=1}^{n} \mu_j\right) dt + \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij}}\, N(0, dt) \\ &\sim \mu dt + \sigma N(0, dt) \end{aligned}$$

so that the convolved process is another Brownian motion

$$dX = \mu dt + \sigma dz$$

where the infinitesimal cumulants of the summed processes are the sum of the infinitesimal means and the sum of the pairwise correlations (the product of the row vector of infinitesimal standard deviations, the correlation matrix, and the column vector of infinitesimal standard deviations) of the component processes

$$\mu = \sum_{j=1}^{n} \mu_j \qquad \text{and} \qquad \sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij}$$

(ii) Convolved Ornstein-Uhlenbeck Processes

By the same reasoning, let the j^{th} Ornstein-Uhlenbeck process have stochastic differential increment $dX = \alpha_j (\mu_j - X_j)dt + \sigma_j\,dz$ and be independent or correlated with each of the other Ornstein-Uhlenbeck processes

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$$\begin{aligned} dX &= \sum_{j=1}^{n} \left(\alpha_j (\mu_j - X_j) dt + \sigma_j dz_j \right) \\ &= \left(\sum_{j=1}^{n} \alpha_j \mu_j - \sum_{j=1}^{n} \alpha_j X_j \right) dt + \sum_{j=1}^{n} \sigma_j dz_j \\ &= \left(\sum_{j=1}^{n} \alpha_j \mu_j - \sum_{j=1}^{n} \alpha_j X_j \right) dt + \sigma dz \end{aligned}$$

where

$$\sigma = \begin{cases} \sqrt{\sum_{j=1}^{n} \sigma_j^2} & \text{if the } dz \text{ terms are independent} \\ \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij}} & \text{if the } dz \text{ terms are correlated} \end{cases}$$

If the characteristic root is homogenous over the set of recurrent Ornstein-Uhlenbeck processes being convolved $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$, the equation can be expressed as

$$\begin{aligned} dX &= \alpha \left(\sum_{j=1}^{n} \mu_j - \sum_{j=1}^{n} X_j \right) dt + \sigma dz \\ &= \alpha \left(\mu - X \right) dt + \sigma dz \end{aligned}$$

where $\mu = \mu_1 + \mu_2 + \ldots + \mu_n = \sum_{j=1}^{n} \mu_j$ and $X(t) = X_1(t) + X_2(t) + \ldots + X_n(t) = \sum_{j=1}^{n} X_j(t)$

and hence, noting that

$$x(0) = x_1(0) + x_2(0) + \ldots + x_n(0)$$

the linear supposition is again an Ornstein-Uhlenbeck process with summed mean reversion levels and squared scale parameters and hence has the same Ornstein-Uhlenbeck Gaussian transition distribution

$$X(t) \sim N\left(x_0 e^{-\alpha t} + \mu(1 - e^{-\alpha t}), \quad \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \right)$$

where

$$\sigma = \begin{cases} \sqrt{\sum_{j=1}^{n} \sigma_j^2} & \text{if the } dz \text{ terms are independent} \\ \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij}} & \text{if the } dz \text{ terms are correlated} \end{cases}$$

If the time correlation parameter is not level across the component Ornstein-Uhlenbeck diffusion, then the process

$$dX = \left(\sum_{j=1}^{n} \alpha_j \mu_j - \sum_{j=1}^{n} \alpha_j X_j \right) dt + \sigma \; dz$$

has the transition distribution, which remains Gaussian,

$$X(t) \sim N\left(\sum_{j=1}^{n} x_0 e^{-\alpha_j t} + \sum_{j=1}^{n} \mu_j (1 - e^{-\alpha_j t}), \; \sigma^2 \sum_{j=1}^{n} (1 - e^{-2\alpha_j t}) / 2\alpha_j \right)$$

Notice that, as expected, adding homogenous additive processes does not produce a new form of process and hence does not create a new Ito-transform pair. These have been included to demonstrate this point as well as to further develop the analytical workings of additive processes.

Convolved Squared Brownian Motion

$$Y = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_i^2 = X_1^2 + X_2^2 + \ldots + X_n^2 \qquad \text{from} \qquad dX_i = \mu dt + \sigma dz$$

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From the square of Brownian motion $Y_i = X_i^2$, the stochastic differential equation is

$$dY_i = 2\mu\left(\frac{\sigma^2}{2\mu} - \sqrt{Y_i}\right)dt + 2\sigma\sqrt{Y_i}\,dz$$

which, as it stands, this does not convolve into a closed form stochastic differential equation because of the $\sqrt{Y_i}\,dt$ term. When underlying the Brownian motion is zero drift $\mu = 0$ (consistent with the next process) the stochastic differential equation simplifies to $dY_j = \sigma^2 dt + 2\sigma\sqrt{Y_j}\,dz$. The convolution of n such processes $Y = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_i^2$ gives $dY = n\sigma^2 dt + 2\sigma\sqrt{Y}\,dz$ which, inclusive of the convolution index, has two-parameters to match with the two coefficients of the reparameterization

$dY = m\,dt + s\sqrt{Y}\,dz$ where $m = n\sigma^2$ and $s = 2\sigma$

Proof

$$d(Y_1 + Y_2 + \ldots + Y_n) = (\sigma^2 + \sigma^2 + \ldots + \sigma^2)dt + 2\sigma\left(\sqrt{Y_1}\,dz_1 + \sqrt{Y_2}\,dz_2 + \ldots + \sqrt{Y_n}\,dz_n\right)$$

Since the stochastic increment is conditional on the current value X_t and the dz_j, terms are independent normal, $dz_j \sim N(0,dt)$, the stochastic part of the equation is

$$\begin{aligned}\sqrt{Y_1}\,dz_1 + \sqrt{Y_2}\,dz_2 + \ldots + \sqrt{Y_n}\,dz_n &\sim N(0, Y_1 dt) + N(0, Y_2 dt) + \ldots + N(0, Y_n dt)\\ &\sim N\left(0, (Y_1 + Y_2 + \ldots + Y_n)dt\right)\\ &\sim N(0, Y dt)\\ &\sim \sqrt{Y}\,dx\end{aligned}$$

giving the stochastic differential equation

$$dY = n\sigma^2 dt + 2\sigma\sqrt{Y}\,dz$$

The proof is complete.

The transition distribution for which is then gamma or central chi-square

$$\begin{aligned}Y &= \sum_{j=1}^{n} Y_j\\ &\sim \sum_{j=1}^{n} (\sigma^2 t)\Gamma(1/2,\ 2)\\ &\sim \sum_{j=1}^{n} \Gamma(1/2,\ 2\sigma^2 t)\\ &\sim \Gamma(n/2,\ 2\sigma^2 t)\\ &\sim (\sigma^2 t)\chi_n^2\end{aligned}$$

and reparameterizing $\sigma = s/2$ and $n = m/\sigma^2 = 4m/s^2$ gives

$$\begin{aligned}Y &\sim (\tfrac{1}{4}s^2 t)\chi^2_{4m/m^2}\\ &\sim \Gamma\left(\frac{2m}{s^2}, \frac{s^2}{2}t\right)\end{aligned}$$

that is, a gamma distribution with shape parameter $n/2 = 2m/s^2$ and scale parameter $2\sigma^2 t = s^2 t/2$. This is a scaled central chi-square on $n = 4m/s^2$ degrees of freedom, the scale factor being the underlying Brownian variance $\sigma^2 t = s^2 t/4$. If the backward variable of the underlying diffusion is non-zero, then the transition distribution of the convolution of squared transforms is non-central chi-square.

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Convolved Squared Ornstein-Uhlenbeck (Three-Parameter Square-Root)

It was shown that the square of an Ornstein-Uhlenbeck process mean reverting to zero was a scaled non-central chi-square distribution. Taking *n* Ornstein-Uhlenbeck processes that are independent and identically parameterized and convolving the squares of each of these processes gives another square-root process with an additional dimension to the parameter set. The definition of this process the sum of *n* independent and identically parameterized squared Ornstein-Uhlenbeck processes, i.e., the transformation

$$Y = \sum_{j=1}^{n} Y_j = \sum_{j=1}^{n} X_j^2$$

obtained from $dX_j = -\alpha X_j\, dt + \sigma dz_j$. From the previous results, for each component which are two parameter square root processes

$$dY_j = 2\alpha\left(\frac{\sigma^2}{2\alpha} - Y_j\right)dt + 2\sigma\sqrt{Y_j}\,dz_j$$

so are each scaled non-central chi-square on a single degree of freedom

$$Y \sim k\chi_1^2(\lambda)$$

The sum of the *n* two parameter square root processes has the stochastic differential

$$d(Y_1 + Y_2 + \ldots + Y_n) = 2\alpha\left(\frac{n\sigma^2}{2\alpha} - (Y_1 + Y_2 + \ldots + Y_n)\right)dt + 2\sigma\left(\sqrt{Y_1}\,dz_1 + \sqrt{Y_2}\,dz_2 + \ldots + \sqrt{Y_n}\,dz_n\right)$$

Like the convolved squared zero drift Brownian motions, since the stochastic increment is conditional on the current value X_t and the dz_j terms are independent normal, $dz_j \sim N(0,dt)$, the stochastic part of the equation is

$$\begin{aligned}\sqrt{Y_1}\,dz_1 + \sqrt{Y_2}\,dz_2 + \ldots + \sqrt{Y_n}\,dz_n &\sim N(0, Y_1 dt) + N(0, Y_2 dt) + \ldots + N(0, Y_n dt)\\ &\sim N\left(0, (Y_1 + Y_2 + \ldots + Y_n)dt\right)\\ &\sim N(0, Y dt)\\ &\sim \sqrt{Y}\,dx\end{aligned}$$

and the stochastic differential increment can be expressed simply as

$$dY = 2\alpha\left(\frac{n\sigma^2}{2\alpha} - Y\right)dt + 2\sigma\sqrt{Y}\,dz$$

Identically parameterized non-central chi-squares (and scaled non-central chi-squares for equivalent scale factors) are summable over the degrees of freedom and the non-centrality parameter. Therefore, the transition distribution is a scaled non-central chi-squared distribution on *n* degrees of freedom with a quantity of *n* of the non-centrality parameter of the two parameter square root process

$$\begin{aligned}Y = Y_1 + Y_2 + \ldots + Y_n &\sim k\chi_1^2(\lambda) + k\chi_1^2(\lambda) + \ldots + k\chi_1^2(\lambda)\\ &\sim k\chi_n^2(n\lambda)\end{aligned}$$

$$Y \sim k\chi_n^2(n\lambda)$$

The scale factor is the variance of the underlying Ornstein-Uhlenbeck process

$$k = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

and the non-centrality parameter is *n* times the ratio of the square of the mean to the variance of the underlying Ornstein-Uhlenbeck diffusion $n\lambda = n(x_0 e^{-\alpha t})^2 / k = n x_0^2 e^{-2\alpha t} / k = y_0^2 e^{-2\alpha t} / k$ where $y_0 = n x_0^2$. Reparameterizing to the required form

$$dY = a(m - Y)dt + s\sqrt{Y}\,dz$$

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where $a = 2\alpha$, $m = n\sigma^2 / 2\alpha$, and $s = 2\sigma$ are the parameters of the required process and $\alpha = a/2$, $n = 4am / s^2$, and $\sigma = s/2$ are the parameters of the underlying process. This gives

$$Y \sim k\chi^2_{4am/s^2}(\lambda)$$

where the scale factor is

$$k = \frac{s^2}{4a}(1 - e^{-at})$$

and the non-centrality parameter is *n* times the ratio of the square of the mean to the variance of the underlying Ornstein-Uhlenbeck processes, $\lambda = y_0 e^{-\alpha t} / k$. This process is the squared modulus, the square of the distance from the origin, of *n* recurrent Ornstein-Uhlenbeck processes that mean revert to zero.

Zero Degrees of Freedom Square-Root (Eccentric Square-Root)

When the three parameter square-root process has zero degrees of freedom (zero convolution index), which has been referred to in the earlier parts of this book as a purely eccentric square-root, the process mean reverts to zero and the transition distribution and density functions are non-central chi-square on zero degrees of freedom, i.e., a purely eccentric non-central chi-square. The stochastic differential increment is

$$dY = -aYdt + s\sqrt{Y}dz$$

which amusingly is the convolution of zero squared Ornstein-Uhlenbeck processes. The transition density function was obtained from solving the forward equation in earlier parts of this work showing that it is a valid process although the transition distribution is not analytic. It can also be thought of as the deconvolution of a central chi-square from a non-central chi-square distribution each with the same number of degrees of freedom, or the related square root processes with these transition distributions. In this case, the resulting distribution of the deconvolution is well defined statistically.

Further Convolution Ito-Transform Pairs

A process *Y* obtained from a transformation *g* of convolved additive processes is also labelled a convolution Ito-transformation pairs in this thesis.

$$Y = g\left(\sum_{i=1}^{n} h(X_i)\right) = g\left(h(X_1) + h(X_2) + ... + h(X_n)\right)$$

Constant Elasticity of Variance[34]

The constant elasticity of variance specification is a process that nests transient Ornstein-Uhlenbeck (with zero basis mean), three-parameter square-root, and geometric Brownian motion. It is defined by the stochastic differential

$$dX = \alpha X dt + \sigma X^{\beta/2} dz$$

Defining a power transformation of the diffusion variable $Y = X^{2-\beta}$ which has the derivatives

$Y_X = (2-\beta)X^{1-\beta} = (2-\beta)YX^{-1}$ and $Y_{XX} = (2-\beta)(1-\beta)X^{-\beta} = (2-\beta)(1-\beta)YX^{-2}$.

From the Taylor expansion for the total derivative

$$dY = Y_X(dX) + \tfrac{1}{2}Y_{XX}(dX)^2$$

[34]Schroder, M., (1989) Computing the Constant Elasticity of Variance Option Pricing Formula. *The Journal of Finance*, *The Journal of Finance*, Vol. 44 No. 1 (Mar., 1989), pp. 211-219.

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$$
\begin{aligned}
dY &= (2-\beta)X^{1-\beta}(\alpha X dt + \sigma X^{\beta/2} dz) + \tfrac{1}{2}(2-\beta)(1-\beta)X^{-\beta}\sigma^2 X^{\beta} dt \\
&= \alpha(2-\beta)X^{2-\beta}dt + \tfrac{1}{2}\sigma^2(2-\beta)(1-\beta)dt + \sigma(2-\beta)X^{1-\beta/2}dz \\
&= \alpha(2-\beta)Y dt + \tfrac{1}{2}\sigma^2(2-\beta)(1-\beta)dt + \sigma(2-\beta)\sqrt{Y}dz \\
&= \alpha(2-\beta)\left(\frac{\sigma^2(1-\beta)}{2\alpha} + Y\right)dt + \sigma(2-\beta)\sqrt{Y}dz
\end{aligned}
$$

This equation is a square-root process that is mean diverging, i.e., a transient square-root. In particular, since there are three-parameters, α, β, and σ, it is a three-parameter square-root process with a positive drift direction. The transition distribution is therefore a scaled non-central chi-square distribution. Letting

$$
2a = -\alpha(2-\beta), \qquad \frac{s^2 n}{2a} = -\frac{\sigma^2(1-\beta)}{2\alpha}, \qquad \text{and} \qquad 2s = \sigma(2-\beta)
$$

so that the square-root is parameterized as per the convolution Ito-transform pair with an Ornstein-Uhlenbeck process

$$
dY = 2a\left(\frac{ns^2}{2a} - Y\right)dt + 2s\sqrt{Y}dz
$$

The transition distribution is then

$$
Y = X^{2-\beta} \sim k\chi^2_n(\lambda)
$$

where the scale factor is the variance of the underlying Ornstein-Uhlenbeck process

$$
k = \frac{s^2}{2a}(1 - e^{-2a\,t})
$$

and the non-centrality parameter is $\lambda = y_0 e^{-2\alpha t}/k$ where $a = -\alpha(1-\beta/2)$, $s = \sigma(1-\beta/2)$, and $n = (1-\beta)(1-\beta/2)$. This gives the $(2-\beta)^{th}$ power of the transition distribution of the constant elasticity of variance process which is

$$
Y = X^{2-\beta} \sim k\chi^2_{(1-\beta)/(1-\beta/2)}(\lambda)
$$

$$
\text{with } k = \frac{\sigma^2(1-\beta/2)}{2\alpha}\left(e^{2\alpha(1-\beta/2)t} - 1\right) \qquad \text{and} \qquad \lambda = y_0 e^{2\alpha(1-\beta/2)t}/k
$$

By Ito-transform pairs, the transition distribution of the constant elasticity of variance process is the inverse power transformation of the scaled non-central chi-square distribution, $X = Y^{1/(2-\beta)}$

$$
X \sim k^{1/(2-\beta)}\left(\chi^2_{(1-\beta)/(1-\beta/2)}(\lambda)\right)^{1/(2-\beta)}
$$

The Jacobian of the transformation is

$$
|J| = \frac{dy}{dx} = (2-\beta)x^{1-\beta}
$$

If $f_Y(\cdot)$ and $F_Y(\cdot)$ are the density and distribution functions of the scaled non-central chi-square, then the distribution function of the constant elasticity of variance process is

$$
F_X(x) = F_Y(x^{2-\beta})
$$

while the density function is

$$
f_X(x) = f_Y(x^{2-\beta})|J| = (2-\beta)x^{1-\beta} f_Y(x^{2-\beta})
$$

Descaling the distribution by k using the standard scaling transformation of $Y = kX$, i.e.,

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$f_Y(y) = D_y F_X(y/k) = k^{-1} f_X(y/k)$ where $D_y = d/dy$

gives

$F_X(x) = F_Y(x^{2-\beta}/k)$ and $f_X(x) = f_Y(x^{2-\beta}/k)\,|J| = k^{-1}(2-\beta)x^{1-\beta} f_Y(x^{2-\beta}/k)$

where $f_Y(\cdot)$ and $F_Y(\cdot)$ are the density and distribution functions of the non-scaled non-central chi-square. Since this is a function of a non-central chi-square distribution, it can be evaluated with a call to any software program that can compute non-central chi-square functions or Bessel functions.

Square-Root of Three-Parameter Square-Root - Bessel process

The Bessel process is defined as the square-root of a three-parameter square-root process, a convolution of squared Ornstein-Uhlenbeck processes, that mean revert to zero. This is the transformation $Y = \sqrt{X}$ obtained from the process with stochastic differential equation $dX = \alpha(\mu - X)dt + \sigma\sqrt{X}\,dz$. The first and second order derivatives of the transformation are $Y_X = 1/2\sqrt{X}$ and $Y_{XX} = -1/4X^{3/2}$. From the Ito formula, the stochastic differential increment is then

$$\begin{aligned} dY &= Y_X\,dX + \tfrac{1}{2}Y_{XX}(dX)^2 \\ &= \frac{1}{2\sqrt{X}}\left(\alpha(\mu - X)dt + \sigma\sqrt{X}\,dz\right) + \tfrac{1}{2}\left(-\frac{1}{4X\sqrt{X}}\right)\sigma^2 X dt \\ &= \tfrac{1}{2}\alpha\left(\frac{\mu - \frac{1}{4}\sigma^2/\alpha}{Y} - Y\right)dt + \tfrac{1}{2}\sigma\,dz \end{aligned}$$

Reparameterizing with $b = \alpha/2$, $\alpha = \mu - \sigma^2/4\alpha$, and $s = \sigma/2$ gives

$$dY = \left(\frac{a}{Y} - bY\right)dt + s\,dz = b\left(\frac{a/b}{Y} - Y\right)dt + s\,dz$$

It was shown that the process upon which this is based, the three-parameter square-root, has transition distribution $X = h^{-1}(Y) = Y^2$

$$X \sim k\chi^2_n(n\lambda)$$

where the scale factor is $k = \sigma^2(1 - e^{-2\alpha t})/2\alpha$ and the non-centrality parameter $n\lambda = y_0 e^{-2\alpha t}/k$. The transition density of the square-root of this process is therefore the square-root $Y = \sqrt{X}$ of a scaled non-central chi-square distribution

$$Y = \sqrt{X} \sim \sqrt{k\chi^2_n(n\lambda)}$$

with the parameters as above. Reparameterizing by solving the three simultaneous equations

$$a = \tfrac{1}{2}\alpha\mu - \tfrac{1}{8}\sigma^2, \quad b = \tfrac{1}{2}\alpha, \text{ and } \quad s = \tfrac{1}{2}\sigma$$

gives

$$\alpha = 2b, \quad \sigma = 2s, \quad \text{and} \quad \mu = (a + \tfrac{1}{2}s^2)/b$$

The transition distribution is then a square-root of a scaled non-central chi-square distribution (a scaled non-central chi-distribution) with scale $\sqrt{k}$ where $k = s^2(1 - e^{-4bt})/b$, degrees of freedom $1 + 2a/s^2$, and non-centrality parameter $\lambda = y_0^2 e^{-4bt}/k$,

$$Y \sim \sqrt{k}\,\chi_{1+2a/s^2}(\lambda)$$

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Table of Ito Transform Pairs

As far as I can ascertain, these are the most important processes of first order diffusion with transition distributions of a known statistical form and closed form formulae for the transition density, up to a normal or non-central chi-square[35] table lookup or function call. All of these are obtainable from the basic model of a cumulative Gaussian basis, standard Brownian motion. Simulation of any of the processes in column one are implemented by simulating the underlying Brownian motion (or Brownian motions) and transforming the simulated path (or paths) with the transformation of column two to the requisite path. The process for first order logistic diffusion is omitted because of the inconvenient form of the stochastic differential.

Each of the processes not containing convolutions, or summations (Σ or $\int$), have forward equations that are transformable to the standard heat equation $f_t = \frac{1}{2} f_{xx}$. Those with limits to dt can alternatively be obtained by deconvolution. Processes that involve summations may be transformable to an equation of a simpler form but not the standard heat equation. Transformation of the forward and backward equations is the more general calculus of diffusion and includes the calculus pertaining to diffusions of arbitrary order and generalized diffusion basis.

[35] or modified Bessel function of the first kind.

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Table 15: Ito-transform and convolution Ito-transform pairs

Differential process $dY(t)$	**Transformation** $Y(t)=h\big(X(t)\big)$	**from the differential process** $dX(t)$
$dY(t)=dB(t)$	$Y(t)=X(t)$	$dX(t)=dB(t)$
$dY(t)=\mu dt+\sigma dB(t)$	$Y(t)=\mu t+\sigma X(t)$	$dX(t)=dB(t)$
$dY(s)=-\alpha Y(s)ds+\sigma dB(s)$	$Y(s)=\sigma e^{-\alpha s}X\left(\frac{e^{2\alpha s}-1}{2\alpha}\right)$	$dX(t)=dB(t)$
$dY(t)=\alpha\big(\mu-Y(t)\big)dt+\sigma dB(t)$	$Y(t)=(\mu/\alpha)t+X(t)$	$dX(t)=-\alpha X(t)dt+\sigma dB(t)$
$dY(t)=\frac{\mu}{Y(t)}dt+\sigma dB(t)$	$Y(t)=\sqrt{\sum_{j=1}^{n}X_j(t)^2}$	$dX_j(t)=\sigma dB_j(t)$
$dY(t)=\alpha\left(\frac{\mu}{Y(t)}-Y(t)\right)dt+\sigma dB(t)$	$Y(t)=\sqrt{\sum_{j=1}^{n}X_j(t)^2}$	$dX_j(t)=-\alpha X_j(t)dt+\sigma dB_j(t)$
$dY(t)=\sigma Y(t)dB(t)$	$Y(t)=\exp\big(X(t)\big)$	$dX(t)=-\frac{1}{2}\sigma^2dt+\sigma dB(t)$
$dY(t)=\mu Y(t)dt+\sigma Y(t)dB(t)$	$Y(t)=\exp\big(X(t)\big)$	$dX(t)=\big(\mu-\frac{1}{2}\sigma^2\big)dt+\sigma dB(t)$
$dY(t)=\alpha Y(t)\log Y(t)dt+\sigma Y(t)dB(t)$	$Y(t)=\exp\big(X(t)\big)$	$dX(t)=\alpha\big(-\frac{1}{2}\sigma^2/\alpha-X(t)\big)dt+\sigma dB(t)$
$dY(t)=\alpha\left(\mu-\log Y(t)\right)Y(t)dt+\sigma Y(t)dB(t)$	$Y(t)=\exp\big(X(t)\big)$	$dX(t)=\alpha\big(\mu-\frac{1}{2}\sigma^2/\alpha-X(t)\big)dt+\sigma dB(t)$
$dY(t)=\mu\left(\frac{\sigma^2}{4\mu}-\sqrt{Y(t)}\right)dt+\sigma\sqrt{Y}dB(t)$	$Y(t)=X(t)^2$	$dX(t)=-\mu dt+\sigma dB(t)$
$dY(t)=\alpha\left(\frac{\sigma^2}{4\alpha}-Y(t)\right)dt+\sigma\sqrt{Y(t)}dB(t)$	$Y(t)=X(t)^2$	$dX(t)=-\alpha X(t)dt+\sigma dB(t)$
$dY(t)=\alpha\left(\frac{\sigma^2}{4\alpha}+\mu\sqrt{Y(t)}-Y(t)\right)dt+\sigma\sqrt{Y(t)}dB(t)$	$Y(t)=X(t)^2$	$dX(t)=\alpha\big(\mu-X(t)\big)dt+\sigma dB(t)$
$dY(t)=\alpha\left(\frac{n\sigma^2}{4\alpha}-Y(t)\right)dt+\sigma\sqrt{Y(t)}dB(t)$	$Y(t)=\sum_{j=1}^{n}X_j(t)^2$	$dX_j(t)=-\alpha X_j(t)dt+\sigma dB_j(t)$
$dY(t)=-\alpha Y(t)dt+\sigma\sqrt{Y(t)}dB(t)$	$Y(t)=\int_0^{dt}X_j(t)^2dj$	$dX_j(t)=-\alpha X_j(t)dt+\sigma dB_j(t)$
$dY(t)=\frac{1}{4}\sigma^2dt+\sigma\sqrt{Y(t)}dB(t)$	$Y(t)=X(t)^2$	$dX(t)=\sigma dB(t)$
$dY(t)=\frac{1}{4}n\sigma^2dt+\sigma\sqrt{Y(t)}dB(t)$	$Y(t)=\sum_{j=1}^{n}X_j(t)^2$	$dX_j(t)=\sigma dB_j(t)$
$dY(t)=\sigma\sqrt{Y(t)}dB(t)$	$Y(t)=\int_0^{dt}X_j(t)^2dj$	$dX_j(t)=\sigma dB_j(t)$

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FOUNDATIONS II: CONTINUOUS TIME PROCESS WITH RANDOM GENERATORS

Introduction

This section, FOUNDATIONS II, discusses variables that are measured, like those of FOUNDATIONS I, on a continuous diffusion variable. Unlike FOUNDATIONS I, the cumulative contributions of normal variables to the aggregate diffusion variable occur at random discrete points in time. As first order diffusions considered by FOUNDATIONS I are the continuous convolution of infinitesimal basis variables, the equivalents in FOUNDATIONS II are the discrete convolution [of continuous variables] of non-infinitesimal basis variables. The number of convolved basis components onto the diffusion variable is determined by the generator which is a discrete counting process in continuous time. These processes are considered continuous time since the process generating the discrete events is defined as such and may produce events on any point in the time continuum of any interval.

There are many reasons why the inclusion of an additional element of randomness, in the time variable in this context, may provide utility to mathematical finance. The most important of these, it is argued, include modeling transaction data, modeling the arrival of news, and the subsequent price volatility as it is disseminated into the market. Further, the modeling may produce mathematical constructs that provide utility in their own right, independently of mathematical finance. In particular, the techniques and processes of FOUNDATIONS II generalize equally spaced time sequences to those that have random spacing between successive variables in diffusion aggregations.

In modeling microstructure transaction data, the times at which transactions take place are random and may conceivably give rise to randomly spaced sequences of observed time invariant homogenous variables. The processes that are discussed in FOUNDATIONS II are well suited to data generated from such variables for a number of reasons. Such reasons include an exact correspondence between the data that are recorded and the assumed random variables generating the data[36]. Therefore, there are no unobserved components. This type of data is considered high-frequency and convergence or passages to limiting processes would generally occur relatively quickly. Estimation and modeling of compound sum diffusions, from transaction data, is convenient and logical and is consistent with ordinary diffusion in the convergence limit. The focus of the practical aspects of FOUNDATIONS II is directed at microstructure transaction data due to the direct applicability and wide scope to provide utility to applied mathematical finance and security modeling.

In summary, random occurrence diffusion provides an additional class of diffusion that adds sophistication to ordinary diffusion and exponential diffusion. The particular form of the increased sophistication is a special form of generalization of conventional diffusion in that it is a special form of stochastic parameter diffusion whereby all the cumulants of the diffusion sum stochastically at the same rate. This added sophistication has exactly nil redundancy in the context of modeling transaction data. The very existence of 'micro-structure' effects are an indicator that financial markets are not statistically fractal and that the modeling requires a process with additional sophistication.

High-Frequency Microstructure Transaction Data

High frequency transaction level data is now available to the financial markets due to advances in the technology required to efficiently record and store the vasts amounts of electronic information that comprise such data. On the Australian Stock Exchange (ASX), traders and market makers transact electronically through links to the SYCOM IV system on an ITS workstation. This system lists current bid, ask and last transaction prices together with the volume on the posted bid and asks, and the most recent trade. The data is updated over second by second time intervals. These intervals are referred to as the tick length. The technological advances in communication systems, computing, and data-warehousing have meant that it has become possible to record and store 'tick-by-tick' data. Such data is stored on continually updated databases, one example being that maintained jointly by the ASX and the Securities Industry Research Centre of Asia-Pacific (SIRCA).

The availability of such data means that researchers are able to zoom in and observe the dynamics at a microscopic level. This form of data is abundant and vast and is yet to be utilized in a manner that allows it, and the

[36] The idea of compound Poisson diffusions to model transaction data is the idea of the author.

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effort that has been made to record it, reach its full potential to contribute to the efficiencies of the financial markets through the greater understanding of market phenomena. This is particularly relevant to market phenomena pertaining to the statistical analysis of security dynamics and the implications for technically accurate and scientifically prudent option quantification and decision making.

In comparison with the estimation techniques concerning daily summary statistics, high frequency transaction microstructure data differs in that it represents a much higher sampling and recording frequency of an observed price variable than that of open and close and the daily maxima and minima. From one statistical and economic point of view, a true market price only exists when a transaction is made; in between the discrete transaction events it is only notional and, if it existed, somewhere between the bid and the ask. Considering transaction data from this theoretical viewpoint, it can be asserted that transaction data is complete regarding the market price signal, rather than censored information such as daily summary statistics [open-close-high-low] or a high frequency sampling of a continuous signal. It is asserted in BOOK 3 that everything in the physical (and digital) world is discrete once some scale threshold is reached and that the granularity/scale threshold in this context is determined by the frequency of the transactions.

If human response and the interaction between the bid and the ask of the supply and demand for securities determines the transaction events according to a Poisson process and the price volatility from transaction to transaction is not dependent upon the duration between these transactions, whether it be the scale threshold of discreteness of the diffusion or the rate at which a notional continuous signal is sampled, then a compound Poisson diffusion is appropriate to model this form of data.

A compound Poisson diffusion, as a stochastic process, comprises two independent diffusions over each of the two dimensions; time and value where diffusion in this context refers to the cumulative aggregation of random variables. The estimation of such a process begins with estimating the Poisson process of the transaction times which is then followed by the estimation of the components of the second dimension, as well as diagnostics to assess the suitability of such a formulation given an observed series. These are considered following the discussion of the compound Poisson process that generates a comound Poisson diffusion.

In modeling the generator, FOUNDATIONS II models the first passage time of the Poisson process completely, in its role as the diffusion generator, including maximum likelihood estimators for all sorts of statistics relating to exponential transaction times; all of the functions that are of interest and utility to one knowledgeable in the art of mathematical statistics.

In an empirical setting, the parameters are estimated at the transaction level, estimated from the maximum likelihood estimators that are derived and provided in the section regarding transaction data. From here, the parameters are combined into the premium calculation for the limiting distribution by letting the transaction intensity λ play the role of h in discrete diffusion, a parameter that would otherwise (if it was not also a rate intensity) simply adjust for the time units adopted. If the forward transition interval is not long enough for the Central Limit Theorem to remove o(h) terms (where $\lambda \propto 1/h$), then the Poisson rate intensity can be summed out of the condition premium with a Poisson density weighted sum.

THE COMPOUND MODEL

A compound model[37] is a random sum of random variables. The discrete count of variables in the sum is the primary distribution while the variables being summed are the secondary variables. The secondary variables that are summed are here assumed to be both independent and homogenous. The primary distribution is discrete as it is used to generate and count discrete additions of the secondary distribution which may be either continuous or discrete, as may be appropriate in a particular application. The primary variable, at time t, is denoted by $N(t)$ and the j^{th} secondary distribution by X_j. The variable of the compound model is denoted by $S(t)$.

The equation of the random sum process, relaing the random sum to the primary and secondary variables (the diffusion variable to the generator and basis variables), is

[37] A compound Poisson model is standard to the actuarial pedagogy for portfolios of short-tailed (usually one-year) non-life business. A text that gives the best contemporary coverage of this model, and the roots of much of this work (particularly the emphasis on generating functions), is the first edition of
Klugman, S. A., Panjer, H. H. and Willmot, G. E., (1998) *Loss Models*. John Wiley & Sons.
It is a suggestion of this thesis to apply a Poisson process (an integrated Poisson distribution) to generate the high frequency compound sum, as a diffusion and analoge to Brownian motion, in order to model high frequency transaction data.

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$$S(t) = X_1 + X_2 + \ldots + X_{N(t)} = \sum_{j=1}^{N(t)} X_j$$

The equivalent algebra of the density of the random variable equation is an $N(t)$-fold convolution of the secondary distribution density functions, a multiple integral containing a random number $N(t)$ of convolution integrals

$$f_{S(t)} = f_{X_1} * f_{X_2} * \ldots * f_{X_{N(t)}}$$

so that the summation operator over the group of random variables maps to the convolution operator over the semi-group of probability density functions. The transition distribution is the distribution of the compound sum and defines the total time aggregation of the secondary random variables that successively convolve onto the process as it moves through time. The moment generating function of the compound process is

$$M_{S(t)}(\theta) = E\left(e^{\theta S(t)}\right) = E\left(e^{\theta X_1 + \theta X_2 + \ldots + \theta X_{N(t)}}\right) = E\left(e^{\theta X_1} e^{\theta X_2} \ldots e^{\theta X_{N(t)}}\right)$$

If the set of secondary variables in the random sum are independent, the moment generating function reduces to

$$M_{S(t)}(\theta) = E\left(e^{\theta X_1} e^{\theta X_2} \ldots e^{\theta X_{N(t)}}\right) = E\left(e^{\theta X_1}\right) E\left(e^{\theta X_2}\right) \ldots E\left(e^{\theta X_{N(t)}}\right) = \prod_{j=1}^{N(t)} E\left(e^{\theta X_j}\right) = \prod_{j=1}^{N(t)} M_{X_j}(\theta)$$

So when the variables are independent and form an orthogonal basis, the summation of the secondary random variables corresponds to the multiplication over moment generating functions of the secondary variables. Alternatively, taking logarithms for the cumulant generating function gives

$$K_{S(t)}(\theta) = \log M_{S(t)}(\theta) = \sum_{j=1}^{N(t)} \log M_{X_j}(\theta) = \sum_{j=1}^{N(t)} K_{X_j}(\theta)$$

which shows that the cumulant generating function of the compound sum is the sum of the cumulant generating functions of the secondary variables. This is a defining feature of diffusion, in particular, first order unit-root diffusion. The first moment of the compound variable is given by

$$E\left(S(t) \mid N(t)\right) = E\left(X_1 + X_2 + \ldots + X_{N(t)}\right)$$

which contains $N(t)$ cumulative contributions of secondary variables X_j with the additional source of variation in the primary distribution $N(t)$ generating and counting the convolved contributions. Conditioning on $N(t)$ so that the sum contains a constant number of terms, the mean of the compound sum is given by the average of the mean of the conditional distribution

$$E\left(S(t)\right) = E\left(E[X_1 + X_2 + \ldots + X_{N(t)} \mid N(t)]\right) = E(\mu_1 + \mu_2 + \ldots + \mu_{N(t)}) = E\left(\sum_{j=1}^{N(t)} \mu_j\right)$$

The stochastic variation remaining in the cumulative expectation, that in excess of an ordinary diffusion, arises from the random number of secondary means. If the moments among the secondary variables are homogenous so that $\mu_j = \mu$, the compound sum contains a quantity $N(t)$ of a homogenous quantity

$$E\left(S(t)\right) = E(\mu + \mu + \ldots + \mu) = E\left(\sum_{j=1}^{N(t)} \mu\right) = E\left(\mu N(t)\right) = \mu E\left(N(t)\right)$$

The second central moment, the variance of the unconditional compound sum, is given by the variance of the mean plus the mean of the variance of the conditional distribution

$$Var\left(S(t)\right) = Var\left(E[S(t) \mid N(t)]\right) + E\left(Var[S(t) \mid N(t)]\right) = Var(\mu_1 + \mu_2 + \ldots + \mu_{N(t)}) + E(\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_{N(t)}^2)$$

The stochastic variation in the variance of the sum of the secondary means, the first term, and the stochastic variation in the mean of the sum of the secondary variances, the second term, again stemming from the random number N of secondary distributions in the compound sum. If both the first and second moments of the secondary variables are homogenous, $(\mu_j, \sigma_j^2) = (\mu, \sigma^2)$, the summations inside the variance and expectation operators again reduce to the quantity $N(t)$ of homogenous terms. The only variable remaining that is random being the generator (the primary distribution) $N(t)$

$$Var\left(S(t)\right) = Var\left(N(t)\mu\right) + E\left(N(t)\sigma^2\right) = \mu^2 Var\left(N(t)\right) + \sigma^2 E\left(N(t)\right)$$

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Similarly, the third central moment, is given by the mean of the skew plus the skew of the mean plus the covariance between the variance and mean of the conditional distribution, making a note of how the relationship between the mean and variance of the conditional distribution adds to the skewness of the unconditional distribution,

$$\begin{aligned} Skew\big(S(t)\big) &= E\left(Skew[S(t)\,|\,N(t)]\right) + 3Cov\left(E[S(t)N(t)],\, Var[S(t)\,|\,N(t)]\right) + Skew\left(E[S(t)\,|\,N(t)]\right) \\ &= Skew(\mu_1+\mu_2+\ldots+\mu_{N(t)}) + 3Cov[(\mu_1+\mu_2+\ldots+\mu_{N(t)}),(\sigma_1^2+\sigma_2^2+\ldots+\sigma_{N(t)}^2)] + E(\kappa_1^3+\kappa_2^3+\ldots+\kappa_{N(t)}^3) \end{aligned}$$

where κ_j^3 is the third central moment of the j^{th} second variable in the sum. If the first three moments of the secondary variables are homogenous, the sums simplify to

$$\begin{aligned} Skew\big(S(t)\big) &= Skew\big(N(t)\mu\big) + 3Cov\left(N(t)\mu,\ N(t)\sigma^2\right) + E\left(N(t)\kappa^3\right) \\ &= \mu^3 Skew\big(N(t)\big) + 3\mu\sigma^2 Var\left(N(t)\right) + \kappa^3 E\left(N(t)\right) \end{aligned}$$

To proceed from here, the primary counting distribution and the secondary distributions must be specified. This work almost exclusively specifies the discrete counting distribution, the distribution of the basis for the discrete generator, as a Poisson variable. The integration of the Poisson basis function of the generator results in the Poisson process. I had intended to include a full mathematical analysis of generator bases distributed as binomial and negative binomial variables but have instead presented a concise discussion of these in the latter parts of FOUNDATIONS II. The Poisson process is included for detailed analysis since the results are algebraically analytic and well manageable due to the full set of level cumulants of the Poisson distribution. The secondary distributions are Gaussian except in the appendices for random walks (**A Compendium of Random Walks**) whereby the Poisson random walks have Bernoulli secondary distributions. Normal secondary distributions are used almost exclusively for two reasons. Firstly, these are the domain of attraction of the Central Limit Theorem which applies to sums and averages of any set of random variables with real and finite mean and variance. Secondly, I have wanted to keep the process discussed in FOUNDATIONS II consistent with those of FOUNDATIONS I, namely Gaussian and first order. Any generalization of the secondary distribution involve what this thesis refers to as the generalized Gaussian distribution, which has no closed form transition distribution but can be calculated numerically by numerical inversion of the moment generating function or by solving the relevant forward and backward equations.

The purpose of this discussion was to provide a rudimentary introduction to the theory of compound random variables and to the exposition of the remainder of the first book including much of (over half) of the Compendium of Random Walks. This sort of development is fundamental to compound random variables and is how the development proceeds, as per the mathematical mode of analysis, for the all of the processes discussed in FOUNDATIONS II.

One should note that the diffusion processes from FOUNDATIONS I are also compound random variables in which the secondary distributions sum (compound) continuously rather than at random and discrete points. Diffusions are, by definition, continuous sums or products (or somewhere in between) of independent sequences of basis variables. The processes discussed here are also sums, and later extended to include products, of independent basis variables but the sums (or products) increment randomly in time, at points determined by a Poisson process, rather than continuously. In the limit, the two forms of accumulations of independent random variables are equivalent. In the context of diffusion, the primary variable is referred to as the generator and the secondary variables as the components of the diffusion basis.

THE POISSON PROCESS

The Generator–the Poisson Process

The focus of this thesis is random phenomena over time and, in particular, many cumulative random moments over time. FOUNDATIONS I of BOOK 1 described processes that cumulate with random movements over successive increments of fixed Newtonian differential width. In contrast, this part of BOOK I, FOUNDATIONS II, considers processes whereby movements may or may not occur over differential increments according to Bernoulli random trials and that cumulatively, the aggregated movements of the process convolve according to an integrated Poisson basis variable, the Poisson process.

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The Bernoulli distributed variable, the distribution that determines whether an increment occurs or does not occur over any infinitesimal differential time increment $N(dt)$, is defined by the single probability $e^{-\lambda\, dt}$, that is,

$$N(dt) \sim B(1, e^{-\lambda dt})$$

Letting the transition density of this variable be $p_n(t)$ where n is the forward diffusion variable and t is the forward time variable.

$$p_n(dt) = \begin{cases} e^{-\lambda dt} = 1 - \lambda dt + o(dt) & if\ n = 0 \\ 1 - e^{-\lambda dt} = \lambda dt + o(dt) & if\ n = 1 \end{cases}$$

and $\quad o(dt) \quad if\ n > 1$

or

$$p_n(dt) = (e^{-\lambda dt})^n (1 - e^{-\lambda dt})^{1-n} = (1 - \lambda dt)^n (\lambda dt)^{1-n} + o(dt) \quad n \forall N[0,1]$$

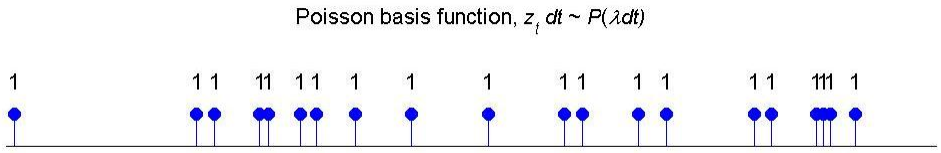
Generator Recurrence, Diffusion Equation, and its Resolution

Letting $N(t)$ be the diffusion variable and z_t be the Poisson basis variable of a Poisson diffusion equation. The scaled basis variable, λdt , is uncorrelated Poisson distributed with mean λdt

$$z_t dt \sim P(\lambda dt)$$

Due to the differential mean, this distribution is also Bernoulli. As such, this variable is either zero or one at any instantaneous point in time, a continuum of zeros with ones interspersed at random times over any non-infinitesimal period

$$z_t dt = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1 - \lambda dt \end{cases} \qquad \text{for } t \geq 0$$



Over any infinitesimal increment, dt , the scaled basis variable is either one or zero and the Poisson diffusion variable $N(t)$ either increments by one are remains unchanged. Mathematically, this can be stated as the Poisson process **forward recurrence**

$$N(t + dt) = z_t dt[N(t) + 1] + (1 - z_t dt)N(t) + o(dt)$$

Rearranging terms gives

$$N(t + dt) - N(t) = z_t dt + o(dt)$$

and dividing both sides by dt for

$$\frac{N(t + dt) - N(t)}{dt} = z_t + \frac{o(dt)}{dt}$$

which, upon taking limits so that $o(dt) / dt \to 0$, results in the **differential form of the unit root Poisson diffusion equation**

$$\frac{dN(t)}{dt} = z_t$$

In practical terms, because the basis variable is a discrete random variable, this can be expressed as

$$dN(t) = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } 1 - \lambda dt \end{cases}$$

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which is follows directly from the fact that $z_t\,dt$ is 1 with probability λdt and 0 with probability $1 - \lambda dt$. Integrating the differential form of the diffusion equation from zero to time t

$$\int_0^t dN(y) = \int_0^t z_y dy$$

$$N(t) = N(0) + \int_0^t z_y dy$$

which is the Poisson diffusion equation in resolved form. $N(0)$ is the backward variable. It is assumed for the time being that that $N(0)$ is zero since it is not required at this stage, so that the **integral form of the Poisson diffusion equation** is

$$N(t) = \int_0^t z_y dy$$

The moment generating function of the scaled basis variable, $z_y\,dy$, is

$$M_{z_y dy}(\theta) = \exp\left(\lambda dt(e^{\theta} - 1)\right)$$

resulting in the transition moment generating function of the resolved Poisson diffusion variable as

$$M_{\int_0^t z_y dy}(\theta) = \exp\left(\int_0^t \lambda dy(e^{\theta} - 1)\right) = \exp\left(\lambda t(e^{\theta} - 1)\right)$$

that is,

$$M_{N(t)}(\theta) = \exp\left(\lambda t(e^{\theta} - 1)\right)$$

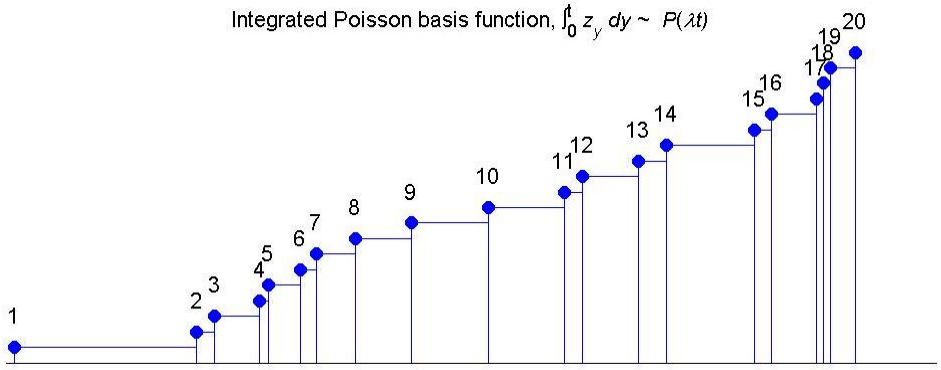
which is the generating function of a Poisson variable with mean $\lambda\, t$. The **transition distribution** is hence

$$N(t) \sim P(\lambda t)$$

Noting that the integral of

$$N(t) = \int_0^t z_y dy$$

is an integration of a series of zeroes and ones, i.e., an integration of **ones interspersed among a continuum of zeroes)**, the total number of ones in the summation equation to the value of the integrated Poisson basis variable and an infinite number of remaining zeroes. The integrated basis, the Poisson process itself, $N(t) \times 1 + [t - N(t)] \times N(t)$ is consequently an accumulated basis of a continuum of zeroes with ones scattered throughout occurring at exponentially spaced increments (Bernoulli on infinitesimals) in time



It can be seen graphically above that each saltus is of fixed length. The scale isn't shown but each step is of size one. There are 20 Poisson increments so the upper value of the Poisson diffusion variable in the graph is also 20.

For comparative purposes, it may be beneficial to compare the mathematics regarding the differential dynamics of the diffusion equation, which is a random variable, with that of the transition density, which assigns probability to the diffusion variable.

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Mathematics of diffusion variable

$$N(t+dt) = [N(t)+1]dN(t) + N(t)[1-dN(t)] + o(dt)$$
$$N(t+dt) = z_t dt[N(t)+1] + (1-z_t dt)N(t) + o(dt)$$
$$N(t+dt) = z_t dtN(t) + z_t dt + N(t) - z_t dtN(t) + o(dt)$$
$$N(t+dt) - N(t) = z_t dt + o(dt)$$
$$\frac{N(t+dt)-N(t)}{dt} = z_t + \frac{o(dt)}{dt}$$
$$\frac{dN(t)}{dt} = z_t \qquad z_t dt \sim P(\lambda dt)$$

Mathematics of transition density

$$p_n(t+dt) = \Pr[dN(t)=1]p_{n+1}(t) + \{1-\Pr[dN(t)=0]\}p_n(t) + o(dt)$$
$$p_n(t+dt) = \lambda dt p_{n+1}(t) + (1-\lambda dt)p_n(t) + o(dt)$$
$$p_n(t+dt) = \lambda dt p_{n+1}(t) + p_n(t) - \lambda dt p_n(t) + o(dt)$$
$$p_n(t+dt) - p_n(t) = \lambda dt\big(p_{n+1}(t) - p_n(t)\big) + o(dt)$$
$$\frac{p_n(t+dt)-p_n(t)}{dt} = \lambda\Big(p_{n+1}(t) - p_n(t)\Big) + \frac{o(dt)}{ddt}$$
$$\frac{dp_n(t)}{dt} = \lambda\Big(p_{n+1}(t) - p_n(t)\Big) \qquad p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

Forward and Backward Equations by Reference to the Chapman-Kolmogorov Equation

On the dt scale, the Poisson process is Bernoulli. The Bernoulli distributed variable, the distribution that determines whether an increment occurs or does not occur over any infinitesimal differential time increment $N(dt)$, is defined by the single probability $e^{-\lambda dt}$, that is,

$$N(dt) \sim B(1, e^{-\lambda dt})$$

Letting the transition density of this variable be $p_n(t)$ where n is the forward diffusion variable and t is the forward time variable.

$$p_n(dt) = \begin{cases} e^{-\lambda dt} = 1 - \lambda dt + o(dt) & if\ n = 0 \\ 1 - e^{-\lambda dt} = \lambda dt + o(dt) & if\ n = 1 \end{cases}$$

and $o(dt) \quad if\ n > 1$

or

$$p_n(dt) = (e^{-\lambda dt})^n (1-e^{-\lambda dt})^{1-n} = (1-\lambda dt)^n (\lambda dt)^{1-n} + o(dt) \quad n \forall N[0,1]$$

Considering both infinitesimal and non-infinitesimal increments. An intuitive relationship linking incremental forward steps to the process at time t is the differential forward recursion equation of the Poisson process is

$$p_n(t+dt) = p_{n-1}(t)p_1(dt) + p_n(t)p_0(dt) + o(dt) = p_{n-1}(t)\lambda dt + p_n(t)(1-\lambda dt) + o(dt)$$

which is the Chapman-Kolmogorov equation, in this case for a discrete two point variable. Subtracting $p_n(t)$ from both sides and dividing by dt gives

$$\frac{p_n(t+dt)-p_n(t)}{dt} = p_{n-1}(t)\lambda - p_n(t)\lambda + \frac{o(dt)}{dt}$$

which, by letting dt approach zero, gives the Kolmogorov forward differential equation for the Poisson process, an equation in t relating successive increments in the forward discrete diffusion variable n;

$$\frac{\partial p_n(t)}{\partial t} = \lambda\Big(p_{n-1}(t) - p_n(t)\Big)$$

with the initial conditions $p_0(0) = 1$ and $p_n(0) = 0$ for $n > 0$ and $p_n(t) = 0$ for $n < 0$ so that

$$\frac{\partial p_0(t)}{\partial t} = \lambda p_0(t)$$

The forward equation can alternatively be expressed as

$$\frac{\partial p_n(t)}{\partial t} + \lambda p_n(t) = \lambda p_{n-1}(t)$$

or in differentially factorized form

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$$\frac{\partial}{\partial t}\left(e^{\lambda t}\ p_n(t) \right) = \lambda e^{\lambda t} p_{n-1}(t)$$

The backward equation requires a variable for an initial value, the backward variable. Letting the transition function with this variable, an initial diffusion value of n_0 at time 0, be denoted $p_n(t \mid n_0)$. The backwards recurrence relation of the process is

$$p_n(t - dt \mid n_0) = p_n(t \mid n_0 + 1)\lambda dt + p_n(t \mid n_0)(1 - \lambda dt) + o(dt)$$

The backward equation is then, after subtracting both sides by $p_n(t \mid n_0)$ and dividing by *dt* and letting *dt* approach zero, by the definition of the backward derivative,

$$-\frac{\partial p_n(t \mid n_0)}{\partial t} = \lambda\left(p_n(t \mid n_0 + 1) - p_n(t \mid n_0) \right)$$

The backward equation can alternatively be expressed as

$$-\frac{\partial p_n(t \mid n_0)}{\partial t} + \lambda p_n(t \mid n_0) = \lambda p_n(t \mid n_0 + 1)$$

or in differentially factorized form

$$-\frac{\partial}{\partial t}\left(e^{-\lambda t}\ p_n(t \mid n_0) \right) = \lambda e^{-\lambda t} p_n(t \mid n_0 + 1)$$

The backward variable is omitted unless explicitly required such as in the context of the backward equation.

Forward Equation by Reference to the Transition MGF

As an alternative from the Chapman-Kolmogorov equation, the forward equation can be obtained from the forward transition mgf of the Poisson process like that for first order unit root Gaussian and generalized Gaussian diffusion. The transition mgf is, by definition

$$M_{N(t)}(\theta) = \sum_{n=0}^{\infty} e^{\theta n} p_t(n)$$

which evaluates to

$$M_{N(t)}(\theta) = e^{\lambda t(e^{\theta} - 1)}$$

and is the mgf of a time integral of homogenous Poisson variables (an infinite product of homogenous Poisson mgfs $M_{dN(t)}(\theta) = e^{\lambda(e^{\theta} - 1)dt}$). The time derivative of the transition mgf is

$$\frac{\partial}{\partial t} M_{N(t)}(\theta) = \lambda(e^{\theta} - 1)e^{\lambda t(e^{\theta} - 1)} = \lambda(e^{\theta} - 1)M_{N(t)}(\theta)$$

i.e.,

$$\frac{\partial}{\partial t} M_{N(t)}(\theta) = \lambda(e^{\theta} - 1)\sum_{n=0}^{\infty} e^{\theta n} p_t(n) = \lambda \sum_{n=0}^{\infty} \frac{\Delta e^{\theta n}}{\Delta n} p_t(n)$$

Using summation by parts with $u(n) = p_t(n)$ and $\Delta v(n) = \Delta e^{\theta n}$ (or alternatively, the formulae can be obtained with differences, i.e., discrete derivatives) so that $\Delta u(n) = \Delta p_t(n)$ and $v(n) = e^{\theta n}$

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$$[e^{\theta n} p_t(n)]_0^\infty - \sum_{j=0}^{\infty} e^{\theta(n+1)} \Delta p_t(n) = 0 - p_t(0) - \sum_{j=0}^{\infty} e^{\theta(n+1)} [p_t(n+1) - p_t(n)]$$

$$= -p_t(0) - \sum_{j=0}^{\infty} e^{\theta(n+1)} p_t(n+1) + \sum_{j=1}^{\infty} e^{\theta n} p_t(n-1)$$

$$= -p_t(0) - \sum_{j=1}^{\infty} e^{\theta n} p_t(n) + \sum_{j=0}^{\infty} e^{\theta n} p_t(n-1)$$

$$= -\sum_{j=0}^{\infty} e^{\theta n} p_t(n) + \sum_{j=0}^{\infty} e^{\theta n} p_t(n-1)$$

$$= -\sum_{j=0}^{\infty} e^{\theta n} [p_t(n) - p_t(n-1)]$$

Hence

$$\frac{\partial}{\partial t} M_{N(t)}(\theta) = -\lambda \sum_{j=0}^{\infty} e^{\theta n} [p_t(n) - p_t(n-1)]$$

that is

$$\frac{\partial}{\partial t} \sum_{j=0}^{\infty} e^{\theta n} p_t(n) = -\lambda \sum_{j=0}^{\infty} e^{\theta n} [p_t(n) - p_t(n-1)]$$

$$\sum_{j=0}^{\infty} e^{\theta n} \frac{\partial}{\partial t} p_t(n) = -\lambda \sum_{j=0}^{\infty} e^{\theta n} [p_t(n) - p_t(n-1)]$$

Inverting the mgf

$$\frac{\partial}{\partial t} p_t(n) = -\lambda [p_t(n) - p_t(n-1)]$$

which is the forward equation for a first order unit root Poisson diffusion, the Poisson process.

Solution to the Forward Equation: The Poisson Process Transition Probability Function

Solving the process for general *t*, initially for zero counts, $p_n(t)$;

$$\frac{\partial p_0(t)}{\partial t} = -\lambda p_0(t)$$

Dividing both sides by $p_n(t)$

$$\frac{\partial \log p_0(t)}{\partial t} = -\lambda$$

and integrating from time 0, up to the forward time variable *t*, over the dummy *y*

$$\int_0^t \partial \log p_0(y) = -\lambda \int_0^t \partial y$$

$$\log p_0(t) - \log p_0(0) = -\lambda t$$

$$p_0(t) = e^{-\lambda t}$$

where $p_0(0) = 1$ giving for the probability of a single count $e^{-\lambda t}$ on the time interval from time zero to time *t*. Solving the process for the first occurrence over the transition interval starting from time zero;

$$\frac{\partial p_1(t)}{\partial t} = \lambda [p_0(t) - p_1(t)] = \lambda [e^{-\lambda t} - p_1(t)]$$

or

$$\frac{\partial p_1(t)}{\partial t} + \lambda p_1(t) = \lambda e^{-\lambda t}$$

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The integrating factor for the equation is

$$IF(t) = \exp\left(\lambda \int dt\right) = e^{\lambda t}$$

Multiplying both sides by the integrating factor

$$e^{\lambda t} \frac{\partial p_1(t)}{\partial t} + \lambda e^{\lambda t} p_1(t) = \lambda$$

giving the exact derivative

$$\frac{\partial}{\partial t}[e^{\lambda t} p_1(t)] = \lambda$$

Integrating again from time 0 up to the forward time variable *t* over the dummy *y*

$$\int_0^t \partial[e^{\lambda y} p_1(y)] = \lambda \int_0^t \partial y$$

$$e^{\lambda t} p_0(t) - \log p_0(0) = \lambda t$$

$$p_1(t) = \lambda t e^{-\lambda t}$$

and similarly for two counts,

$$\frac{\partial p_2(t)}{\partial t} + \lambda p_2(t) = \lambda p_1(t) = \lambda^2 t e^{-\lambda t}$$

or

$$\frac{\partial}{\partial t}[e^{\lambda t} p_2(t)] = \lambda^2 t$$

which gives $p_2(t) = e^{-\lambda t}(\lambda t)^2 / 2$ upon integrating from 0 to *t.* Solving in recursion in an identical manner for the solution for general *n*, gives the discrete transition density and solution to the forward convolution equation

$$p_n(t) = e^{-\lambda t}(\lambda t)^n / n! \quad n \forall N[0, \infty)$$

which is the Poisson probability density function. The random variable for the process count distribution is denoted by $N(t)$ to which $p_n(t) = e^{-\lambda t}(\lambda t)^n / n!$ is the density, hence

$$N(t) \sim P(\lambda t)$$

That is, counts are Bernoulli on differential time increments and Poisson on any increment in time larger than *dt* (but less than reciprocals of time infinitesimals, $1 / dt$). The Poisson process is a first order unit-root diffusion with a Poisson distributed basis.

Moment analysis

The moment generating function of the Poisson process is

$$M_{N(t)}(\theta) = e^{\lambda t(\exp(\theta) - 1)}$$

with a corresponding cumulant generating function

$$K_{N(t)}(\theta) = \lambda t(e^{\theta} - 1)$$

All derivatives of the cumulant generating function for the Poisson process are equal, $K_{N(t)}^{(r)}(\theta) = \lambda t e^{\theta}$, and hence all cumulants above the first are equal and each are linear functions of time. The mean, variance, and skewness and all higher order cumulants are therefore each λt and equal. This is the result of a linear integration of a basis with a level set of cumulants, in this case, λ.

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Correlation of Non-Overlapping Increments – Independence of Poisson Increments

Increments of the Poisson process are independent over non-overlapping time intervals. It has been pointed out and is further shown that a Poisson process is a discrete, first order diffusion with a unit-root, and accordingly, is a Markov process which implies the property of independent increments since the increments are the components of the time contiguous sum of the first order aggregation of the basis function. The Poisson process, therefore, shares this commonality with Brownian motion, first passage times of Brownian motion, and Poisson process compound sums and all other unit root Markov processes, i.e., those that are a result of direct integration of a basis function of arbitrary uncorrelated component distributions. If N_t is a Poisson process so that $N_t \sim P(\lambda t)$ and $N_{t+s} \sim P(\lambda(t+s))$, then $N_{t+s} - N_t \sim P(\lambda s)$ independently of $N_t \sim P(\lambda t)$.

Proof

$$\begin{aligned} Cov(N_{t+s} - N_t, N_t) &= Cov(N_{t+s}, N_t) - Cov(N_t, N_t) \\ &= Cov(N_{t+s}, N_t) - Var(N_t,) \\ &= \big((t+s) \wedge t\big)\lambda - t\lambda \\ &= t\lambda - t\lambda \\ &= 0 \end{aligned}$$

The proof is complete.

Autocovariance and Autocorrelation Functions of the Poisson Process

Due to the independent increment property of the Poisson process, the autocovariance function of the process takes a specific form unique to processes with independent increments. Namely, the per unit time variance parameter or function of parameters, times the smaller of the set of two time increments starting at where the processes was last observed or known at the given value of the backward variable. This is the length of the overlap times the unit variance parameter,

$$Cov(N_t, N_s) = \lambda(s \wedge t) = \lambda \min(s,t) \qquad \text{or, equivalently,} \qquad Cov(N_{t+s}, N_s) = \lambda s$$

The autocorrelation function is then

$$Corr(N_t, N_s) = \frac{(s \wedge t)}{\sqrt{t}\sqrt{s}} = \frac{\min(s,t)}{\sqrt{t}\sqrt{s}}$$

or equivalently

$$Corr(N_{t+s}, N_s) = \frac{s}{\sqrt{(t+s)}\sqrt{s}} = \sqrt{\frac{s}{t+s}}$$

Limiting Moment Features of the Poisson Process

The kurtosis is

$$K^{(4)}_{N(t)}(0) + 3[K^{(2)}_{N(t)}(0)]^2 = \lambda t + 3(\lambda t)^2$$

Looking at the standardized measures of shape for third and fourth degree, the standardized skew and kurtosis, the central moments scaled by the appropriate power of the variance

$$\xi_r(N(t)) = E[\{N(t) - \mu_{N(t)}\}^r] / \sigma^r_{N(t)}$$

The standardized skewness of the Poisson process is

$$\xi_3 = (\lambda t)/(\lambda t)^{3/2} = 1/\sqrt{\lambda t}$$

which vanishes for large *t*, that is, $\xi_3 \to 0$ as $\sqrt{t} \to \infty$. This implies the skewness tends to zero relative to the distributional spread. The standardized kurtosis of the Poisson process is

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$$\xi_4 = [\lambda t + 3(\lambda t)^2]/(\lambda t)^2 = 3 + \frac{1}{\lambda t}$$

which approaches 3 as *t* becomes large, that is $\xi_4 \to 3$ as $\sqrt{t} \to \infty$. This implies that the 'shape', in terms of the sharpness of the modal peak and thickness of the distributional tails, tends to a mesokurtic state and the distributional shape of a normal bell curve.

These two behaviors over *t* imply skew and heavy-tailedness for early and moderate *t*, while for large *t*, the effects of the Central Limit Theorem dominate resulting in an increasingly normally like distributed random variable for the aggregate standardized diffusion variable. The form of the forward equation is consequently the heat equation in the limit as $o(1/t)$ terms, reciprocal infinities, become neglibible. This is proved shortly.

Poisson Process Limit Theorem

The limiting values of the third and fourth scaled measures of shape, the coefficients of skewness and kurtosis, suggest a convergence to a normal curve. The normal distribution is specified by two-parameters, one for center and one for scale, and this suggests that the transition distribution of the Poisson processes, adjusted for these two-parameters, approaches a unit-normal. The mean and variance of the Poisson process are both λt. The moment generating function for the standardized Poisson process, $[N(t) - \lambda t] / \sqrt{\lambda t}$ is

$$M_{[N(t)-\lambda t]/\sqrt{\lambda t}}(\theta) = E\left(e^{\frac{N(t)-\lambda t}{\sqrt{\lambda t}}\theta}\right) = e^{-\theta\sqrt{\lambda t}} E\left(e^{\frac{N(t)}{\sqrt{\lambda t}}\theta}\right) = e^{-\theta\sqrt{\lambda t}} M_{N(t)}\left(\frac{\theta}{\sqrt{\lambda t}}\right) = e^{-\theta\sqrt{\lambda t}} e^{\lambda t(\exp(\theta/\sqrt{\lambda t})-1)}$$

giving a cumulant generating function

$$K_{[N(t)-\lambda t]/\sqrt{\lambda t}}(\theta) = -\theta\sqrt{\lambda t} + \lambda t(e^{\theta/\sqrt{\lambda t}} - 1)$$

Expanding the exponential in a power series gives

$$\begin{aligned} K_{[N(t)-\lambda t]/\sqrt{\lambda t}}(\theta) &= -\theta\sqrt{\lambda t} + \lambda t\left(\frac{\theta}{\sqrt{\lambda t}} + \frac{\theta^2}{2(\sqrt{\lambda t})^2} + \frac{\theta^3}{2(\sqrt{\lambda t})^3} + \ldots\right) \\ &= \tfrac{1}{2}\theta^2 + O(1/\sqrt{t}) \\ &\to \tfrac{1}{2}\theta^2 \end{aligned}$$

as $t \to \infty$ and the higher order cumulants are dominated by the second, and is the cumulant generating function of a unit normal variable;

$$\lim_{t\to\infty}[N(t) - \lambda t]/\sqrt{\lambda t} \sim N(0,1)$$

Re-standardizing by adding back the mean and rescaling by the variance gives the limiting transition distribution

$$\lim_{t\to\infty} N(t) \sim N(\lambda t, \lambda t)$$

which is an arithmetic Brownian motion[38]. This Brownian motion, the limiting process, has equivalent drift and volatility and therefore has a unit signal to noise ratio and a unit coefficient of variation.

In summary, ignoring the backward variable, the Poisson process on different scales of time is

(1) Bernoulli[39] on infinitesimal intervals, $N(dt) \sim B(1, \lambda dt)$

(2) Poisson on general finite intervals, and $N(t) \sim P(\lambda t)$

---

[38] there is some degree of overlap of notation for those that are not familiar with distribution theory and stochastic processes; $N(\cdot)$ denotes the Poisson process, a count process from stochastic processes, and $N(\cdot,\cdot)$ denotes a normal variable from distribution theory. I hope that the context makes this clear throughout this thesis.

[39] precisely, this is $N(dt) \sim B(1, 1 - e^{-\lambda dt})$,i.e, $N(dt) \sim B\big(1, \lambda dt + o(dt)\big)$, so that $N(dt) \sim B(1, dt) + o(dt)$

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(3) normal over large or extended periods of time[40] $N(t) \sim N(\lambda t, \lambda t)$

The large *t* stationary transition density function is

$$p_n(t) = \frac{1}{\sqrt{2\pi\lambda t}} \exp\left(- \frac{[n - \lambda t]^2}{2\lambda t} \right)$$

The Poisson process is time homogenous so that the density of the continuous variable, inclusive of the backward variables (inclusive of an initial value n_0 at time t_0) is

$$p_{N(t)|N(t_0)}(n \mid n_0) = \frac{1}{\sqrt{2\pi\lambda(t - t_0)}} \exp\left(- \frac{[n - n_0 - \lambda(t - t_0)]^2}{2\lambda(t - t_0)} \right)$$

and prior to complete convergence to this form, from ordinary statistics, this can be improved by the continuity correction to accommodate the replacement of a discrete variable with a continuous one.

The Forward and Backward Equations of the Asymptotic Process: Brownian Motion

The requirement for an asymptotic convergence and for the resulting diffusion equation to hold is that a single unit step in the process is asymptotically equivalent, for a large number of steps, to a Newtonian differential so that it is proportional to the infinitesimal time differential *dt* . Letting the transition probability function at time *t*, with a backward variable n_0 at time 0, be denoted by the notational shorthand

$p_{N(t)|N(t_0)}(n \mid n_0) = p_n(t \mid n_0)$.

Taking the forward equation of the Poisson process

$$\frac{\partial p_n(t \mid n_0)}{\partial t} = \lambda \left(p_{n-1}(t \mid n_0) - p_n(t \mid n_0) \right)$$

and expanding the first bracketed term on the diffusion side in a univariate Taylor series about the forward variable *n* for a unit backward increment in *n*

$$\frac{\partial p_n(t \mid n_0)}{\partial t} = \lambda \left(p_n(t \mid n_0) - \frac{\partial}{\partial n} p_n(t \mid n_0) + \tfrac{1}{2} \frac{\partial^2}{\partial n^2} p_n(t \mid n_0) - \tfrac{1}{3!} \frac{\partial^2}{\partial n^2} p_n(t \mid n_0) + \ldots \right) - \lambda p_n(t \mid n_0)$$

gives the forward equation of a unit-cumulant-ratio generalized heat equation with terms to all orders

$$\frac{\partial p_n(t \mid n_0)}{\partial t} = -\lambda \frac{\partial}{\partial n} p_n(t \mid n_0) + \tfrac{1}{2} \lambda \frac{\partial^2}{\partial n^2} p_n(t \mid n_0) - \tfrac{1}{3!} \lambda \frac{\partial^2}{\partial n^2} p_n(t \mid n_0) + \ldots$$

As the lower order properties start to dominate by the Central Limit Theorem, to second order, the equation becomes that of a unit-signal-to-noise-ratio Brownian motion,

$$\frac{\partial p_n(t \mid n_0)}{\partial t} = \tfrac{1}{2} \lambda \frac{\partial^2}{\partial n^2} p_n(t \mid n_0) - \lambda \frac{\partial}{\partial n} p_n(t \mid n_0)$$

Further convergence, because of the unit snr, the equation becomes that of an asymptotic Dirac density and trace as a further convergence occurs to one order above that of the Central Limit Theorem

$$\frac{\partial p_n(t \mid n_0)}{\partial t} = -\lambda \frac{\partial}{\partial n} p_n(t \mid n_0)$$

Similarly, taking the backward equation of the Poisson process

---

[40] which is formally $N(1/dt) \sim N(\lambda/dt, \lambda/dt)$ or $\lim_{t \to \infty} N(t) \sim N(\lambda t, \lambda t)$ which is then, for all intensive purposes, linear time *t*.

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$$-\frac{\partial p_n(t \mid n_0)}{\partial t} = \lambda\left(p_n(t \mid n_0 + 1) - p_n(t \mid n_0) \right)$$

and expanding the first bracketed term on the diffusion side in a univariate Taylor series about the backward variable n_0 for a unit forward increment in n_0

$$-\frac{\partial p_n(t \mid n_0)}{\partial t} = \lambda\left(p_n(t \mid n_0) + \frac{\partial}{\partial n_0} p_n(t \mid n_0) + \tfrac{1}{2}\frac{\partial^2}{\partial n_0^2} p_n(t \mid n_0) - p_n(t \mid n_0) \right)$$

gives the backward equation of a unit signal to noise ratio Brownian motion as the second order properties become dominant

$$-\frac{\partial p_n(t \mid n_0)}{\partial t} = \tfrac{1}{2}\lambda \frac{\partial^2}{\partial n_0^2} p_n(t \mid n_0) + \lambda \frac{\partial}{\partial n_0} p_n(t \mid n_0)$$

The convergence is relative and not absolute as the higher order cumulants only diminish in importance, rather than vanishing completely, relative to the two lower order cumulants, the mean and variance and eventually, only the mean (a linear function of *t*). The higher the rate intensity λ, the more quickly the lower order effects dominate.

First Passage Time of the Poisson Process

The time between increments in the Poisson process is exponentially distributed with rate λ and the time between multiple increments is gamma (Erlang) distributed, the sum of independent exponentials since the increments are independent, the exponentials are independent. This additionally implies that the first passage time distribution, the distribution of the waiting time for the n^{th} recurrence, is gamma distributed. The gamma distribution has shape parameter n and scale parameter $1/\lambda$. Defining the random variable T_n as the waiting time for the n^{th} recurrence, the variable that measures the duration required for the Poisson process to produce n events. The key identity of this argument equates equivalent probabilities for equivalent events, namely, that the waiting time for at least the n^{th} recurrence of a discrete monotone diffusion is equivalent to the occurrence of *n* or more recurrences by that time

Logical identity $\Pr(T_n < t) = \Pr(N_t \geq n)$

This identity can be stated for the discrete non-decreasing variable $N(t)$ as

$$F_{T_n}(t) = \sum_{j=n}^{\infty} p_j(t) = 1 - \sum_{j=0}^{n-1} p_j(t)$$

For the first recurrence, the distribution of the waiting time is equal to the complement of zero Poisson events by time *t*

$$F_{T_1}(t) = 1 - p_0(t)$$

or

$$F_{T_1}(t) = 1 - e^{-\lambda t}$$

Differentiating for the density function of the first recurrence $DF_{T_1}(t)$,

$$f_{T_1}(t) = \lambda e^{-\lambda t} \text{ for } t \geq 0$$

This is the density function of an exponential variable, or a gamma variable with unit shape parameter and scale parameter λ. Hence

$$T_1 \sim \Gamma(1, 1/\lambda)$$

Similarly, for the second recurrence

$$F_{T_2}(t) = 1 - p_0(t) - p_1(t) = 1 - e^{-\lambda t} - (\lambda t)e^{-\lambda t}$$

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and differentiating for the density of the second recurrence $DF_{T_2}(t)$

$$f_{T_2}(t) = \lambda^2 t e^{-\lambda t}$$

which is the density function of a gamma variable with shape parameter of 2

$$T_2 \sim \Gamma(2,1/\lambda)$$

and so on. The density function of the convolved recurrence times, T_n, is then

$$f_{T_n}(t) = \lambda\Big(p_{n-1}(t) - p_{n-2}(t) \Big) = \lambda \Pr(N_t = n-1) = \frac{\lambda^n}{\Gamma(n)} t^{n-1} e^{-\lambda t} \qquad \text{for } t \geq 0$$

which is the density of a gamma distributed variable with shape parameter n and scale parameter $1/\lambda$,

$$T_n \sim \Gamma(n,1/\lambda)$$

The waiting time of the first recurrence is $T_1 \sim \Gamma(1, 1/\lambda)$, the waiting time of the second recurrence $T_2 \sim \Gamma(2, 1/\lambda)$, the third $T_3 \sim \Gamma(3, 1/\lambda)$, and the n^{th}, $T_n \sim \Gamma(n, 1/\lambda)$. By the convolution properties of the Erlang distribution (gammas with integer shape parameters), the time between each of these variables is exponentially distributed as per the waiting time for the first Poisson occurrence,

$$T_n - T_{n-1} \sim \Gamma(1,1/\lambda)$$

In terms of distribution functions, this relation linking the renewal (or first passage time) density function to the differences in the count probabilities can be interpreted as a restatement of the Poisson process forward differential equation

$$\frac{\partial p_n(t)}{\partial t} = \lambda\Big(p_{n-1}(t) - p_n(t) \Big)$$

To observe this, for n Poisson occurrences, the key identity can be expressed as

$$F_{T_n}(t) = 1 - p_{n-1}(t) = \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

and differentiating for the density function of T_n gives

$$f_{T_n}(t) = -\frac{\partial}{\partial t} p_{n-1}(t) = \frac{\partial}{\partial t} \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

$$= \sum_{j=n}^{\infty} j \times e^{-\lambda t} \frac{\lambda^j t^{j-1}}{j!} - \lambda \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} = \lambda \sum_{j=n-1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} - \lambda \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

$$= \lambda\Big([1 - p_{n-2}(t)] - [1 - p_{n-1}(t)] \Big) = \lambda\Big(p_{n-1}(t) - p_{n-2}(t) \Big)$$

or equivalently,

$$\frac{\partial p_{n-1}(t)}{\partial t} = \lambda\Big(p_{n-2}(t) - p_{n-1}(t) \Big) \qquad \text{i.e.,} \qquad \frac{\partial p_n(t)}{\partial t} = \lambda\Big(p_{n-1}(t) - p_n(t) \Big)$$

which is the restatement of the forward differential equation of the Poisson process.

The density of this distribution is therefore the solution to the forward equation for the first passage time, or time over state, transition distribution of the Poisson process. The Poisson process is a non-decreasing diffusion, since it is the integration of a basis variable with positive support, that of independent Poissons, and hence the first passage time distribution is well posed and integrates to one. This is in contrast to some of the distributions of first passage times of ordinary diffusions which do not.

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Simulation Algorithm for a Poisson Process

Generate a vector of exponential random deviates with the required shape parameter. These can be generated by first generating a vector of unit uniforms, random numbers from $U_j \sim U(0,1)$, and then calculating the exponential variates from the formula $t_j = -(1 / \lambda) \log(1 - U_j) \sim \Gamma(1,1 / \lambda)$, the inverse function of the cumulative distribution function of the exponential distribution. Sum these until the length of time over which the required simulated path is obtained. Alternatively, specify a fixed number of increments, i.e., a first passage time to, say, n Poisson steps. At each point of increase of the vector of cumulative exponential variates there is a unit step in increase in the simulated Poisson process. Streamlined MATLAB code to do this is

```
b=1/λ; n=n;                                          %initialize parameters
dt=exprnd(b,n,1); t=cumsum([0;dt)];   %simulate exp basis variable and sum
stairs(t);                                                          %graph
```

note here that `b` plays a similar role as `h` in the simulation code of FOUNDATIONS 1.

Simulations

It is clear from the simulated paths in the following graphics that the Poisson process is a diffusion that

i. is monotone;
ii. has a unit signal-to-noise ratio (unit-SNR); and
iii. closely resembles linear time for a large number of Poisson increments.

These features arise from the fact that an aggregation of a Poisson basis variable results in **i.** a monotone variable because the support of the scaled basis is zero-one and the support of its unit root aggregation is on the positive integers, **ii.** a unit-SNR diffusion due to the level set of cumulants of the Poisson basis variable, and **iii.** increasingly resembles linear time due to the dominance of the first cumulant with the passage of time for unit-SNR diffusion (**iii.** is a consequence of of **ii.** and, to some extent, so is **i.**).

The importance of these features in relation to this thesis regards the fact that
(1) the first passage time of a Poisson process is well defined, a gamma of summed exponentials, and consequently integrates to one and that
(2) compound Poisson aggregations become increasingly like Brownian motions with the passage of time since the transition distribution is Brownian in the diffusion variable of the Poisson process (the cumulants sum linearly with this variable) which becomes increasingly linear and void of variation.

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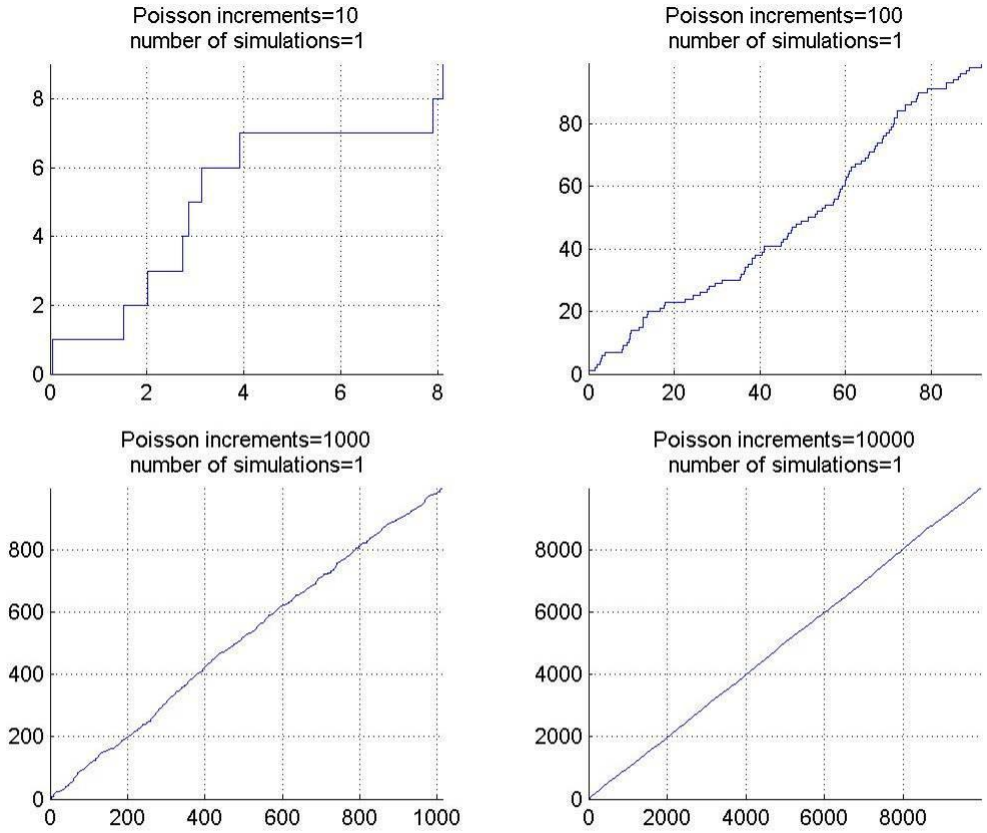


Figure 24: single simulated path of a Poisson process; 10, 100, 1000, and 10000 steps

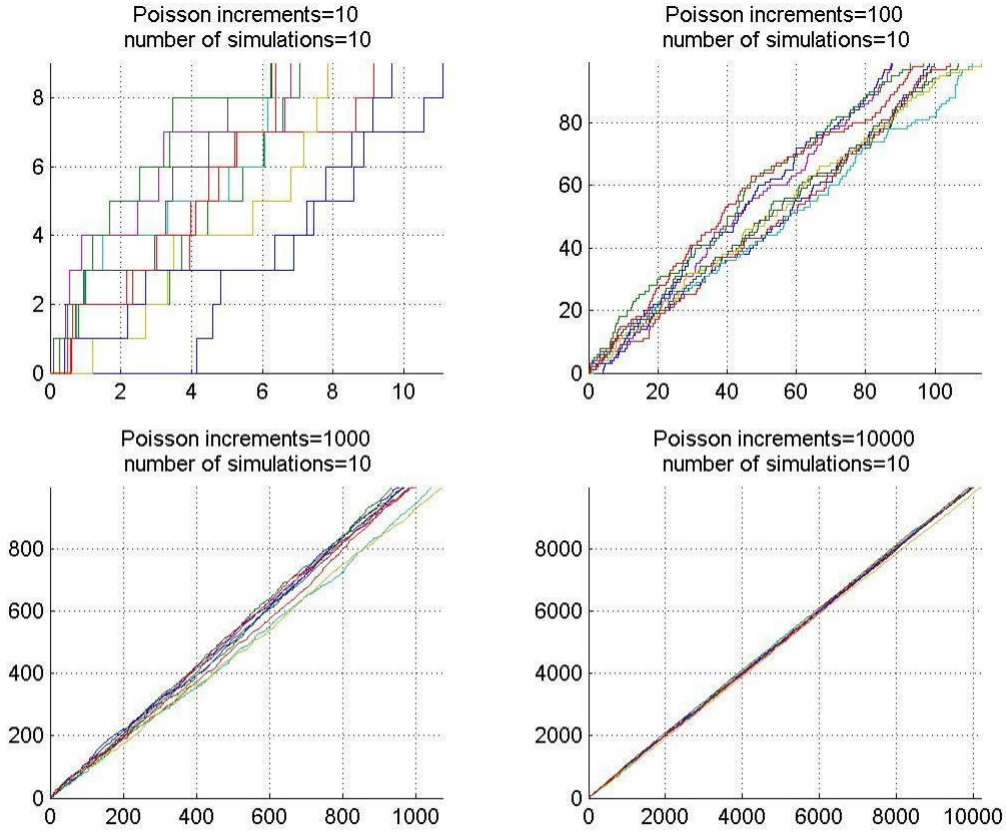


Figure 25: 10 simulated paths of a Poisson process; 10, 100, 1000, and 10000 steps

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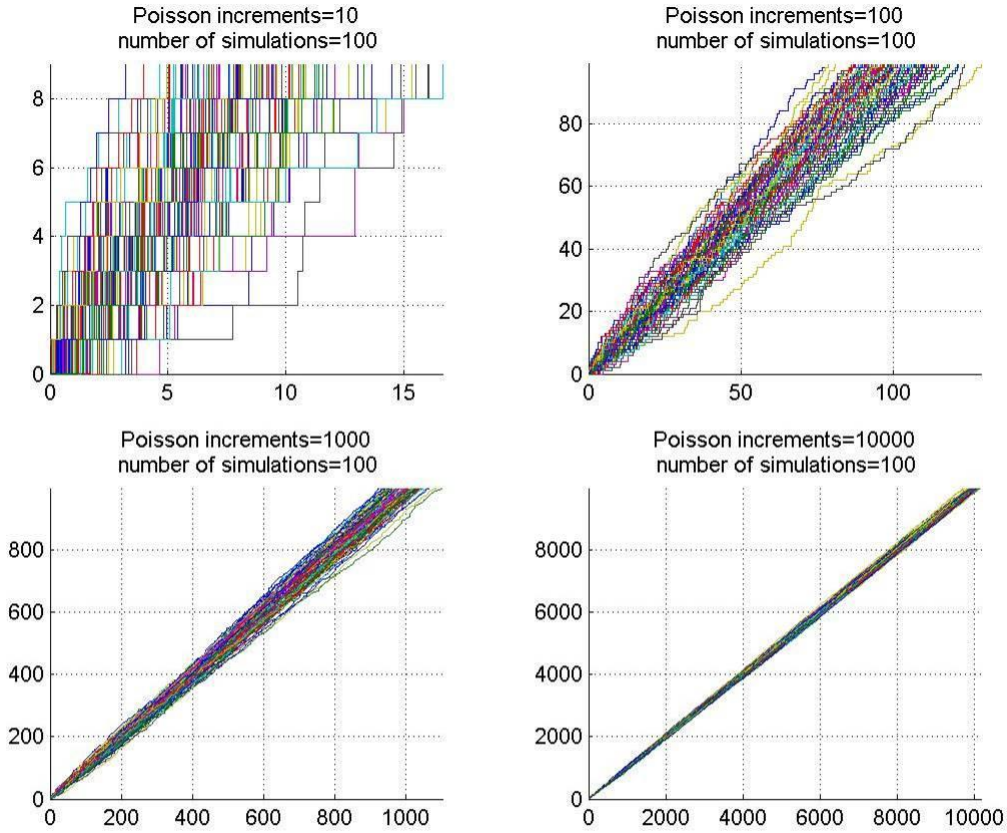


Figure 26: 100 simulated paths of a Poisson process; 10, 100, 1000, and 10000 steps

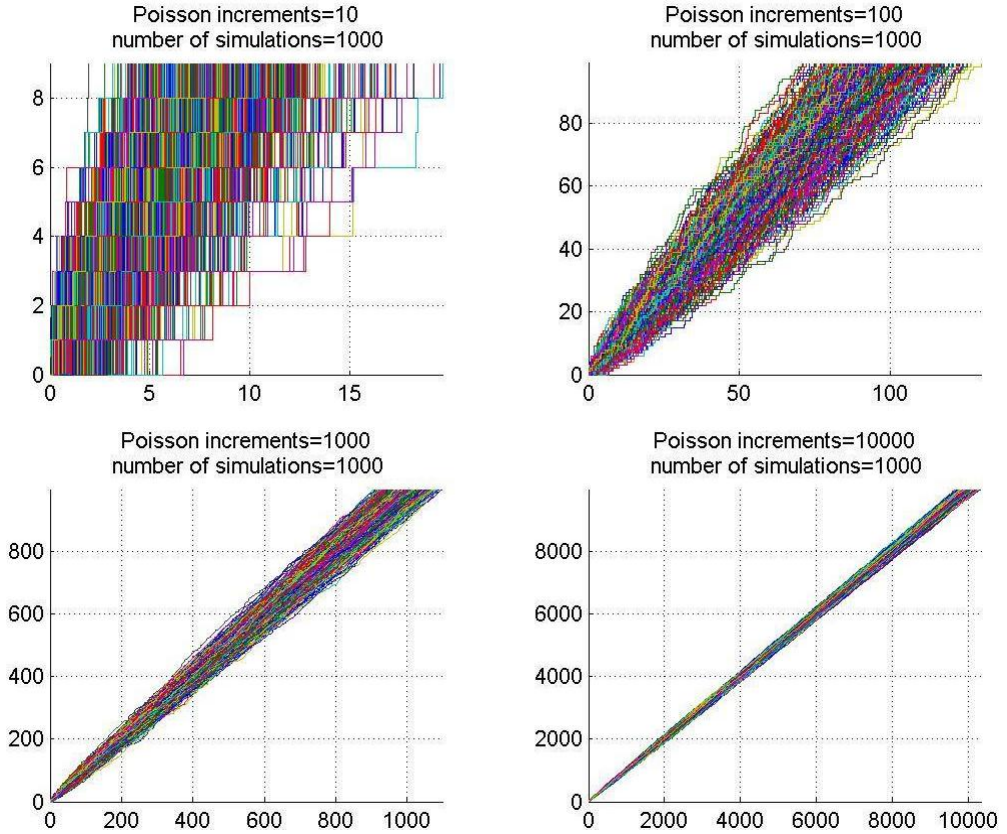


Figure 27: 1000 simulated path of a Poisson process; 10, 100, 1000, and 10000 steps

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Transformations of a Poisson Variable (Aaron's Lemma)

From the distribution of dN_t as a Poisson variable with mean λdt

$dN_t = z_t dt \sim P(\lambda dt)$

which has level cumulants at all orders, $\kappa_r[dN_t] = \lambda dt$, it is fact that

(1) dN_t has mean $E(dN_t) = \lambda dt$ and stochastic variation $Var(dN_t) = \lambda dt$

and further, since $E[(dN_t)^4] = \lambda dt + o(dt)$, it is also a fact that

(2) $(dN_t)^2$ has mean $E[(dN_t)^2] = \lambda dt + o(dt)$ and stochastic variation $Var[(dN_t)^2] = \lambda dt + o(dt)$

More generally, since $E[(dN_t)^{2n}] = \lambda dt + o(dt)$

(3) $(dN_t)^n$ has mean $E[(dN_t)^n] = \lambda dt + o(dt)$ and stochastic variation $Var[(dN_t)^n] = \lambda dt + o(dt)$ for all $n > 1$.

It is therefore an assertion [of this thesis] that powers of dN_t have the same distribution as dN_t within $o(dt)$, i.e., that $(dN_t)^n \to dN_t$ as $o(dt)$ becomes negligible. This makes sense when it is remembered that

$$dN_t = \begin{cases} 1 & \text{if a Poisson event occurs on } dt \\ 0 & \text{if a Poisson event doesn't occur on } dt \end{cases} \quad \text{so} \quad (dN_t)^n = \begin{cases} 1 & \text{if a Poisson event occurs on } dt \\ 0 & \text{if a Poisson event doesn't occur on } dt \end{cases}$$

From the Taylor series expansion for derivatives, in this case for the total derivative peculiar to stochastic calculus

$$\frac{dg(X)}{dt} = \frac{\partial g(X)}{\partial X}\frac{dX}{dt} + \tfrac{1}{2}\frac{\partial^2 g(X)}{\partial X^2}\frac{(dX)^2}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(X)}{\partial X^3}\frac{(dX)^3}{dt} + \dots$$

and for the transformation $g(\cdot)$ of a first order Poisson diffusion N_t which has cumulants to all orders (rather than, say, a first order normal diffusion for which has only two cumulants)

$$\frac{dg(N_t)}{dt} = \frac{\partial g(N_t)}{\partial N_t}\frac{dN_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial N_t^2}\frac{(dN_t)^2}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial N_t^3}\frac{(dN_t)^3}{dt} + \dots$$

which has an infinite number of terms because of the infinite number of cumulants of the Poisson diffusion basis function. Incorporating the assertion that, because $(dN_t)^n = dN_t + o(dt)$, the powers of Poisson differentials of all orders revert to that of first order $(dN_t)^n \to dN_t$ as dt becomes negligible

$$\begin{aligned}
\frac{dg(N_t)}{dt} &= \frac{\partial g(N_t)}{\partial N_t}\frac{dN_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial N_t^2}\frac{(dN_t)^2}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial N_t^3}\frac{(dN_t)^3}{dt} + \dots \\
&= \frac{o(dt)}{dt} + \frac{\partial g(N_t)}{\partial N_t}\frac{dN_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial N_t^2}\frac{dN_t}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial N_t^3}\frac{dN_t}{dt} + \dots \\
&= \frac{dN_t}{dt}\left(\frac{\partial g(N_t)}{\partial N_t} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial N_t^2} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial N_t^3} + \dots\right) + \frac{o(dt)}{dt}
\end{aligned}$$

and letting $dt \to 0$ become infinitesimal so that $o(dt)/dt \to 0$ results in the formula for derivatives of transformed first order Poisson diffusion

$$\frac{dg(N_t)}{dt} = \frac{dN_t}{dt}\left(\frac{\partial g(N_t)}{\partial N_t} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial N_t^2} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial N_t^3} + \dots\right)$$

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which is an infinite series[41]. Alternatively expressed in differential form as

$$dg(N_t) = \left(\frac{\partial g(N_t)}{\partial N_t} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial {N_t}^2} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial {N_t}^3} + \ldots \right) dN_t$$

The effect of this is that derivatives of Poisson variables, which are discrete variables on *N* (naturals), act consistently with the discrete calculus; continuous derivatives of first order Poisson diffusion are analogues to discrete differences of the finite calculus on ordinary numbers (which differ only because the Poisson derivative occurs with probability λdt or does not occur with probability $1 - \lambda dt$ over dt and retains its status as a continuous time infinitesimal derivative). For example, the derivative of any power of a Poisson process results in a factorial power of one degree lower as for the sum (discrete) calculus.

Example 1: As an example, considering an exponential function of a Poisson process

$$g(N_t) = e^{N_t}$$

The partial time derivative is zero and the partial derivatives with respect to N_t are each equal

$$\frac{\partial g(N_t)}{\partial N_t} = e^{N_t} = g(N_t) \qquad \frac{\partial^2 g(N_t)}{\partial {N_t}^2} = e^{N_t} = g(N_t) \quad \frac{\partial^3 g(N_t)}{\partial {N_t}^3} = e^{N_t} = g(N_t) \quad \text{and so on.}$$

The expansion is consequently

$$\frac{d}{dt}e^{N_t} = \frac{dN_t}{dt}\left(e^{N_t} + \tfrac{1}{2}e^{N_t} + \tfrac{1}{3!}e^{N_t} + \ldots\right) = e^{N_t}\frac{dN_t}{dt}\left(1 + \tfrac{1}{2} + \tfrac{1}{3!} + \ldots\right) = e^{N_t}(e-1)\frac{dN_t}{dt}$$

which is stated alternatively as

$$\frac{d}{dN_t}e^{N_t} = e^{N_t}(e-1) \qquad\qquad d\,e^{N_t} = e^{N_t}(e-1)dN_t$$

and should be compared with the derivative of ordinary calculus $\frac{d}{dt}e^t = e^t$ and $\frac{d}{dt}e^{g(t)} = e^t\frac{d}{dt}g(t)$.

Example 2: Extending this example, letting

$$g(N_t) = e^{aN_t}$$

$$\frac{\partial g(N_t)}{\partial N_t} = ae^{aN_t} = ag(N_t) \quad \frac{\partial^2 g(N_t)}{\partial {N_t}^2} = a^2e^{aN_t} = a^2g(N_t) \quad \frac{\partial^3 g(N_t)}{\partial {N_t}^3} = a^3e^{aN_t} = a^3g(N_t) \text{ and so on,}$$

the expansion has an infinite number of terms which are summable as an exponential function

$$\frac{d}{dt}e^{aN_t} = \frac{dN_t}{dt}\left(ae^{aN_t} + \tfrac{1}{2}a^2e^{aN_t} + \tfrac{1}{3!}a^3e^{aN_t} + \ldots\right) = e^{aN_t}\frac{dN_t}{dt}\left(a + \tfrac{1}{2}a^2 + \tfrac{1}{3!}a^3 + \ldots\right) = e^{aN_t}(e^a - 1)\frac{dN_t}{dt}$$

that is,

---

[41] compare this with that for standard Brownian motion for which the only cumulant that is non-zero is the second

$$\begin{aligned}\frac{dg(B_t)}{dt} &= \frac{\partial g(B_t)}{\partial B_t}\frac{dB_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(B_t)}{\partial {B_t}^2}\frac{(dB_t)^2}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(B_t)}{\partial {B_t}^3}\frac{(dB_t)^3}{dt} + \ldots \\ &= \frac{\partial g(B_t)}{\partial B_t}\frac{dB_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(B_t)}{\partial {B_t}^2}\frac{dt}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(B_t)}{\partial {B_t}^3}\frac{dB_t dt}{dt} + \ldots \\ &= \frac{\partial g(B_t)}{\partial B_t}\frac{dB_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(B_t)}{\partial {B_t}^2}\frac{dt}{dt} + \frac{o(dt)}{dt}\end{aligned}$$

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$$\frac{d}{dN_t}e^{aN_t} = e^{aN_t}(e^a - 1) \qquad d\,e^{aN_t} = e^{aN_t}(e^a - 1)dN_t$$

which clearly has more in common with a discrete derivative that the infinitesimal derivative $\frac{d}{dt}e^{at} = a\,e^{at}$.

For verification purposes, testing the right and left hand sides when a Poisson event does and does not occur;

- if a Poisson event occurs on dt, $dN_t = 1$ and $de^{aN_t} = e^{aN_t + a} - e^{aN_t}$ as it should
- if a Poisson event does not occur on dt, $dN_t = 0$ and $de^{aN_t} = 0$ as it should.

Example 3: Derivative of a square of a Poisson process, $g(N_t) = {N_t}^2$. The partial derivatives are

$$\frac{\partial g(N_t)}{\partial N_t} = 2N_t \qquad \frac{\partial^2 g(N_t)}{\partial {N_t}^2} = 2 \qquad \frac{\partial^3 g(N_t)}{\partial {N_t}^3} = 0$$

giving the terminating series with two terms

$$\frac{d}{dN_t}{N_t}^2 = \left(2N_t + \tfrac{1}{2}\times 2 + \tfrac{1}{3!}\times 0 + \ldots\right)$$

i.e., a derivative of

$$\frac{d}{dN_t}{N_t}^2 = 2N_t + 1 \quad \text{if } dN_t = 1 \text{ then } d{N_t}^2 = (N_t + 1)^2 - {N_t}^2 = {N_t}^2 + 2N_t + 1 - {N_t}^2 = 2N_t + 1$$

which makes sense because N_t is a discrete variable.

A Peculiarity of Transformations of a Discrete Stochastic Variable

All of these derivatives have been forward derivatives. Because Poisson variables are discrete, the calculus has the additional peculiarity that backward derivatives are different from forward derivatives. This is similar to discrete calculus.

From the Taylor series expansions of $f(x + h)$ and $f(x - h)$

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2!}h^2 f''(x) + \tfrac{1}{3!}h^3 f'''(x) + \ldots + \tfrac{1}{n!}h^n f^{(n)}(x) + \ldots$$
$$f(x-h) = f(x) - hf'(x) + \tfrac{1}{2!}h^2 f''(x) - \tfrac{1}{3!}h^3 f'''(x) + \ldots + \tfrac{1}{n!}(-h)^n f^{(n)}(x) + \ldots$$

For a forward derivative,

$$f(x+h) - f(x) = hf'(x) + \tfrac{1}{2!}h^2 f''(x) + \tfrac{1}{3!}h^3 f'''(x) + \ldots + \tfrac{1}{n!}h^n f^{(n)}(x) + \ldots$$
$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{\tfrac{1}{2!}h^2 f''(x) + \tfrac{1}{3!}h^3 f'''(x) + \ldots + \tfrac{1}{n!}h^n f^{(n)}(x) + \ldots}{h}$$

and for a backward derivative

$$f(x) - f(x-h) = hf'(x) - \tfrac{1}{2!}h^2 f''(x) + \tfrac{1}{3!}h^3 f'''(x) - \ldots - \tfrac{1}{n!}(-h)^n f^{(n)}(x) + \ldots$$
$$\frac{f(x) - f(x-h)}{h} = f'(x) - \frac{\tfrac{1}{2!}h^2 f''(x) - \tfrac{1}{3!}h^3 f'''(x) + \ldots + \tfrac{1}{n!}(-h)^n f^{(n)}(x) + \ldots}{h}$$

Hence, forward and backward derivatives of the stochastic calculus differ because of the non-negligible effects of higher orders of stochastic variation, in the most prominent examples of Brownian motion and the Poisson process when $x = B_t$ and $x = N_t$ in which $h = dB_t$ and $h = dN_t$ for the transformation f. This is most notable and relevant for Poisson variables $h = dN_t$ because they are discrete and the discrete calculus makes a clear distinction between forward and backward differences. For Brownian motion, this effect is null since $-dB_t$ has the same distribution as dB_t.

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Backward Calculus of Functions of the Poisson Process

Because the Taylor series expansion of $f(x+h)-f(x)$ differs from $f(x)-f(x-h)$ and the fact that the Poisson diffusion variable is discrete and has odd cumulants as well as even, the forward derivative of a function of a Poisson process differs from that of the related backward derivative. Therefore, in some circumstances the forward calculus of a Poisson process must be distinguished from the backward calculus. The symbols Δ and ∇ will denote forward and backward differentials in this context. The calculus of a Poisson variable closely resembles that of discrete calculus. This has material consequences regarding how a Poisson diffusion is defined in terms of differential operators. If the expansion of the forward total derivative of the function of the Poisson diffusion variable is

$$\begin{aligned}\frac{dg(N_t)}{dt} &= \frac{\partial g(N_t)}{\partial N_t}\frac{\Delta N_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial {N_t}^2}\frac{(\Delta N_t)^2}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial {N_t}^3}\frac{(\Delta N_t)^3}{dt} + \dots \\ &= \frac{o(dt)}{dt} + \frac{\partial g(N_t)}{\partial N_t}\frac{\Delta N_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial {N_t}^2}\frac{\Delta N_t}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial {N_t}^3}\frac{\Delta N_t}{dt} + \dots \\ &= \frac{\Delta N_t}{dt}\left(\frac{\partial g(N_t)}{\partial N_t} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial {N_t}^2} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial {N_t}^3} + \dots\right) + \frac{o(dt)}{dt}\end{aligned}$$

then the equivalent backward total derivative is

$$\begin{aligned}\frac{\nabla g(N_t)}{dt} &= \frac{\partial g(N_t)}{\partial N_t}\frac{\nabla N_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial {N_t}^2}\frac{(\nabla N_t)^2}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial {N_t}^3}\frac{(d\square\nabla N_t)^3}{dt} + \dots \\ &= \frac{o(dt)}{dt} + \frac{\partial g(N_t)}{\partial N_t}\frac{\nabla N_t}{dt} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial {N_t}^2}\frac{\nabla N_t}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial {N_t}^3}\frac{\nabla N_t}{dt} + \dots \\ &= \frac{\nabla N_t}{dt}\left(\frac{\partial g(N_t)}{\partial N_t} + \tfrac{1}{2}\frac{\partial^2 g(N_t)}{\partial {N_t}^2} + \tfrac{1}{3!}\frac{\partial^3 g(N_t)}{\partial {N_t}^3} + \dots\right) + \frac{o(dt)}{dt}\end{aligned}$$

by invoking the assertion that $(\nabla N_t)^n = \nabla N_t + o(dt)$ as a direct analogy to $(\Delta N_t)^n = \Delta N_t + o(dt)$.

This is material when there is a modulating factor in a compound Poisson diffusion which requires more exactitude in defining the components of differential change. The forward and backward derivatives that are of most interest to this thesis are

Table 18: forward and backward derivatives of an exponential function of a Poisson process

Function $g(N_t)$	**Forward derivative**	**Backward derivative**
e^{aN_t}	$\frac{\Delta e^{aN_t}}{\Delta N_t} = e^{aN_t}(e^a - 1)$	$\frac{\nabla e^{aN_t}}{\nabla N_t} = e^{aN_t}(1 - e^{-a})$
e^{-aN_t}	$\frac{\Delta e^{-aN_t}}{\Delta N_t} = e^{-aN_t}(e^{-a} - 1)$	$\frac{\nabla e^{-aN_t}}{\nabla N_t} = e^{-aN_t}(1 - e^{a})$
Function $g(N_t)$	**Forward derivative**	**Backward derivative**
a^{N_t}	$\frac{\Delta a^{N_t}}{\Delta N_t} = a^{N_t}(a - 1)$	$\frac{\nabla a^{N_t}}{\nabla N_t} = a^{N_t}(1 - a^{-1})$
a^{-N_t}	$\frac{\Delta a^{-N_t}}{\Delta N_t} = a^{-N_t}(a^{-1} - 1)$	$\frac{\nabla a^{-N_t}}{\nabla N_t} = a^{-N_t}(1 - a)$

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MAXIMUM LIKELIHOOD ESTIMATION OF EXPONENTIAL TRANSACTION TIMES

A. Maximum Likelihood Estimation of the Poisson Intensity – the Mean Number of Trades per Unit Time

With transactions occurring according to a Poisson process $N(t) \sim P(\lambda t)$, the time between transactions is the successive deconvolutions of Erlang distributions with shape parameters increasing in unit steps corresponding to the respective recurrence number. Hence the times between transactions are distributed as independent exponential variables[42]

$$\begin{aligned} D_n &= T_n - T_{n-1} \\ &\sim \Gamma(n,1/\lambda) - \Gamma(n-1,1/\lambda) \\ &\sim \Gamma(1,1/\lambda) \\ &\sim Exp(\lambda) \end{aligned}$$

The density function of an arbitrary duration between transaction $j-1$ and transaction j is then

$$f_{D_j}(t) = \lambda e^{-\lambda t_j} \qquad t \geq 0$$

with distribution function

$$F_{D_j}(t) = 1 - e^{-\lambda t_j} \qquad t \geq 0$$

The data for the individual renewals is then the first differences of the transaction times. The likelihood equation L is

$$L(\lambda) = \prod_{j=1}^{n} f_{D_j}(t) = \lambda^n e^{-\lambda \sum_{j=1}^{n} t_j}$$

and log-likelihood l

$$l(\lambda) = \log L(\lambda) = \sum_{j=1}^{n} \log f_{D_j}(t) = n \log \lambda - \lambda \sum_{j=1}^{n} t_j$$

Differentiating with respect to the Poisson arrival rate λ for the log-likelihood gradient l_λ

$$l_\lambda(\lambda) = \sum_{j=1}^{n} \log f_{D_j}(t) = \frac{n}{\lambda} - \sum_{j=1}^{n} t_j$$

for which the maximum likelihood estimator solves the equation for the single stationary point (a maxima) of the log-likelihood

$$l_\lambda(\hat{\lambda}) = 0 = \sum_{j=1}^{n} \frac{1}{f_{D_j}(t)} \frac{df_{D_j}(t;\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{j=1}^{n} t_j$$

The maximum likelihood estimator is then the reciprocal of the arithmetic mean of the recurrence time between the transactions

$$\hat{\lambda} = 1/\bar{t}$$

This estimator is simple because of the unit root.

---

[42] Note that the set of exponentials themselves form a diffusion basis; an exponential diffusion basis for which the forward transition distribution of the resolved unit root process, from the linearly integrated exponentials, is gamma with shape parameter (and hence all cumulants) proportional to time *t*.

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B. Log-Likelihood Curvature, Information, and the Sampling Distribution of the Maximum Likelihood Estimator of the Poisson Intensity

i. Asymptotic Sampling Distribution of the Poisson Intensity Maximum Likelihood Estimator

Differentiating the log-likelihood again for the second derivative gives the likelihood curvature $l_{\lambda\lambda}$

$$l_{\lambda\lambda}(\lambda) = -\frac{n}{\lambda^2}$$

which is constant and independent of the random variable components of the likelihood. The negative gives the information

$$I(\lambda) = n/\lambda^2$$

The Cramer-Rao lower bound is the inverse of the information $I^{-1}(\lambda) = \lambda^2/n$. Hence, the asymptotic sampling distribution of the estimator of the per unit time Poisson rate intensity is unbiased normal with variance λ^2/n

$$\hat{\lambda} = 1/\bar{t} \sim AN\left(\lambda, \frac{\lambda^2}{n}\right)$$

ii. The Exact Sampling Distribution of the Poisson Intensity Maximum Likelihood Estimator

Each duration, or recurrence time, between two trades is exponentially distributed since trades are events generated by a Poisson process, and the sum of the n observed trades is a realization of a gamma distribution with a shape parameter *n*

$$t_j \sim \Gamma(1, 1/\lambda)$$

$$\sum_{j=1}^{n} t_j \sim \Gamma(n, 1/\lambda)$$

Scaling the gamma to give the sampling distribution of the mean,

$$\bar{t} = \frac{1}{n}\sum_{j=1}^{n} t_j \sim \Gamma\left(n, \frac{1}{n\lambda}\right)$$

which is a gamma distribution with shape parameter n and scale parameter $1/n\lambda$. This is the exact sampling distribution of the maximum likelihood estimator of the inverse of the Poisson rate intensity which is the scale of the exponential transaction times as well the mean duration between transactions. The inverse of this distribution, then, is the exact sampling distribution of the associated estimator of the Poisson intensity

$$\frac{1}{\bar{t}} \sim inv\Gamma\left(n, \frac{1}{n\lambda}\right)$$
$$\sim 1 \Big/ \Gamma\left(n, \frac{1}{n\lambda}\right)$$

which is an inverse gamma variable. Exact calculations regarding the sampling distribution can proceed with this distribution. Since the gamma variable has a non-zero probability at zero, the inverse gamma variable has a asymptote in probability and therefore has complexity in the set of distributional moments. The density function of this distribution is

$$f_{1/\bar{t}}(x) = \frac{(n\lambda)^n}{\Gamma(n)} \frac{e^{-n\lambda/x}}{x^{n+1}} \quad x \geq 0$$

The general k^{th} raw moment, for $k < n$, is

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$$E(1/\bar{t}^{\,k}) = \frac{\lambda^k n^k \Gamma(n-k)}{\Gamma(n)}$$

so that the mean and variance are

$$E(1/\bar{t}) = \frac{\lambda n}{n-1} \qquad \text{and} \qquad E(1/\bar{t}^{\,2}) = \frac{\lambda^2 n^2}{(n-1)^2(n-2)}$$

which are equal to the asymptotic mean and variance, λ and λ^2/n, only as n becomes large. The moments exist up to, but excluding, order n. Also, the skewness and kurtosis are

$$Skew(1/\bar{t}) = \frac{4\lambda^3 n^3}{(n-1)^3(n-2)(n-3)} \qquad \text{and} \qquad Kurt(1/\bar{t}) = \frac{3\lambda^4 n^4 (n+5)}{(n-1)^4(n-2)(n-3)(n-4)}$$

giving coefficients of skewness and kurtosis of

$$\xi_3(1/\bar{t}) = \frac{\sqrt{n-2}}{n-3} \qquad \text{and} \qquad \xi_4(1/\bar{t}) = 3\frac{(n-2)(n+5)}{(n-3)(n-4)}$$

so that the distribution is skew and leptokurtic. These two statistics approach zero and three, respectively, as n becomes large. Therefore, the skewness vanishes and the shape becomes mesokurtic leaving a mesokurtic bell. Therefore, the asymptotic distribution of the maximum likelihood estimator of the Poisson arrival rate is normal[43] with mean $\lambda n/(n-1) \to \lambda$ and variance $\lambda^2 n^2/(n-1)^2(n-2) \to \lambda^2/n$. This convergence occurs as the inverse gamma distribution gets a full set of real and finite moments as the mode of the gamma in the denominator moves further and further away from the origin as the shape parameter increases with subsequent exponentials convolved onto the time progression of the summed transaction times in the observation set.

The density function of the generalized inverse Gaussian distribution is

$$f(x) = \frac{(b/a)^{\lambda}}{2K_{\lambda}(ab)} x^{\lambda-1} \exp\left(-\tfrac{1}{2}(a^2/x + b^2 x) \right)$$

which gives a moment generating function of the inverse gamma distribution

$$M_{1/\bar{t}}(\theta) = \frac{2}{\Gamma(n)} (\sqrt{-n\lambda\theta})^n K_n(2\sqrt{-n\lambda\theta})$$

where $K_{\lambda}(x)$ is the modified Bessel function of the third kind[44].

Given the analytical difficulties concerning the inverse gamma distribution, namely a distribution with complex moments, it may be good practice to work with the inverse of the estimator of the Poisson intensity which is gamma distributed and the scale of the exponential spacings, rather than the estimator itself, whenever feasible.

MODELLING AND QUANTIFYING EXPONENTIAL TRANSACTION TIMES

Before calculating bounds for transaction times and rates, the transaction time data should be visualized graphically with empirical PDFs and CDFs (histograms and ogives) as well price and volume over transaction time. The PDF should be exponential and can be supplemented with a QQ Plot for the exponential distribution.

---

[43] This is very interesting because both $\bar{t}$ and its reciprocal $1/\bar{t}$ are both asymptotically normal. The distribution of $\bar{t}$ is closer to standard normal than $1/\bar{t}$ so it is preferable to work in $1/\bar{\lambda} = \bar{t}$.

[44] The density function of the generalised inverse Gaussian distribution is given in
i. Schoutens, W., (2003) Levy Processes in Finance: Pricing financial derivatives, John Wiley & Sons.
The formulae for the modified Bessel function of the third kind is contained in
ii. Abramowitz, M. and Stegun, I. A. (1968) *Handbook of Mathematical Functions*. Dover.

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Preliminary Result: the Gamma Distribution as a Scaled Chi-Square Distribution

A gamma distribution is also a scaled chi-square distribution so that cumulative density functions and percentiles of gamma distributions can be expressed in terms of cumulative density functions and percentiles of chi-square distributions;

$$\begin{aligned} X &\sim \Gamma(n,1) \\ &\sim \tfrac{1}{2}\Gamma(n,2) \\ &\sim \tfrac{1}{2}\Gamma(2n/2,2) \\ &\sim \tfrac{1}{2}\chi^2_{2n} \end{aligned}$$

By the scaling properties of the gamma function, a gamma function with shape parameter n and unit scale is one half a chi-square variable with $2n$ degrees of freedom. As such, an incomplete gamma function can be expressed as a cumulative density function of a chi-square variable

$$F_{\Gamma(n,1)}(x) = F_{\chi^2_{2n}}(2x)$$

the mathematics for which is,

$$\begin{aligned} F_{\Gamma(n,1)}(x) &= \Pr\left(\Gamma(n,1) < x\right) \\ &= \Pr\left(\tfrac{1}{2}\Gamma(2n/2,2) < x\right) \\ &= \Pr\left(\Gamma(2n/2,2) < 2x\right) \\ &= \Pr\left(\chi^2_{2n} < 2x\right) \\ &= F_{\chi^2_{2n}}(2x) \end{aligned}$$

Hence, the inverse function of an incomplete gamma function can be expressed as a percentile of a chi-square distribution

$$F^{-1}_{\Gamma(n,1)}(p) = \frac{1}{2}F^{-1}_{\chi^2_{2n}}(p)$$

tables and statistical functions in software being widely available.

A. Mean Number of Trades

(i) Per Unit Time

For transactions occurring according to a Poisson process, the mean recurrence time between subsequent transactions is $1/\lambda$ and the mean rate at which transactions occur is λ per unit of time.

The exact sampling distribution of the estimator of the mean number of trades per unit time, the Poisson rate intensity, is inverse gamma with shape parameter n and scale $1/n\lambda$, . The density function is

$$f_{1/\Gamma(n,1/n\lambda)}(x) = \frac{(n\lambda)^n e^{-n\lambda/x}}{x^{n+1}\Gamma(n)}$$

For a 95% probability interval, with 2.5% probability in each tail, the lower bound LB solves the equation

$$0.025 = \int_0^{LB} f_{1/\Gamma(n,1/n\lambda)}(x)dx \,,$$

i.e.,

$$0.025 = \int_0^{LB} \frac{(n\lambda)^n e^{-n\lambda/x}}{x^{n+1}\Gamma(n)}dx$$

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With a change of variable to $y = n\lambda / x$, the integral equation becomes

$$0.025 = \int_{n\lambda / LB}^{\infty} \frac{y^{n-1}e^{-y}}{\Gamma(n)} dy = 1 - F_{\Gamma(n,1)}(n\lambda / LB)$$

where $F_{\Gamma(n,1)}(\cdot)$ is the incomplete gamma function, the cumulative density function of a gamma density with shape parameter n and unit scale. Solving the equation for the lower bound of the confidence interval

$$F_{\Gamma(n,1)}(n\lambda / LB) = 0.975$$

or

$$n\lambda / LB = F^{-1}_{\Gamma(n,1)}(0.975)$$

gives

$$LB = \frac{n\lambda}{F^{-1}_{\Gamma(n,1)}(0.975)}$$

where $F_{\Gamma(n,1)}{}^{-1}(0.975)$ is the 97.5^{th} percentile of the gamma function with unit scale. The upper bound of the probability interval obtained by solving

$$0.975 = \int_{UB}^{\infty} f_{1/\Gamma(n,,n\lambda)}(x)dx$$

is, by symmetric arguments,

$$UB = \frac{n\lambda}{F^{-1}_{\Gamma(n,1)}(0.025)}$$

The maximum likelihood estimators of the bounds for are

$$LB = \frac{n}{\bar{t}\ F^{-1}_{\Gamma(n,1)}(0.975)} \qquad \text{and} \qquad UB = \frac{n}{\bar{t}\ F^{-1}_{\Gamma(n,1)}(0.025)}$$

In terms of chi-square probabilities, for which tables are common, these can be expressed as

$$LB = \frac{2n}{\bar{t}\ F^{-1}_{\chi^2_{2n}}(0.975)} \qquad \text{and} \qquad UB = \frac{2n}{\bar{t}\ F^{-1}_{\chi^2_{2n}}(0.025)}$$

where $F_{\chi^2_{2n}}(x)$ is the cumulative density function and $F^{-1}\chi^2_{2n}(p)$ is the p^{th} percentile of a chi-square distribution with $2n$ degrees of freedom.

As was shown, the estimator is unbiased normal with λ and variance λ^2 / n

$$\hat{\lambda} = 1/\bar{t} \sim AN\left(\lambda, \frac{\lambda^2}{n}\right)$$

The asymptotic confidence interval for the mean number of transactions per unit time is

$$\hat{\lambda} \pm F^{-1}_{N(0,1)}(\alpha/2)\frac{\hat{\lambda}}{\sqrt{n}} = \frac{1}{\bar{t}} \pm F^{-1}_{N(0,1)}(\alpha/2)\frac{1}{\bar{t}\sqrt{n}}$$

and for 95% probability,

$$\frac{1}{\bar{t}} \pm 1.96\frac{1}{\bar{t}\sqrt{n}}$$

Given that the number of transactions cannot be negative, the lower boundary cannot be below zero – considering the high frequency at which trades occur (with small durations in between), the parameter is likely to be small and the number of observations very large. The sampling variance is therefore likely to be small and the

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point estimate of the mean is likely to be very close to the correct mean for the duration between any two transactions. One may want to consider calculating the bounds based upon the exact sampling distribution of inverse gamma and compare the two intervals, i.e., compare the asymptotic interval to that based on the exact sampling distribution.

(ii) Per *k* Units of Time

Say one is now interested in the mean number of transactions over k successive units of time; due to homogeneity of the distribution between trades and the sampling distribution of the parameter estimator, the distribution is again asymptotically unbiased normal with k additions of variance, i.e., a sum of k homogenous random variables of the sampling distribution

$$\sum_{j=1}^{k} \hat{\lambda} \sim AN\left(k\lambda, \frac{k\lambda^2}{n}\right)$$

The probability interval for the mean number of trades over k successive units of time, based on the point estimate of the parameter, is

$$k\hat{\lambda} \pm F_{N(0,1)}^{-1}(\alpha/2)\frac{k\hat{\lambda}}{\sqrt{n}} = \frac{k}{\bar{t}} \pm F_{N(0,1)}^{-1}(\alpha/2)\frac{k}{\bar{t}\sqrt{n}}$$

and for 95% probability

$$\frac{k}{\bar{t}} \pm 1.96\frac{k}{\bar{t}\sqrt{n}}$$

The exact sampling distribution for k time periods is not practical since it is not possible to sum inverse gamma variables into a statistical distribution with a known and convenient form. One could alternatively take the view that the k periods of time is simply a rescaling of the estimator, say,

$$\hat{\lambda} = 1/\bar{t} \sim 1/\Gamma(n, 1/n\lambda)$$

$$\begin{aligned} k\hat{\lambda} = k/\bar{t} &\sim \frac{1}{(1/k)\Gamma(n, 1/n\lambda)} \\ &\sim \frac{1}{\Gamma(n, 1/nk\lambda)} \\ &\sim \frac{1}{\Gamma(n, 1/n\lambda')} \end{aligned}$$

where $\lambda' = k\lambda$. The bounds are then simply k times the bounds for the interval for a single Poisson time unit as for the asymptotic interval. This is a conservative estimate since scaling the inverse gamma results in a distribution that has greater variance than that which would result from convolving k inverse gammas.

B. Mean Recurrence Time Between Successive Trades

The mean duration between occurrences β is the reciprocal of the frequency at which trades occur, i.e., the inverse of the Poisson intensity,

$$\beta = h(\lambda) = 1/\lambda$$

By the invariance property of maximum likelihood estimators, the maximum likelihood estimator of inverse Poisson intensity is the inverse of the maximum likelihood estimator of the Poisson intensity itself. This is the average of the transaction times

$$\hat{\beta} = \bar{t}$$

The exact sampling distribution of this estimator was found in obtaining the exact sampling distribution of the Poisson intensity and is gamma distributed

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$$\hat{\beta} = \bar{t} = \frac{1}{n}\sum_{j=1}^{n} t_j \sim \Gamma\left(n, \frac{1}{n\lambda}\right)$$

and the asymptotic sampling distribution of the estimator, from the gamma Central Limit Theorem, is normal

$$\hat{\beta} = \frac{1}{\hat{\lambda}} = \bar{t} \sim AN\left(\frac{1}{\lambda}, \frac{1}{n\lambda^2}\right) \sim AN\left(\beta, \frac{\beta^2}{n}\right)$$

(i) Duration Between Any Two Trades

The exact sampling distribution of the estimator of the Poisson event recurrence time, the inverse of the Poisson rate intensity and scale of the exponential recurrence, is gamma with shape parameter n and scale $1 / n\lambda$. The density function is

$$f_{\Gamma(n,1/n\lambda)}(t) = \frac{(n\lambda)^n t^{n-1} e^{-n\lambda t}}{\Gamma(n)}$$

For a 95% probability interval, with 2.5% probability in each tail, the lower bound LB solves the equation

$$0.025 = \int_0^{LB} f_{\Gamma(n,1/n\lambda)}(x)dx$$

i.e.,

$$0.025 = \int_0^{LB} \frac{(n\lambda)^n t^{n-1} e^{-n\lambda t}}{\Gamma(n)} dt$$

With a change of variable $s = n\lambda t$, the equation becomes

$$0.025 = \int_0^{n\lambda \times LB} \frac{s^{n-1} e^{-s}}{\Gamma(n)} ds = F_{\Gamma(n,1)}(n\lambda \times LB)$$

where $F_{\Gamma(n,1)}(\cdot)$ is the incomplete gamma function, the cumulative density function of a gamma density with shape parameter n and unit scale. Solving the equation by inverting the incomplete gamma function,

$$n\lambda \times LB = F^{-1}_{\Gamma(n,1)}(0.025)$$

or

$$LB = \frac{F^{-1}_{\Gamma(n,1)}(0.025)}{n\lambda}$$

where $F^{-1}{}_{\Gamma(n,1)}(0.025)$ is the tail probability of a unit scale gamma with shape parameter n. By symmetrical mathematics, the upper bound of the probability interval, obtained by solving

$$0.975 = \int_{UB}^{\infty} f_{\Gamma(n,,n\lambda)}(x)dx$$

is

$$UB = \frac{F^{-1}_{\Gamma(n,1)}(0.975)}{n\lambda}$$

The, the bounds for the average time between trades, are reciprocals of those for the mean number of transactions. The bounds for are estimated by inserting the maximum likelihood estimator into the parameter ,

$$LB = \frac{\bar{t}}{n} F^{-1}_{\Gamma(n,1)}(0.025) \qquad \text{and} \qquad UB = \frac{\bar{t}}{n} F^{-1}_{\Gamma(n,1)}(0.975)$$

In terms of chi-square probabilities, these can be expressed as

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$$LB = \frac{\bar{t}}{2n} F^{-1}_{\chi^2_{2n}}(0.025) \qquad \text{and} \qquad UB = \frac{\bar{t}}{n} F^{-1}_{\chi^2_{2n}}(0.975)$$

The asymptotic probability interval for the average duration between any two transactions, with a 5% type I error rate, is

$$\begin{aligned}\frac{1}{\hat{\lambda}} \pm 1.96 \frac{1}{\hat{\lambda}\sqrt{n}} &= \bar{t} \pm 1.96 \bar{t} / \sqrt{n} \\ &= \bar{t}\left(1 \pm 1.96 / \sqrt{n}\right)\end{aligned}$$

(ii) Duration Between Any *k* + 1 Trades

Say now one is interested in the mean duration between k successive trades. Again due to homogeneity of the independent increments, the Poisson process that compose the distribution between $k + 1$ successive trades is the sum of k variables of the renewal distribution between any two trades (k durations for $k + 1$ renewals including the first).

The exact sampling distribution of the estimator of the k-fold convolution of Poisson recurrence times, also the convolution of the scale of the individual exponential recurrence times, is gamma with shape parameter kn and scale $1 / n\lambda$ by the convolution properties of gammas with homogenous scale

$$\begin{aligned}\sum_{j=1}^{k} \hat{\beta}_j &= \sum_{j=1}^{k} \bar{t} \\ &\sim \sum_{j=1}^{k} \Gamma\left(n, \frac{1}{n\lambda}\right) \\ &\sim \Gamma\left(kn, \frac{1}{n\lambda}\right)\end{aligned}$$

The density function is

$$f_{\Gamma(kn,1/n\lambda)}(t) = \frac{(n\lambda)^{kn} t^{kn-1} e^{-n\lambda t}}{\Gamma(kn)}$$

The 95% probability bounds for 2.5% in each tail solve the equations

$$0.025 = \int_0^{LB} f_{\Gamma(kn,1/n\lambda)}(t)dt \qquad \text{and} \qquad 0.975 = \int_{UB}^{\infty} f_{\Gamma(kn,1/n\lambda)}(t)dt$$

And by similar mathematics as for a single renewal, percentiles now for a gamma with shape parameter nk rather than n, the bounds of the interval are

$$LB = \frac{F^{-1}_{\Gamma(kn,1)}(0.025)}{n\lambda} \qquad \text{and} \qquad UB = \frac{F^{-1}_{\Gamma(kn,1)}(0.975)}{n\lambda}$$

and estimated by maximum likelihood as

$$LB = \frac{\bar{t}}{n} F^{-1}_{\Gamma(kn,1)}(0.025) \qquad \text{and} \qquad UB = \frac{\bar{t}}{n} F^{-1}_{\Gamma(kn,1)}(0.975)$$

These can be expressed in terms of percentiles of a chi-square distribution with $2nk$ degrees of freedom

$$LB = \frac{\bar{t}}{2n} F^{-1}_{\chi^2_{2kn}}(0.025) \qquad \text{and} \qquad UB = \frac{\bar{t}}{2n} F^{-1}_{\chi^2_{2n}}(0.975)$$

The asymptotic distribution of the estimator is unbiased normal with k additions of variance

$$\sum_{j=1}^{k} \frac{1}{\hat{\lambda}} \sim AN\left(\frac{k}{\lambda}, \frac{k}{n\lambda^2}\right)$$

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and a probability interval for the average recurrence time between any $k + 1$ trades, the k successive recurrence times between say the m^{th} trade and the $(m + k + 1)^{th}$ trade, is

$$\frac{k}{\hat{\lambda}} \pm 1.96 \frac{1}{\hat{\lambda}\sqrt{n}} = k\bar{t} \pm 1.96\, \bar{t}\sqrt{k/n}$$
$$= \bar{t}\left(k \pm 1.96\sqrt{k/n}\right)$$

C. Tail Probabilities for Transaction Times

From the distribution function of the time between successive trades that compose Poisson process increments, the percentile function is

$$\pi(t_j) = 1 - e^{-\lambda t_j}$$
$$e^{-\lambda t_j} = 1 - \pi(t_j)$$
$$t_j(\pi) = -\frac{1}{\lambda}\log\left(1 - \pi(t_j)\right)$$

i.e.,

$$t(\pi) = -\frac{\log(1-\pi)}{\lambda} = -\beta\log(1-\pi)$$

For instance, immediately following a transaction event there is a 2.5% probability that the next transaction occurs within $t(0.025) = -\log(0.975) / \lambda = 0.0253 / \lambda$ time units and a 97.5% probability that it will occur within $t(0.975) = -\log(0.025) / \lambda = 3.6889 / \lambda$ time units. This function $t(\pi)$ is proportional to the inverse of the Poisson parameter, the average rate at which transactions occur per unit time. Consequently, it is proportional to the mean number of transactions per unit time and is simply a constant scaling of the mean duration between Poisson occurrences of transactions. The maximum likelihood estimator of the percentile is

$$\hat{t}(\pi) = \frac{-1og(1-\pi)}{\hat{\lambda}} = -\bar{t}\log(1-\pi)$$

Since the renewals are exponentially distributed, and the exponential distribution has an analytic inverse of its cumulative distribution function, tail probabilities π of any renewal length are also analytic. The tail probabilities are simply rescalings of the sampling distribution of the scale of the renewal distribution

$$-\hat{\beta}\log(1-\pi) \sim -\log(1-\pi)\Gamma\left(n, \frac{1}{n\lambda}\right)$$
$$\sim \Gamma\left(n, -\frac{\log(1-\pi)}{n\lambda}\right)$$
$$\sim \Gamma\left(n, \frac{1}{n\lambda'}\right)$$

where $\lambda' = -[\log(1 - \pi)] / \lambda$. From the intervals of the renewal time of the next transaction, the bounds of the 95% probability interval for the percentile π are

$$LB = -\log(1-\pi)\frac{F^{-1}_{\Gamma(n,1)}(0.025)}{n\lambda} \quad \text{and} \quad UB = -\log(1-\pi)\frac{F^{-1}_{\Gamma(n,1)}(0.975)}{n\lambda}$$

and the maximum likelihood estimators are

$$LB = -\log(1-\pi)\frac{\bar{t}}{n}F^{-1}_{\Gamma(n,1)}(0.025) \quad \text{and} \quad UB = -\log(1-\pi)\frac{\bar{t}}{n}F^{-1}_{\Gamma(n,1)}(0.975)$$

These are alternatively expressed in terms of chi-square probabilities

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$$LB = -\log(1-\pi)\frac{\bar{t}}{2n}F^{-1}_{\chi^2_{2n}}(0.025) \qquad \text{and} \qquad UB = -\log(1-\pi)\frac{\bar{t}}{2n}F^{-1}_{\chi^2_{2n}}(0.975)$$

The asymptotic sampling distribution of the estimator of the percentile π is asymptotically normal

$$t(\pi) = \frac{-1og(1-\pi)}{\hat{\lambda}} = -\bar{t}\log(1-\pi) \sim AN\left(\frac{-\log(1-\pi)}{\lambda}, \frac{\log^2(1-\pi)}{n\lambda^2}\right)$$

and a 95% probability interval for an estimated percentile π of the time between Poisson occurrences is

$$-\log(1-\pi)/\hat{\lambda} \pm 1.96\frac{\log^2(1-\pi)}{\hat{\lambda}\sqrt{n}} = -\bar{t}\log(1-\pi) \pm 1.96\,\bar{t}\,\frac{\log^2(1-\pi)}{\sqrt{n}}$$
$$= -\bar{t}\log(1-\pi)\left(1 \pm 1.96\frac{\log(1-\pi)}{\sqrt{n}}\right)$$

For the time between $k + 1$ Poisson occurrences, the statistical technique can be extended like that for the mean duration between any $k + 1$ trades.

In the diffusion limit, the width of these intervals approach zero. *The rate at which the width of these intervals approach zero is also approximately the rate at which a compound Poisson diffusion approaches an ordinary diffusion.*

THE COMPOUND POISSON PROCESS: ARBITRARY ADDITIVE BASIS COMPONENTS

Recurrence Dynamics, the Diffusion Equation, and its Resolution

For any differential increment in the generator, $dN(t)$, the compound Poisson diffusion either increases with a sultus of a convolved scaled basis component $X_t h$ or remains unchanged. These two outcomes depend on whether the Poisson process increments or does not increment. The equation expressing this is the **forward recurrence** of the compound Poisson diffusion

$$S(t+dt) = [S(t) + X_t h]dN(t) + S(t)[1 - dN(t)] + o(dt)$$

From the differential form of the diffusion equation of the Poisson process, $dN(t) = z_t dt$, this can be expressed as

$$S(t+dt) = z_t dt[S(t) + X_t h] + (1 - z_t dt)S(t) + o(dt)$$

Rearranging terms gives

$$S(t+dt) - S(t) = z_t X_t h dt + o(dt)$$

Dividing by dt

$$\frac{S(t+dt) - S(t)}{dt} = z_t X_t h + \frac{o(dt)}{dt}$$

and letting $dt \to 0$ so that $o(dt)/dt \to 0$ results in the **differential form of the diffusion equation** for an ordinary compound Poisson diffusion

$$\frac{dS(t)}{dt} = z_t X_t h$$

where h is proportional to the inverse of the Poisson rate intensity, $h \propto 1/\lambda$. This is an ordinary first order unit root diffusion with an ordinary basis variable $X_t h$ dotted with a Poisson basis variable z_t . Dividing both sides by the Poisson basis variable z_t

$$\frac{dS(t)}{z_t dt} = X_t h$$

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and whenever $z_t dt$ is held fixed at one, $z_t dt = 1$, for all values of t and the span variable h is proportional to dt, $h \propto dt$, the differential diffusion equation and all of the derived equations simplify to those of ordinary Brownian motion $dS(t) \propto X_t dt$

Generator for ordinary diffusion, $z_t\, dt = 1$ for $t \geq 0$



Reconsidering the differential form of the diffusion equation of the compound Poisson process, noting again from the Poisson process that $dN(t) = z_t\, dt$, the differential form can alternatively be expressed as

$$\frac{dS(t)}{dN(t)} = X_t h \qquad \text{or} \qquad \frac{1}{h}\frac{dS(t)}{dN(t)} = X_t$$

which expresses the diffusion equation as an ordinary unit root diffusion that changes differentially with respect to a generating Poisson process. Resolving the compound Poisson diffusion equation from the form, $dS(t) = X_t h\, z_t\, dt$, by integrating over time from time zero (the backward time variable) to the forward time variable

$$\int_{y=0}^{t} dS(y) = h\int_{y=0}^{t} X_y z_y dy$$

results in one form of the **integral expression** of the diffusion variable $S(t)$

$$S(t) = S(0) + h\int_{y=0}^{t} X_y z_y dy$$

Letting the times at which the Poisson events of the generator basis z_y occur be t_j . The scaled Poisson basis is one at the times t_j and zero everywhere else

$$z_y dy = \begin{cases} 1 & \text{if } y = t_j \\ 0 & \text{if } y \neq t_j \end{cases} \qquad \text{for } j = 1,\ 2,\ ,3,\ \ldots,\ N(t) \text{ and } 0 \leq y \leq t$$

Poisson basis function, $z_t\, dt \sim P(\lambda dt)$



so that

$$X_y z_y dy = \begin{cases} 1 \times X_y & \text{if } y = t_j \\ 0 \times X_y & \text{if } y \neq t_j \end{cases} = \begin{cases} X_y & \text{if } y = t_j \\ 0 & \text{if } y \neq t_j \end{cases} \qquad \text{for } j = 1,\ 2,\ ,3,\ \ldots,\ N(t) \text{ and } 0 \leq y \leq t$$

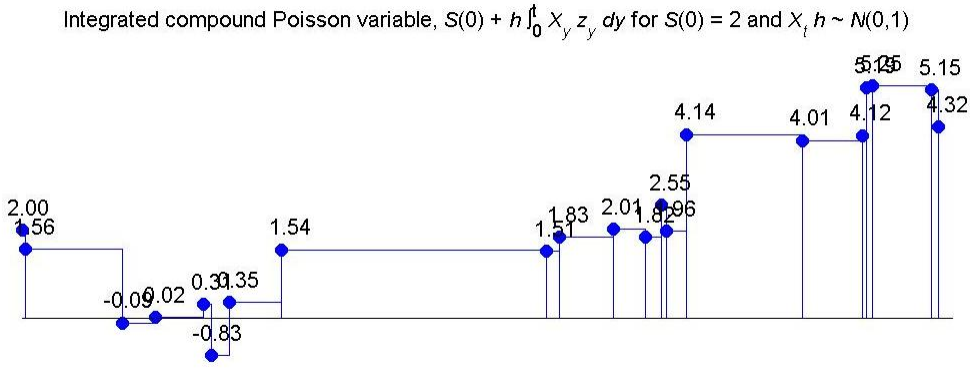
and there is a total of $N(t)$ times t_j at which the Poisson generator produces ones and resulting convolutions of diffusion basis variables. Therefore, the integral part of the resolved diffusion equation of the compound Poisson process can alternatively be written as a discrete sum

$$\int_{y=0}^{t} X_y z_y dy = \sum_{j=1}^{N(t)} X_{t_j}$$

and the **resolved diffusion variable** can alternatively be written as

$$S(t) = S(0) + \sum_{j=1}^{N(t)} X_{t_j} h$$

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and due to the homogeneity and time invariance of the diffusion basis variable, $X_{t_j} = X_j$, so that

$$S(t) = S(0) + \sum_{j=1}^{N(t)} X_j h$$

This is exactly the form of the integral expression of discrete first order unit root diffusion from PART II of BOOK 3 with the forward time index n replaced by a Poisson process $N(t)$. The diffusion variable, at time T, unconditional on the Poisson generator $N(T)$ is then

$$S(T) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \left(S(0) + \sum_{j=1}^{n} X_j h \right)$$

where, for convenience, it is assumed that a summation from 1 to zero, $\sum_1^0$ is a null sum so that

$$S(T) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times e^{-\alpha h n} S(0) + \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \left(e^{-\alpha h n} \sum_{j=1}^{n} e^{\alpha h j} X_j h \right)$$

that is

$$S(T) = S(0) + \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \left(\sum_{j=1}^{n} X_j h \right)$$

Limiting Form as h Becomes Infinitesimal, h → 0

From the moment generating function of the Poisson process $N(t) \sim P(\lambda t)$

$$M_{P(t\lambda)}(\theta) = \exp\left(\lambda t (e^{\theta} - 1)\right)$$

Letting the Poisson parameter be equal to the inverse of $\lambda = 1 / h$, the generating function becomes

$$M_{P(t/h)}(\theta) = \exp\left(t(e^{\theta} - 1)/h\right)$$

and scaling the variable by h, i.e., $h\,N(t) \sim h\,P(\lambda t)$ results in

$$M_{P(t/h)\times h}(\theta) = \exp\left(t(e^{\theta} - 1)h\right) = M_{P(t/h)}(\theta h) = \exp\left(t(e^{\theta h} - 1)/h\right)$$

Expanding the exponential in the index in a Maclaurin series

$$\begin{aligned} M_{hN(t)}(\theta) &= \exp\left(t[\theta h + o(h)]/h\right) \\ &= \exp\left(t\theta + o(h)/h\right) \end{aligned}$$

which limits to $\exp(t\theta)$ as $h \to 0$

$$\lim_{\substack{\lambda = 1/h \\ h \to 0}} M_{hN(t)}(\theta) \to \exp(t\theta)$$

The limit of the scaled Poisson process, therefore, is linear time

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$$\lim_{h\to 0} hN(t) \sim \lim_{h\to 0} hP(t/h) \to t$$

Since $j = 1, 2, 3, \ldots, N(t)$ in the discrete sum

$$\sum_{j=1}^{N(t)} X_{jh} h$$

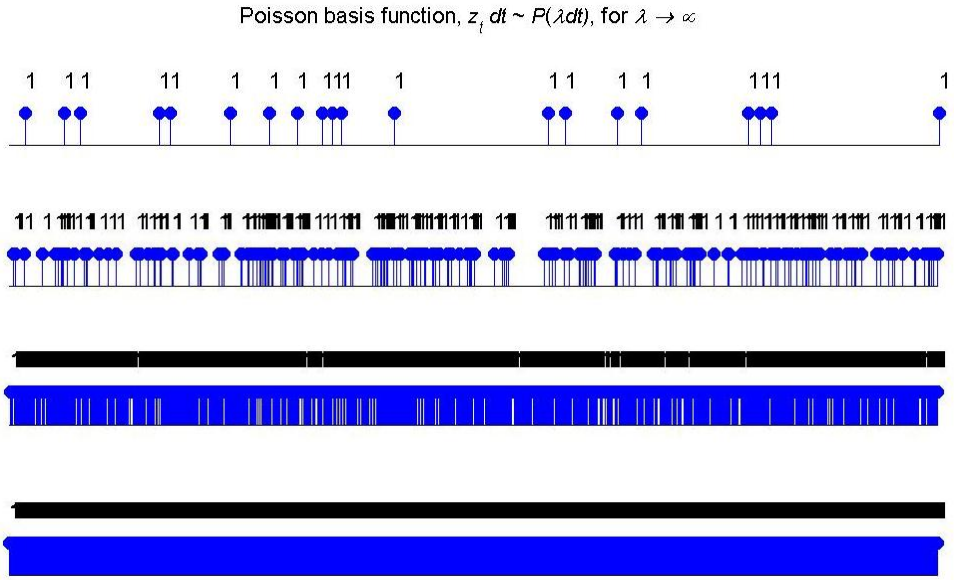
the time indexing argument of the basis function X is $jh = h, 2h, 3h, \ldots, h\,N(t)$. The infinitesimal limit of the upper limit of the integration of the secondary basis components in the integral solution is consequently proportional to the forward time variable t; with $\lambda = 1/h$, taking the limits $h \to dy \to 0$, subject to $hj \to y$, so that hj becomes a time continuum from 0 to $h\,N(t)$, and utilizing the limit

$$\lim_{h\to 0} hN(t) = t$$

results in the resolved diffusion variable of first order unit root diffusion, Brownian motion,

$$S(t) = S(0) + \int_{y=0}^{t} X_y dy$$

where the limiting form of the Poisson basis function



becomes an orthogonal continuous basis function, i.e., a Gaussian-Hilbert space. This, together with the theory for modulated roots and multiplicatively summed basis functions, provides a complete foundation for compound Poisson diffusions pertaining to BOOK 3, the Diffusion Calculus of Arbitrary Order, to model high frequency transaction data.

Transition Density of a Compound Poisson Diffusion

From the discrete sum form of the resolved diffusion equation of the compound Poisson variable, it is seen that $N(t)$ counts the number of random contributions of independent copies of the time invariant basis distribution X onto the aggregate diffusion variable, the transition distribution conditional on the starting or initial value of the compound sum, $S(t) \mid S(t_0)$ or $S(t) \mid S(t_0) = x_0$ where it is usual to assume for convenience that the backward time variable is zero, $t_0 = 0$. The notation $S(t)$ is shorthand for the conditional variable $S(t) \mid S(t_0)$ and $f_{S(t)}(x \mid x_0)$ or $f_{S(t)}(x)$ is shorthand for the transition density function $f_{S(t)\mid S(t_0)}(x \mid x_0)$. The transition variable of the compound Poisson process sum from the resolved diffusion equation is

$$S(t) = x_0 + X_1 h + X_2 h + \ldots + X_{N(t)} h = x_0 + \sum_{j=1}^{N(t)} X_j h$$

where $S(0) = x_0$ and since it is assumed that the secondary basis variables convolving onto the random sum are independent and identically distributed, the density and distribution function of the process are $N(t)$ fold convolutions of the secondary basis density and distribution functions

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$$f_{S(t)|S(0)} = f_{x_0+X_1+X_2+\ldots+X_{N(t)}} = f_{x_0} * f_{X_1} * f_{X_2} *\ldots* f_{X_{N(t)}}$$

and

$$F_{S(t)|S(0)} = F_{x_0+X_1+X_2+\ldots+X_{N(t)}} = F_{x_0} * F_{X_1} * F_{X_2} *\ldots* F_{X_{N(t)}}$$

where it is assumed that the span variable h has been factored into the cumulants of the X_j to keep the equations uncluttered. The transition density is hence

$$f_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times f_{x_0} * f_{X_1} *\ldots* f_{X_n}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times f_{X^{*n}}(x \mid x_0)$$

where $f_{X^{*0}}(x) = \delta_{x_0}(x)$ is a Dirac delta function, and the transition distribution function is

$$f_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times F_{x_0} * F_{X_1} *\ldots* F_{X_n}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times F_{X^{*n}}(x \mid x_0)$$

If the scaled basis variable is normal with mean μh and variance $\sigma^2 h$, $X \sim N(\mu h, \sigma^2 h)$, then the transition density and distribution functions are

$$f_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \frac{1}{\sigma\sqrt{h\,n}\sqrt{2\pi}} \exp\left(-\frac{[x - x_0 - \mu h n]^2}{2\sigma^2 h\,n}\right)$$

and

$$F_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{h\,n}\sqrt{2\pi}} \exp\left(-\frac{[y - x_0 - \mu h n]^2}{2\sigma^2 h\,n}\right) dy$$

$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \Phi\left(\frac{x - x_0 - \mu h n}{\sigma\sqrt{h\,n}}\right)$$

where $\Phi(x)$ is the cumulative distribution function of a standard normal $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy$.

Transition MGF and Moment Analysis

The moment generating functions of the component distributions of the diffusion basis function are each $M_X(\theta)$ since the components are time invariant and homogenous. The moment generating function of the unit root compound distribution, conditional on the Poisson generator, can be expressed as

$$M_{S(t)|N(t)}(\theta) = M_{x_0+X_1+X_2+\ldots+X_{N(t)}}(\theta) = e^{\theta x_0} M_X(\theta)^{N(t)}$$

or in terms of cumulant generating functions

$$K_{S(t)|N(t)}(\theta) = \theta x_0 + N(t) K_X(\theta)$$

It is seen from the cumulant generating function that each of the cumulants fall out upon differentiation as linear functions of the Poisson generating variable $N(t)$. The moment generating function of the unconditional compound variable is then the expectation of the conditional moment generating function with regard to the Poisson generator

$$M_{S(t)}(\theta) = E\left(M_{S(t)|N(t)}(\theta)\right) = E\left(M_{x_0+X_1+X_2+\ldots+X_{N(t)}}(\theta)\right) = E\left(e^{\theta x_0} M_X(\theta)^{N(t)}\right) = e^{\theta x_0} E\left(e^{N(t)\log M_X(\theta)}\right)$$

that is,

$$M_{S(t)}(\theta) = e^{\theta x_0} M_{N(t)}\left(\log M_X(\theta)\right)$$

This also implies that the unconditional moments of the diffusion variable are expectations of the conditional moments. In the case of a compound Poisson process, the Poisson generator is Poisson with mean λt,

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$N(t) \sim P(\lambda t)$, so that the moment generating function of the distribution which counts the random number of contributions onto the compound sum of cumulative noise is

$$M_{N(t)}(\theta) = \exp\left(\lambda t(e^{\theta} - 1)\right)$$

The moment generating function of the diffusion variable is whence

$$M_{S(t)}(\theta) = \exp\left(\theta x_0 + \lambda t[M_X(\theta) - 1]\right)$$

and the cumulant generating function $K_{S(t)}(\theta) = \log M_{S(t)}(\theta)$ is then

$$K_{S(t)}(\theta) = \theta x_0 + \lambda t[M_X(\theta) - 1]$$

The r^{th} cumulant of the compound Poisson process, ignoring the initial value (the backward diffusion variable) which only effects the first[45], from the r^{th} derivative of the cumulant generating function,

$$K^{(r)}_{S(t)}(\theta) = \lambda t M^{(r)}_X(\theta)$$

which is, simply and conveniently, proportional to time with the constant of proportionality equal to product of the Poisson rate intensity per unit time and the r^{th} raw moment of the basis distribution due to the unit root. Zeroing the mgf parameter

$$K^{(r)}_{S(t)}(0) = \lambda t M^{(r)}_X(0)$$

gives

$$\kappa_r[S(t)] = \lambda t \mu_r(X)$$

The cumulants for all orders, therefore, sum linearly with time and characterize the property of independent increments. This is in exact correspondence with an ordinary Gaussian unit-root diffusion, Brownian motion. The difference being that there is a full set of non-zero cumulants which implies a Brownian motion with a generalized Gaussian basis.

Seeing that all cumulants are linear functions of time, it can then be reasonably inferred from inspection that this process converges to a Brownian motion so long as the raw moments of the secondary distribution, the distribution of the cumulative increments being added by the Poisson process, are relatively well behaved in that the cumulants are not complex or increasing in order of magnitude greater than the square-root of the power of the order of time. The fact that a Poisson process compound sum normalizes and converges to a normal diffusion with a dominant Brownian mean and variance is proven mathematically by reference to the moments in relation to one another and consequently, the Central Limit Theorem. This is an example of the generalized heat equation, that for diffusion with a generalized Gaussian basis, converging to that of the standard heat equation from the effects of the Central Limit Theorem.

Mathematical Preliminary: Addition and Subtraction of Random Variables and the Additive Convolution and Deconvolution of Density Functions

Convolution Formula: the Convolution Integral

The distribution of the summation of two independent random variables is obtained either by multiplying their moment generating functions or convolving their density functions. The former, $M_{S(t)}(\theta) M_X(\theta) = M_{S(t)+X}(\theta)$, is a result that has been used repeatedly throughout this work. The latter is given by the convolution formulae

[45] The first cumulant is $K^{(1)}_{S(t)}(0) = x_0 + \lambda t M^{(1)}_X(0)$, that is, $\kappa_1[S(t)] = x_0 + \lambda t \mu_1(X)$.

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$$
\begin{aligned}
F_{S+X}(y) &= \Pr(S+X \le y) \\
&= \int_{-\infty}^{\infty} \Pr(X = x)\Pr(S + x \le y \mid X = x)dx \\
&= \int_{-\infty}^{\infty} f_X(x)\Pr(S \le y - x \mid X = x)dx \\
&= \int_{-\infty}^{\infty} f_X(x)F_{S|X}(y - x \mid x)dx
\end{aligned}
$$

This is the convolution formula for the distribution function. The convolution formula for the density function is obtained by differentiating the formula for the convolution of the distribution function $DF_{S+X}(y)$

$$f_{S+X}(y) = \int_{-\infty}^{\infty} f_X(x)\, f_{S|X}(y - x \mid x)dx$$

In discrete terms, this is equivalent to polynomial multiplication. When the domains of the variables being summed are not defined over the entire real line, in relation to specifying the limits of integration, the integration may have to be split into several separate and possibly overlapping regions of integration.

Deconvolution Formula: the Deconvolution Integral

The distribution of the difference between two independent random variables is obtained either by dividing one moment generating function by the other or by deconvolving their density functions. The former is $M_{S(t)}(\theta) / M_X(\theta) = M_{S(t)-X}(\theta)$ while the latter is given by the deconvolution formulae

$$
\begin{aligned}
F_{S-X}(y) &= \Pr(S - X \le y) \\
&= \int_{-\infty}^{\infty} \Pr(X = x)\Pr(S - x \le y \mid X = x)dx \\
&= \int_{-\infty}^{\infty} f_X(x)\Pr(S \le y + x \mid X = x)dx \\
&= \int_{-\infty}^{\infty} f_X(x)F_{S|X}(y + x \mid x)dx
\end{aligned}
$$

which is the deconvolution formula for the distribution function. The formula for deconvolution of the density functions is obtained by differentiating the deconvolution formulae for the distribution function $DF_{S-X}(y)$

$$f_{S-X}(y) = \int_{-\infty}^{\infty} f_X(x)\, f_{S|X}(y + x \mid x)dx$$

In discrete terms, deconvolution is equivalent to polynomial division.

Forward Integro-Differential Equation of a Compound Poisson Process

Considering the transition moment generating function of the Poisson compound diffusion

$$M_{S(t)|S(t_0)}(\theta) = \exp\left(\theta x_0 + \lambda t[M_X(\theta) - 1]\right)$$

Differentiating the transition mgf with respect to the time variable[46] gives

$$\frac{\partial}{\partial t} M_{S(t)|S(t_0)}(\theta) = \lambda\left(M_X(\theta) - 1\right) M_{S(t)|S(t_0)}(\theta)$$

or in terms of convolution of the basis variable onto the diffusion variable

$$\frac{\partial}{\partial t} M_{S(t)|S(t_0)}(\theta) = \lambda\left(M_{S(t)+X\ |S(t_0)}(\theta) - M_{S(t)|S(t_0)}(\theta)\right)$$

Expressing the differential equation in terms of the definition of the moment generating function

---

[46] intuition and analytics similar to
Feller, W., (1966) An Introduction to Probability Theory and Its Applications, Volume 2. John Wiley & Sons.

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$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{\theta x} f_{S(t)|S(t_0)}(x \mid x_0)dx = \lambda\int_{-\infty}^{\infty} e^{\theta x} f_{S(t)+X|S(t_0)}(x \mid x_0)dx - \lambda\int_{-\infty}^{\infty} e^{\theta x} f_{S(t)+X|S(t_0)}(x \mid x_0)dx$$

and by noting the convolution formula for the density function of the sum of two random variables is

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(y) f_Y(x-y)dy = \int_{-\infty}^{\infty} f_X(x-y) f_Y(y)dy$$

this becomes

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0)dx = \lambda\int_{-\infty}^{\infty} e^{\theta x} \int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x-y \mid x_0)dydx - \lambda\int_{-\infty}^{\infty} e^{\theta x} f_{S(t)+X|S(t_0)}(x \mid x_0)dx$$

Inverting the moment generating function on both sides of the equation gives the Kolmogorov forward integro-differential equation for the Poisson process compound sum, in terms of the secondary basis distribution,

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0)dx = \lambda\int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x-y \mid x_0)dy - \lambda\, f_{S(t)+X|S(t_0)}(x \mid x_0)$$

This is the analogue forward equation to the diffusions discussed in FOUNDATIONS I that concerned continuous first order linear aggregations of infinitesimal diffusion basis variables, i.e., continuous time and continuous additions of the basis variables.

The forward convolution equation[47] can alternatively be obtained, rather than by backward induction from the moment generating function, from the forward recurrence relation of the Poisson process generating the successive incremental convolutions. On any forward differential increment in the forward time variable (an increase in the transition interval) a Poisson event occurs or it does not occur, with the probabilities λdt and $1 - \lambda dt$. If **(i)** a Poisson event occurs, an additional secondary distribution is added to the compound sum while if **(ii)** no Poisson event occurs, the number of secondary distributions in the compound sum stays fixed and the transition distribution is unchanged. Mathematically, this is stated as

$$f_{S(t+dt)|S(t_0)}(x \mid x_0) = \lambda dt \times f_{S(t)+X|S(t_0)}(x \mid x_0) + (1-\lambda dt) \times f_{S(t)|S(t_0)}(x \mid x_0) + o(dt)$$

and in terms of a convolution integral,

$$f_{S(t+dt)|S(t_0)}(x \mid x_0) = \lambda dt\int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x-y \mid x_0)dy + (1-\lambda dt) f_{S(t)|S(t_0)}(x \mid x_0) + o(dt)$$

Subtracting $f_{S(t)}(x)$ from both sides

$$f_{S(t+dt)|S(t_0)}(x \mid x_0) - f_{S(t)|S(t_0)}(x \mid x_0) = \lambda dt\int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x-y \mid x_0)dy - \lambda f_{S(t)|S(t_0)}(x \mid x_0)dt + o(dt)$$

and dividing through by dt

$$\frac{f_{S(t+dt)|S(t_0)}(x \mid x_0) - f_{S(t)|S(t_0)}(x \mid x_0)}{dt} = \lambda\int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x-y \mid x_0)dy - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0) + \frac{o(dt)}{dt}$$

Letting dt approach zero so that $o(dt) / dt \to 0$, gives the forward equation

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda\int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x-y \mid x_0)dy - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0)$$

Backward Integro-Differential Equation of a Compound Poisson Process

The backward equation involveds differential changes in the the backward variables, x_0 and t_0, which describe the the initial conditions of the transition inveral. In a similar manner to the latter derivation of the forward equation, the backward equation is derived by the intuitive arguments[48] related to the Poisson process generating the convolution sum. On any forward differential increment in the backward time variable (a reduction in the transi-

[47] Intuition and work of the author of this thesis (although well understandable to any good actuary).
[48] Again, those of the author.

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tion interval) a Poisson event has either occurred or not occurred, and the probabilities of each of these two outcomes is again λdt and $1 - \lambda dt$. **(i)** If no Poisson events occurred, then the convolution sum will have the same number of components and the transition distribution is unchanged. **(ii)** If a Poisson event did occur, then the convolution sum would have had one less component in the convolution sum and therefore needs to be deconvolved of one variable from the back end of the sum, i.e., the first increment of the compound sum is removed;

$$f_{S(t)|S(t_0-dt)}(x\,|\,x_0) = \lambda dt \times f_{S(t)|S(t_0)-X}(x\,|\,x_0) + (1-\lambda dt) \times f_{S(t)|S(t_0)}(x\,|\,x_0) + o(dt)$$

From the deconvolution formula and remembering that the process is time homogenous,

$$f_{S(t)|S(t_0-dt)}(x\,|\,x_0) = \lambda dt \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x\,|\,x_0+y)dy + (1-\lambda dt)\, f_{S(t)|S(t_0)}(x\,|\,x_0) + o(dt)$$

Subtracting $f_{S(t)}(x)$ from both sides

$$f_{S(t)|S(t_0-dt)}(x\,|\,x_0) - f_{S(t)|S(t_0)}(x\,|\,x_0) = \lambda dt \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x\,|\,x_0+y)dy - \lambda\, f_{S(t)|S(t_0)}(x\,|\,x_0)dt + o(dt)$$

and dividing through by dt

$$\frac{f_{S(t)|S(t_0-dt)}(x\,|\,x_0) - f_{S(t)|S(t_0)}(x\,|\,x_0)}{dt} = \lambda \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x\,|\,x_0+y)dy - \lambda\, f_{S(t)|S(t_0)}(x\,|\,x_0) + \frac{o(dt)}{dt}$$

Letting dt approach zero so that $o(dt) / dt \to 0$, gives the backward equation,

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x\,|\,x_0) = \lambda \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x\,|\,x_0+y)dy - \lambda\, f_{S(t)|S(t_0)}(x\,|\,x_0)$$

For backward convolution, rather than deconvolution, the variable of the time derivative changes to forward running time and the integral becomes that of backward convlution

$$\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x\,|\,x_0) = \lambda \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x\,|\,x_0-y)dy - \lambda\, f_{S(t)|S(t_0)}(x\,|\,x_0)$$

which can be obtained by the same line of reasoning with a forward increment in the backward time variable and a convolution of the backward diffusion variable

$$f_{S(t)|S(t_0+dt)}(x\,|\,x_0) = \lambda dt \times f_{S(t)|S(t_0)+X}(x\,|\,x_0) + (1-\lambda dt) \times f_{S(t)|S(t_0)}(x\,|\,x_0) + o(dt)$$

The Property of Independent Increments of Poisson Process Compound Sums

The Poisson process has the property of independent increments since it is a linear integration of independent Poisson variables and a Markov process. A compound Poisson process is a linear summation of the independent secondary basis variables and, therefore, also a Markov process and consequently also has the property of independent increments like the Poisson process generating the sum. In terms of distributions, the property of independent increments says that the difference between the process at, say, $t + s$ and that at time s depends only on the difference between the two times,

$$N(t+s) - N(s) \overset{d}{=} N(t) \text{ i.e., } P(\lambda(t+s)) - P(\lambda s) \sim P(\lambda t) \text{ independently of } N(s) \sim P(\lambda s)$$

Allowing for this property in random compounding sums, a compound sum has itself independent increments if and only if the counting distribution of the generator has independent increments. A Poisson process compounding sum of general secondary distributions, therefore, has independent increments so long as the secondary distributions are independent.

Proof

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$$\begin{aligned} S(t+s)-S(s) &= \sum_{j=1}^{N(t+s)} X_j - \sum_{j=1}^{N(s)} X_j \\ &= \sum_{j=1}^{N(t+s)-N(s)} X_j \\ &= \sum_{j=1}^{N(t)} X_j \\ &= X^{*N(t)} \end{aligned}$$

This is a convolution of $N(t)$ independent distributions of type X and hence only depends on t (and not s). The Poisson process compounding sum therefore has independent increments. This can be expressed as zero correlation between increments, $Cov[S(t+s)-S(s), S(t)] = 0$ $Cov[S(t+s)-S(s), S(t)] = 0$.

The proof is complete.

Autocovariance and Autocorrelation of a Poisson Process Compound Sum of Independent and Homogeneous Secondary Distributions (Orthogonal Basis)

Since the compound Poisson process has independent increments, the autocovariance of the process between two overlapping time intervals, say t and s, is proportional to the length of the overlapping transition intervals running from a common backward variable, $\min(t, s) = (t \wedge s)$,

$$Cov\big(S(t), S(s)\big) \propto (t \wedge s) \qquad \text{or} \qquad Cov\big(S(t+s), S(t)\big) \propto t \qquad \text{so} \qquad Cov\big(S(t+s), S(t)\big) = \lambda\left(\mu_1^2(X) + \sigma_2(X)\right)t$$

the constant of proportionality being the product of the Poisson arrival rate per unit time and the second raw moment of the secondary distribution $\lambda(\mu_X^2 + \sigma_X^2)$.

Proof

$$\begin{aligned} Cov\big(S(t+s), S(t)\big) &= Cov\left(\sum_{j=1}^{N(t+s)} X_j, \sum_{i=1}^{N(t)} X_i\right) \\ &= Cov\left(\sum_{j=1}^{N(t)} X_j + \sum_{j=N(t)+1}^{N(t+s)} X_j, \sum_{i=1}^{N(t)} X_i\right) \\ &= Cov\left(\sum_{j=1}^{N(t)} X_j, \sum_{i=1}^{N(t)} X_i\right) + Cov\left(\sum_{j=N(t)+1}^{N(t+s)} X_j, \sum_{i=1}^{N(t)} X_i\right) \\ &= Var\left(\sum_{i=1}^{N(t)} X_i\right) \quad \text{because the basis components are independent} \\ &= \kappa_2[S(t)] \\ &= \lambda\mu_2(X)t \\ &= \lambda\left(\mu_1^2(X) + \sigma_2(X)\right)t \end{aligned}$$

The proof is complete.

The autocorrelation function is identical to the Poisson process generating the recurring additions of secondary distributions and that of any Markov process, i.e., first order unit-root diffusion; the autocorrelation function is

$$Corr(N_t, N_s) = \frac{(s \wedge t)}{\sqrt{t}\sqrt{s}} = \frac{\min(s,t)}{\sqrt{t}\sqrt{s}}$$

or equivalently

$$Corr(N_{t+s}, N_s) = \frac{s}{\sqrt{(t+s)}\sqrt{s}} = \sqrt{\frac{s}{t+s}}$$

Since the mean, variance, and correlation of a Poisson process compound diffusion are each Brownian, it is reasonable to assume that the limit of this process for sufficiently regular secondary distributions with finite lower moments is a Brownian motion, as t becomes large.

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Moment Analysis and Long Run Stationary Behavior

From

$$K_{S(t)}^{(r)}(\theta) = \lambda t M_X^{(r)}(\theta)$$

the r^{th} order cumulant is

$$\kappa_r[S(t)] = \lambda t \mu_r(X)$$

The r^{th} standardized cumulant is

$$\frac{\kappa_r[S(t)]}{\kappa_2[S(t)]^{r/2}} = \frac{\lambda t \mu_r(X)}{[\lambda t \mu_2(X)]^{r/2}} = \frac{1}{(\lambda t)^{r/2-2}} \frac{\mu_r(X)}{[\mu_2(X)]^{r/2}}$$

which approaches zero as t becomes large so long as $\mu_r(X)$ are finite and do not increase with order r faster than $t^{r/2-2}$ and is then the limiting process found by the limiting arguments of the forward and backward equations, a Brownian motion with infinitesimal parameters $\lambda\mu$ and $\lambda(\mu^2 + \sigma^2)$.

Notice here that λ acts as an adjustment for time scale as the parameter for increment length - its function changes from the time scale parameter, per unit time, of the Poisson process to the time scale parameter of the limiting Brownian motion and consequently time scales the cumulants so that they are in the appropriate time units of the time aggregation and ensure that the increments are in-scale and produce only local movements that are in proportion to the increment length. The equivalent parameter for a discrete diffusion with equally spaced increments is h.

Integro-Differential Equation of Put and Call Option Premia for Compound Poisson Diffusion

If one is interested in numerical solutions to equations, one may wish to experiment with the integro-differential equation of the option premium. Considering the backward integro-differential equation for additive compound Poisson diffusion

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x \mid x_0 + y) dy - \lambda f_{S(t)|S(t_0)}(x \mid x_0)$$

and defining the put and call option premia as discounted expected values of the maturity intrinsic value at the forward time variable

$$V(k,t;x_0,t_0) = \begin{cases} e^{-r(t-t_0)} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx & \text{for a call} \\ e^{-r(t-t_0)} \int_{-\infty}^{k} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx & \text{for a put} \end{cases}$$

Multiplying both sides of the equation by the options intrinsic value, for a call for example, and integrating over the moneyness

$$-\int_k^{\infty} (x-k) \frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) dx = \lambda \int_k^{\infty} (x-k) \int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x \mid x_0 + y) dy dx - \lambda \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$

Taking the integrals inside the time partial derivative and deconvolution integral since the option integration is independent of the backward variables

$$-\frac{\partial}{\partial t_0} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx = \lambda \int_{-\infty}^{\infty} f_X(y) \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0 + y) dx dy - \lambda \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$

and discounting to the backward time variable at the risk free rate $e^{-r(t-t_0)}$

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$$-e^{-r(t-t_0)}\frac{\partial}{\partial t_0}\int_k^\infty (x-k)\,f_{S(t)|S(t_0)}(x\,|\,x_0)dx$$

$$=\lambda e^{-r(t-t_0)}\int_{-\infty}^{\infty} f_X(y)\int_k^\infty (x-k)\,f_{S(t)|S(t_0)}(x\,|\,x_0+y)dxdy-\lambda e^{-r(t-t_0)}\int_k^\infty (x-k)\;f_{S(t)|S(t_0)}(x\,|\,x_0)dx$$

Taking the discount factor inside the time partial derivative by the converse of the product rule[49]

$$re^{-r(t-t_0)}\int_k^\infty (x-k)\,f_{S(t)|S(t_0)}(x\,|\,x_0)dx-\frac{\partial}{\partial t_0}e^{-r(t-t_0)}\int_k^\infty (x-k)\,f_{S(t)|S(t_0)}(x\,|\,x_0)dx$$

$$=\lambda\int_{-\infty}^{\infty} f_X(y)e^{-r(t-t_0)}\int_k^\infty (x-k)\,f_{S(t)|S(t_0)}(x\,|\,x_0+y)dxdy-\lambda e^{-r(t-t_0)}\int_k^\infty (x-k)\;f_{S(t)|S(t_0)}(x\,|\,x_0)dx$$

giving the equation of the option premium for an additive compound Poisson diffusion

$$rV(k,t;x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t;x_0,t_0)=\lambda\int_{-\infty}^{\infty} f_X(y)V(k,t;x_0+y,t_0)dy-\lambda V(k,t;x_0,t_0)$$

with an arbitrary distribution of the basis components X. When the basis components are normal, $X \sim N(\mu, \sigma^2)$, the equation is

$$rV(k,t;x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t;x_0,t_0)=\frac{\lambda}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\mu)^2/\sigma^2}V(k,t;x_0+y,t_0)dy-\lambda V(k,t;x_0,t_0)$$

When the basis components are exponential, $X \sim \Gamma(1, \beta)$, the equation is

$$rV(k,t;x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t;x_0,t_0)=\frac{\lambda}{\beta}\int_{-\infty}^{\infty} e^{-y/\beta}V(k,t;x_0+y,t_0)dy-\lambda V(k,t;x_0,t_0)$$

Prudent foresight and commonsense suggests specifying the distribution of the basis components X and writing the pde in terms of the density of X, such as those for normal and exponential basis distributions. Then only restricted ranges of the convolution integrals need to be taken into consideration, for example, over ranges of $\pm 3.5\sigma$ either side of the mean μ when $X \sim N(\mu, \sigma^2)$.

Limiting Form of the PDE of Put and Call Option Premia for Additive Compound Poisson Diffusion

From the Taylor series expansion

$$f(x+y)=f(x)+y\frac{\partial}{\partial x}f(x)+\tfrac{1}{2!}y^2\frac{\partial^2}{\partial x^2}f(x)+\tfrac{1}{3!}y^3\frac{\partial^3}{\partial x^2}f(x)+\ldots$$

Expanding the option premium in the convolution integral of the pde

$$rV(k,t;x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t;x_0,t_0)=\lambda\int_{-\infty}^{\infty} f_X(y)\left\{1+y\frac{\partial}{\partial x_0}+\tfrac{1}{2}y^2\frac{\partial^2}{\partial x_0^2}+\ldots\right\}V(k,t;x_0,t_0)dy-\lambda V(k,t;x_0,t_0)$$

Say that $X \sim \log N(\mu h, \sigma^2 h)$ so that $\log X \sim N(\mu h, \sigma^2 h)$ and

$$\int_{-\infty}^{\infty} y\,f_X(y)dy=\mu h \qquad \text{and} \qquad \int_{-\infty}^{\infty} y^2 f_X(y)dy=\sigma^2 h+\mu^2 h^2$$

$$rV(k,t;x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t;x_0,t_0)=\lambda\left\{\mu h\frac{\partial}{\partial x_0}+\tfrac{1}{2}(\sigma^2 h+\mu^2 h^2)\frac{\partial^2}{(\partial x_0)^2}+o(h)\right\}V(k,t;x_0,t_0)$$

and further setting $\lambda = 1 / h$

[49] see BOOK 2, Replicating Portfolios.

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$$rV(k,t;x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t;x_0,t_0)=\left\{\mu\frac{\partial}{\partial x_0}+\tfrac{1}{2}\sigma^2\frac{\partial^2}{(\partial x_0)^2}+\frac{o(h)}{h}\right\}V(k,t;x_0,t_0)$$

Letting $h\to 0$ so that **(i)** the Poisson events occur infinitely often, i.e., an infinitesimal waiting time between each, and **(ii)** the mean magnitude and variation in the sultus of each convolved basis component scales down with the exponential increments. This has the desired effect as to leave the ratio of the two both finite and non-zero. The limit $h\to 0$ consequently results in

$$rV(k,t;x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t;x_0,t_0)=\tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x_0^2}V(k,t;x_0,t_0)+\mu\frac{\partial}{\partial x_0}V(k,t;x_0,t_0)$$

or $\quad rV-\dfrac{\partial V}{\partial t_0}=\tfrac{1}{2}\sigma^2\dfrac{\partial^2 V}{\partial x_0^2}+\mu\dfrac{\partial V}{\partial x_0}\qquad$ where $\qquad V=V(k,t;x_0,t_0)$

which is the pde of an option on the limit of the additive compound Poisson diffusion, an option on an ordinary Brownian motion. If the basis had have been gamma distributed, then the option pde would have taken the form of the generalized heat equation, with the addition of a sink, as *h* became negligible.

COMPOUND POISSON PROCESS SUM OF GAUSSIANS

Now assuming that the secondary distributions are independent and homogeneous normal variables with non-zero mean and non-unit volatility

$$X_j\sim N(\mu,\sigma^2)$$

The compound distribution conditional on $N(t)$ is a sum of independent normals and since, as is utilized repeatedly in this work, a set of normal distributions forms a semi-group under the binary operation of convolution, the random sum is distributed normally with mean $\mu N(t)$ and variance $\sigma^2 N(t)$

$$S(t)\,|\,N(t)\sim N\left(x_0+\mu N(t),\ \sigma^2 N(t)\right)\qquad\text{where}\qquad N(t)\sim P(\lambda t)$$

This is a normal variable with a Poisson distributed mean and a Poisson distributed variance and hence is also a standard Brownian motion on the metric $N(t)$. This is also a special form of stochastic volatility model, whereby the mean is also stochastic, synchronously with the volatility, and retains a constant mean-variance ratio (a fixed signal to noise ratio). This latter aspect of a compound sum diffusion relating to stochastic cumulants is elaborated on at a later stage in FOUNDATIONS II.

Conditional and Unconditional Density Functions

The compound sum of normals given the Poisson generator process $N(t)$ has a normal transition density function

$$f_{S(t)|N(t)}(x\,|\,x_0)=\frac{1}{\sigma\sqrt{2\pi N(t)}}\exp\left(-\frac{[x-x_0-\mu N(t)]^2}{2\sigma^2 N(t)}\right)\quad\text{where}\quad N(t)\sim P(\lambda t)$$

and when the process takes on a specific value, say *n*, the density is

$$f_{S(t)|N(t)=n}(x\,|\,x_0,n)=\frac{1}{\sigma\sqrt{2\pi n}}\exp\left(-\frac{[x-x_0-\mu n]^2}{2\sigma^2 n}\right)$$

which is Brownian in the variable *n*. Considering here that the mean value and the variance of the random variable $N(t)$ are both proportional to *t*, the constant of proportionality being λ, it seems plausible from this alone that there is a very close resemblance between the continuous Gaussian diffusions and the Gaussian compound Poisson sums. This is also an intuitive argument or motivation for the result concerning the limiting, long run stationary distribution. Unconditionally, the density function is the Poisson weighted average of the above function over the variable *n*,

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$$
\begin{aligned}
f_{S(t)}(x \mid x_0) &= \sum_{n=0}^{\infty} \; p_n(t) f_{S(t)|N(t)=n}(x \mid x_0) \\
&= \sum_{n=0}^{\infty} \; [e^{-\lambda t}(\lambda t)^n / n!] f_{S(t)|N(t)=n}(x \mid x_0) \\
&= \sum_{n=0}^{\infty} \; \frac{e^{-\lambda t}(\lambda t)^n}{n!\sigma\sqrt{2\pi n}} \exp\left(-\frac{[x - x_0 - \mu n]^2}{2\sigma^2 n} \right)
\end{aligned}
$$

Kolmogorov Forward Integro-Differential Equation for the Gaussian Compound Poisson Diffusion

$$
\frac{\partial}{\partial t} f_{S(t)}(x) = \frac{\lambda}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\mu)^2/\sigma^2} f_{S(t)}(x-y)dy - \lambda \; f_{S(t)}(x)
$$

or with notation inclusive of the backward diffusion variable,

$$
\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \frac{\lambda}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\mu)^2/\sigma^2} f_{S(t)|S(t_0)}(x-y \mid x_0)dy - \lambda \; f_{S(t)|S(t_0)}(x \mid x_0)
$$

Kolmogorov Backward Integro-Differential Equation for the Gaussian Compound Poisson Diffusion

$$
\frac{\partial}{\partial t} f_{S(t)|S(0)}(x \mid x_0) = \frac{\lambda}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\mu)^2/\sigma^2} f_{S(t)|S(0)}(x \mid x_0 + y)dy - \lambda \; f_{S(t)|S(0)}(x \mid x_0)
$$

or inclusive of a backward time variable,

$$
-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \frac{\lambda}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-\mu)^2/\sigma^2} f_{S(t)|S(t_0)}(x \mid x_0 + y)dy - \lambda \; f_{S(t)|S(t_0)}(x \mid x_0)
$$

The Relationship Between Modeling, Simulation, and Estimation

Assertion: there is a close connection between the activities of simulation and estimation. This connection lies in the necessity to correctly define the random variables and how the random variables interact.

This process is well suited to transaction micro-structure data both in terms of the resemblance to the actual conditions which give rise to the sequence of variables and the correspondence between the model variables and the variables generating the data. In this case there is statistical variation in both time and state dimensions. This is a desirable although not necessary feature of a mathematical representation of a physical phenomenon. An implication of a correspondence between the mathematical representation and the actual phenomenon is that estimation and fitting of the process, to a particular observed sequence of transaction data in the case of interest here, can be done directly and without recourse to further simplifying or abstracting assumptions that would result in a divergence between the model process and the physical process. In other words, there is

a close correspondence between the variables generating the sequence with the variables whose parameters are being estimated by reference to the sequence in order to represent the future sequence of the process.

The degree of this correspondence is a measure of the applicability of the mathematical process to the physical process of interest. In particular, when the sequence of random variables of the mathematical process corresponds exactly to the sequence of observed random variables (perfect correspondence) then the model is the process generating the actual real world outcomes.

A heuristic model test then, assuming that there is a one-to-one correspondence between the simulated random variables and the observed random variables of the process being modeled, is to simulate it and have the simulated process accurately behave like the process being modeled, for both individual simulated paths and large numbers of simulated paths.

In the context of microstructure transaction data, the modeling, estimation of the parameters and option premia, and simulation takes place in both dimensions. The parameters of the process generating the occurrence of transactions and the parameters of the process of cumulative noise, or compound of secondary distributions, are esti-

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mated separately and then combined into the joint model, the compound Poisson-normal diffusion or log-diffusion, for which simulation and estimation of the best estimate of option premia can follow. It is believed that such simulations bear a close resemblance to observed microstructure transaction series.

Simulation: Algorithm for Generating Paths of Compound Poisson Diffusion

Generate a vector of times at which Poisson events occur with the required rate intensity (vector of abscissa), as per the discussion of the Poisson process. Generate a corresponding vector of normal variates with the required mean and variance and calculate a vector of the cumulative sum (vector of ordinates). At each point in time at which a Poisson event occurs, plot the corresponding coordinate, i.e., plot the vector of the cumulative sum of normals verse the times of Poisson events. Streamlined MATLAB code to do this is

```
x0=x_0; mu=μ; sigma=σ; h=h; b=1/λ; n=n;              %initialize parameters
dt=exprnd(b,n,1); t=cumsum([0;dt)];      % simulate exp basis variable & sum
z=mu.*h+sigma.*randn(n,1).*sqrt(h);x=cumsum([x0;z]);%sim diff'n basis & sum
stairs(t,x);                                                        %plot
```

Simulated Paths of Compound Poisson Diffusion

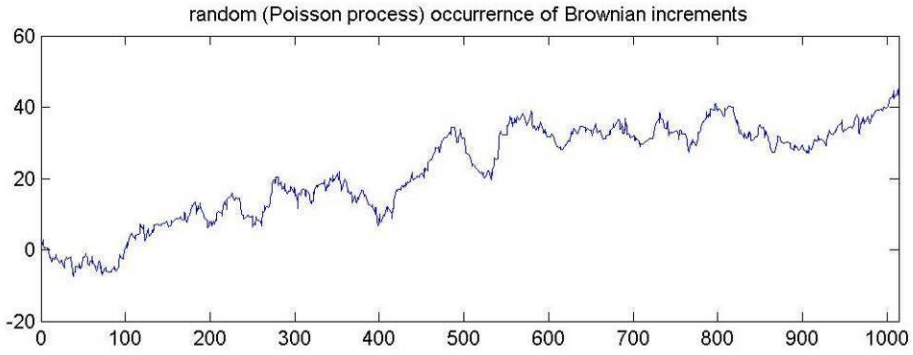


Figure 28: single simulated path of a compound Poisson Gaussian diffusion; 1000 steps

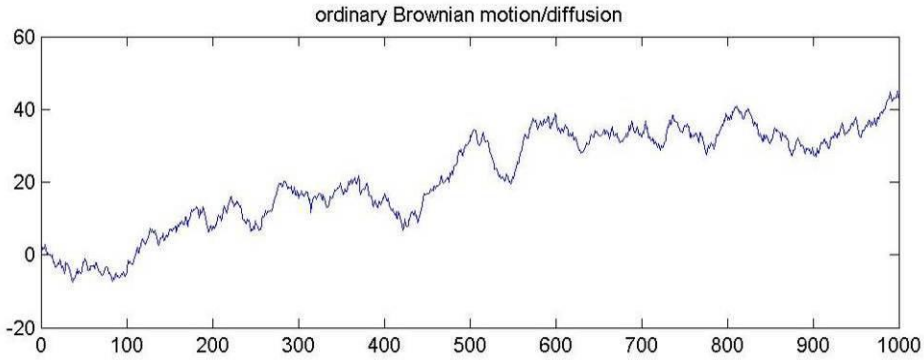


Figure 29: single simulated path of an ordinary additive Gaussian diffusion; 1000 steps

It is interesting to contrast this with the corresponding regular diffusion generated with the same sequence of random numbers. The second path is an ordinary Brownian path which concludes at time index 1000 exact while the compound Poisson process path, which is the first series, concludes at time index 1013 to the nearest integer. The Poisson process was generated with 1000 unit exponentials (x0=0, mu=0, sigma=1, h=1, b=1, n=1000 which is good to get started; a next step would be, say, x0=2, mu=0.1, sigma=0.5, h=0.1, b=0.2, n=50).

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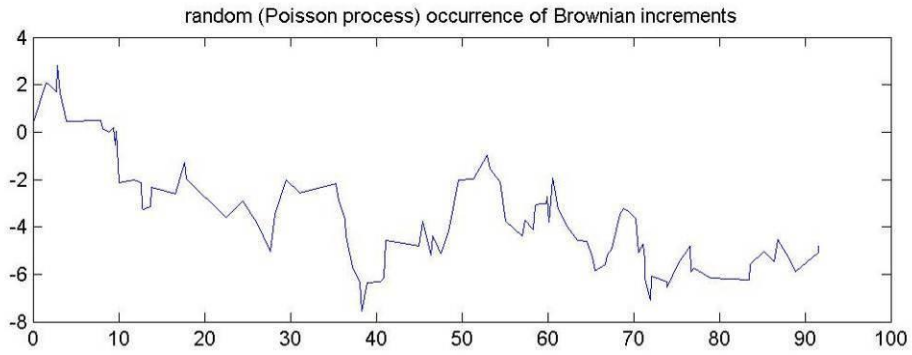


Figure 30: single simulated path of a compound Poisson Gaussian diffusion; 100 steps

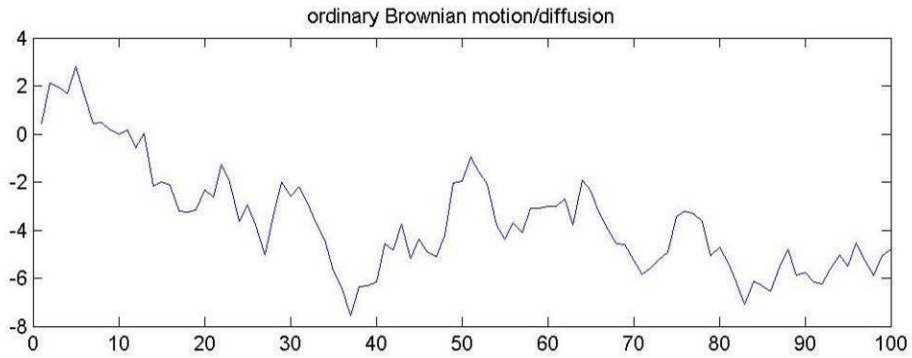


Figure 31: single simulated path of an ordinary additive Gaussian diffusion; 100 steps

Observe the similarity between the ordinary Brownian motion and the compound Poisson sample paths over the larger time scale, in the graphics, 1000 increments. At the smaller time scale of the first 100 increments in the last two graphics, the differences between the two processes becomes apparent with obvious differences with regard to the randomly occurring points for which determine the length of each increment in the compound Poisson sample path. The Poisson path seems much less regular than that with fixed increment length for small numbers of increments.

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Further Simulations

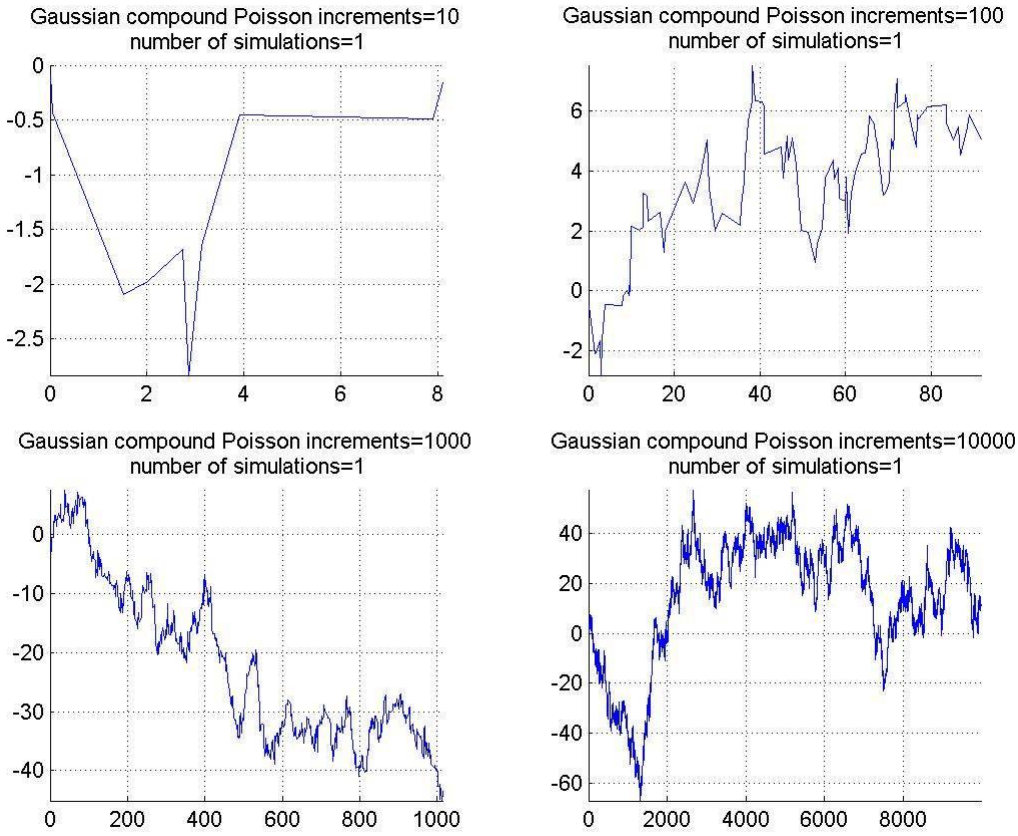


Figure 32: single simulated path of a compound Poisson diffusion; 10, 100, 1000, and 10000 steps

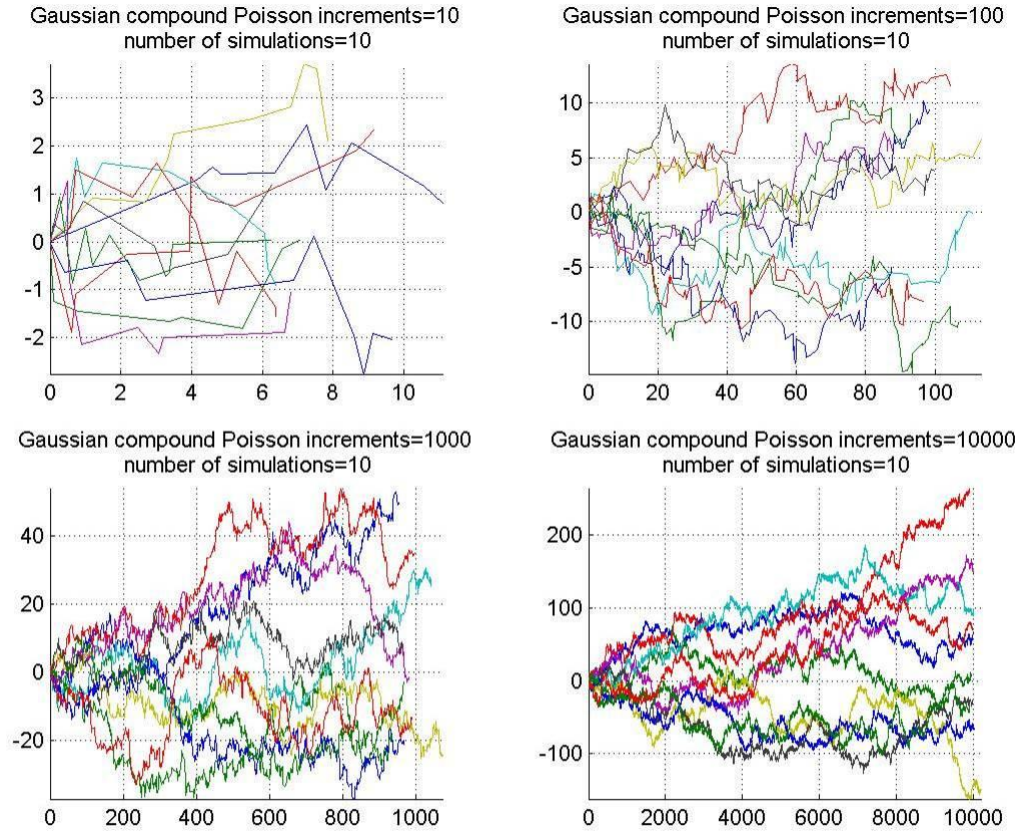


Figure 33: 10 simulated paths of a compound Poisson diffusion; 10, 100, 1000, and 10000 steps

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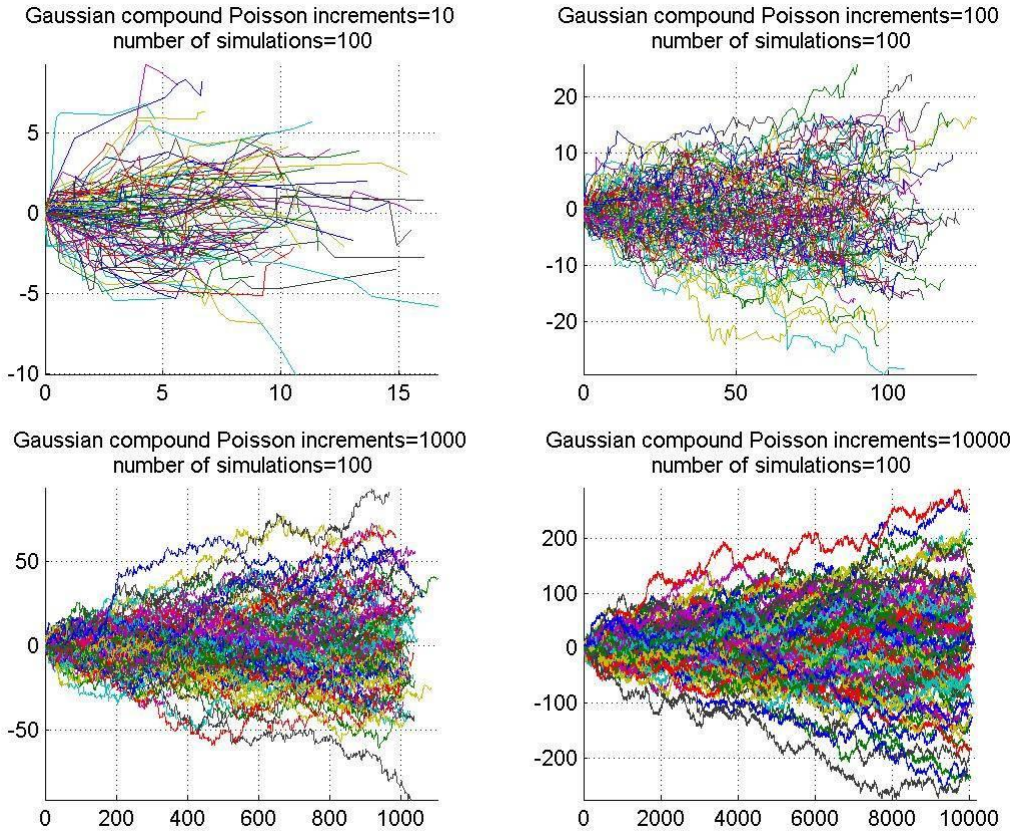


Figure 34: 100 simulated paths of a compound Poisson diffusion; 10, 100, 1000, and 10000 steps

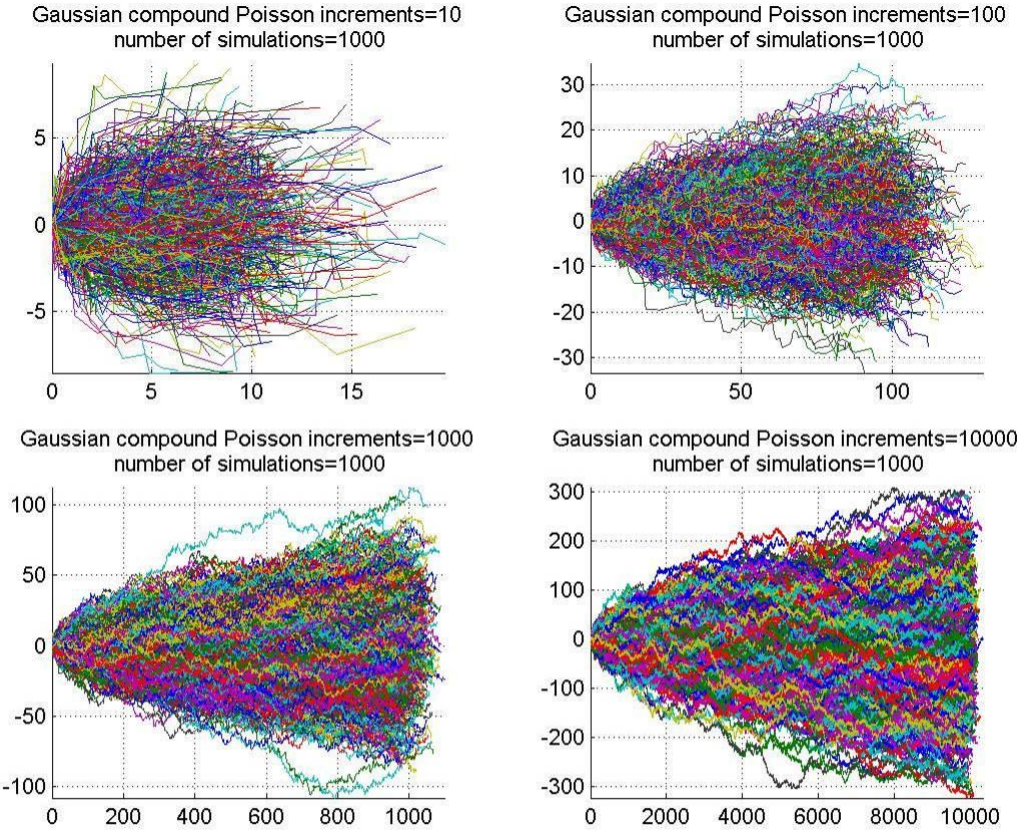


Figure 35: 1000 simulated paths of a compound Poisson diffusion; 10, 100, 1000, and 10000 steps

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Ancillary Statistical Results for the Analysis of Moments: Conversion Between Moment Statistics

The raw moments of any distribution can be expressed in terms of cumulants. From these expressions, the cumulants can be obtained in terms of the raw moments by backward solving the recursive formulae. Letting μ_r and κ_r be the r^{th} raw moment and the r^{th} cumulant of the random variable X. The cumulant generating function is the logarithm of the moment generating function, $K_X(\theta) = \log M_X(\theta)$. Differentiating once and multiplying both sides by $M_X(\theta)$ gives $M_X^{(1)}(\theta) = M_X(\theta)\, K_X^{(1)}(\theta)$. Applying Leibniz' rule for the n^{th} derivative of a product of two functions,

$$D^n u(x)v(x) = \sum_{j=0}^{n} {}^{n}C_j \, D^j u(x) D^{n-j} v(x)$$

gives

$$\begin{aligned} M_X^{(n)}(\theta) &= D^{n-1} M_X^{(1)}(\theta) \\ &= D^{n-1} M_X(\theta) K_X^{(1)}(\theta) \\ &= \sum_{j=0}^{n-1} {}^{n-1}C_j \, D^j M_X(\theta) D^{n-1-j} K_X^{(1)}(\theta) \\ &= \sum_{j=0}^{n-1} {}^{n-1}C_j \, M_X^{(j)}(\theta) K_X^{(n-j)}(\theta) \end{aligned}$$

Setting the moment generating function parameter to zero gives the recursive conversion formula linking the *nth* order raw moment to the n^{th} order cumulant in terms of the lower order raw moments and cumulants;

$$\mu_n = \sum_{j=0}^{n-1} {}^{n-1}C_j \, \mu_j \kappa_{n-j} \qquad \textbf{(raw moments from cumulants)}$$

From this formula, any cumulant can be derived from the raw moments and visa-versa ($\mu_0 = 1, \kappa_0 = 1$, and $\kappa_1 = \mu_1$). To obtain the cumulants from raw or central moments for higher orders, software should probably be used. The conversion formulae for central moments in terms of raw moments are derived with a binomial expansion of the definition of the central moment as a centered expectation

$$\begin{aligned} \sigma_n &= E[(X-\mu)^n] \\ &= E\left[\sum_{j=0}^{n} {}^{n}C_j \; X^j (-\mu)^{n-j}\right] \\ &= \sum_{j=0}^{n} {}^{n}C_j \, (-1)^{n-j} E(X^j) \mu^{n-j} \\ &= \sum_{j=0}^{n} {}^{n}C_j \, (-1)^{n-j} \mu_j \mu^{n-j} \end{aligned}$$

that is, by the formula

$$\sigma_n = \sum_{j=0}^{n} {}^{n}C_j \, (-1)^{n-j} \mu_j \mu^{n-j} \qquad \textbf{(central moments from raw moments)}$$

where σ_n is the n^{th} order central moment, μ_j is the j^{th} order raw moment, $\mu_1 = \mu$ is the mean, and $\mu_0 = 1$. The conversion formulae for raw moments in terms of central moments are derived also by a binomial expansion,

$$\begin{aligned} \mu_n &= E[X^n] \\ &= E[(X-\mu+\mu)^n] \\ &= E\left[\sum_{j=0}^{n} {}^{n}C_j \, (X-\mu)^j (\mu)^{n-j}\right] \\ &= \sum_{j=0}^{n} {}^{n}C_j \; E[(X-\mu)^j] \mu^{n-j} \\ &= \sum_{j=0}^{n} {}^{n}C_j \; \sigma_j \mu^{n-j} \end{aligned}$$

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that is, by the formula

$$\mu_n = \sum_{j=0}^{n} {}^{n}C_j \, \sigma_j \mu^{n-j} \qquad \textbf{(raw moments from central moments)}$$

where $\mu_1 = \mu$ is the mean, $\sigma_0 = 1$, and $\sigma_1 = 0$.

Analysis of Moments

The conditional moment generating function is

$$M_{S(t)|N(t)}(\theta) = \exp\left(x_0\theta + \mu N(t)\theta + \tfrac{1}{2}\sigma^2 N(t)\theta^2 \right)$$

giving the cumulant generating function of

$$K_{S(t)|N(t)}(\theta) = x_0\theta + \mu N(t)\theta + \tfrac{1}{2}\sigma^2 N(t)\theta^2 = x_0\theta + (\mu\theta + \tfrac{1}{2}\sigma^2\theta^2)\, N(t)$$

This is the same cumulant generating function as a Brownian motion with the exception that linear time t is replaced with a compound Poisson process $N(t)$. This is the key difference between the processes of FOUNDATIONS I and the processes of FOUNDATIONS II, and is sometimes formally referred to as the randomization of time. The relevant moment measures then arise directly from the properties of the normal distribution,

(1) $E[S(t) \mid N(t)] = x_0 + \mu N(t)$

(2) $Var[S(t) \mid N(t)] = \sigma^2 N(t)$

(3) $Skew[S(t) \mid N(t)] = 0$

(4) $Kurt[S(t) \mid N(t)] = 3\sigma^4 N(t)^2$

The unconditional moment generating function, given by nested expectation, is

$$M_{S(t)}(\theta) = M_{N(t)}\left(\log M_X(\theta)\right) = \exp\left(x_0\theta + \lambda t[M_X(\theta) - 1]\right)$$

and with the normal secondary distribution $M_X(\theta) = e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2}$, this is

$$M_{S(t)}(\theta) = \exp\left(x_0\theta + \lambda t[e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2} - 1]\right)$$

The cumulant generating function is then

$$K_{S(t)}(\theta) = \log M_{N(t)}\left(\log M_X(\theta)\right) = x_0\theta + \lambda t\left(M_X(\theta) - 1\right)$$

and with the normal secondary distribution, this is

$$K_{S(t)}(\theta) = \log M_{N(t)}\left(\log M_X(\theta)\right) = x_0\theta + \lambda t\left(e^{\mu\theta + \frac{1}{2}\sigma^2\theta^2} - 1\right)$$

Hence, for a Poisson process counting distribution and ignoring the backward variable, all derivatives of the cumulant generating function are the mean rate of occurrence of the primary count distribution of the generator times the derivatives of the same order of the moment generating function of the secondary distribution of the basis variables;

$$K_{S(t)}^{(r)}(\theta) = \lambda t \times M_X^{(r)}(\theta) \quad r = 1,2,3,\ldots$$

This means the cumulants of the unconditional compound distribution are each proportional to the raw moments of the secondary distribution; the distribution that determines the size of the additional contributions to the random sum;

$$K_{S(t)}^{(r)}(0) = \lambda t \times M_X^{(r)}(0) \quad r = 1,2,3,\ldots$$
$$\kappa_{S(t)}^{(r)} = \lambda t \times \mu_X^{(r)}$$

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Since the secondary distributions are normal and

$$\mu_r(X) = \int_{-\infty}^{\infty} (\mu + \sigma y)^r \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy\,,$$

the first four cumulants of $S(t)$ are

(1) $\kappa_1 = x_0 + \mu\lambda t$ (**a Brownian mean**)

(2) $\kappa_2 = (\mu^2 + \sigma^2)\lambda t$ (**a Brownian variance**)

(3) $\kappa_3 = (\mu^3 + 3\sigma^2\mu)\lambda t$ (**non-zero skew**)

(4) $\kappa_4 = (\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4)\lambda t$ **(excess kurtosis)**

The mean, variance, and skewness of $S(t)$ are given by the first three cumulants, while the kurtosis is

$$Kurt[S(t)] = \kappa_4 + 3\kappa_2^2 = (\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4)\lambda t + 3(\mu^2 + \sigma^2)^2(\lambda t)^2$$

Autocovariance Function

The autocovariance function takes the same form as that for any linear summation of an independent set of homogenous basis variables with finite mean and variance,

$$Cov\big(S(t+s), S(t)\big) = \lambda\Big(\mu_1^2(X) + \sigma_2(X)\Big)t = \lambda\Big(\mu^2 + \sigma^2\Big)t$$

i.e., the autocovariance of any two points in time is simply the variance of the process over which the intervals from the backward variable of the diffusion overlap.

Moment Analysis for Large Time Distributional Behavior

Because all of the cumulants of the Poisson compound distribution are a function of the parameters of the secondary distribution, which are constant and here equal to μ and σ^2, multiplied by the mean rate λt, so that they are proportional to λt, the same arguments regarding limiting values of the Poisson process apply to the Poisson process compound sum. So, again, looking at the standardized measures of shape for third and fourth degree, for the Poisson process compound sum,

$$\xi_r(S(t)) = E[\{S(t) - \mu_{S(t)}\}^r] / \sigma_{S(t)}^r$$

for r = 3 and 4. The standardized skew is

$$\xi_3 = \frac{(\mu^3 + 3\sigma^2\mu)(\lambda t)}{[(\mu^2 + \sigma^2)(\lambda t)]^{3/2}} = \frac{\mu^3 + 3\sigma^2\mu}{(\mu^2 + \sigma^2)^{3/2}} \frac{1}{\sqrt{\lambda t}} \propto \frac{1}{\sqrt{\lambda t}}$$

which vanishes for large t, that is, $\xi_3 \to 0$ as $\sqrt{t} \to \infty$ and therefore, the transition variable tends to zero asymptotic skewness in relation to the distributional spread in the limit. The standardized kurtosis is

$$\xi_4 = \frac{(\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4)\lambda t + 3(\mu^2 + \sigma^2)^2(\lambda t)^{2^2}}{[(\mu^2 + \sigma^2)\lambda t]^2} = 3 + \frac{\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4}{(\mu^2 + \sigma^2)^2} \frac{1}{\lambda t}$$

which, as the second term vanishes, approaches 3 as t becomes large, that is $\xi_4 \to 3$ as $\sqrt{t} \to \infty$ and therefore tends to, asymptotically, a mesokurtic normal bell.

Gaussian Compound Poisson Process Limit Theorem

The vanishing skewness and approach of the kurtosis to the value 3 for large t imply a convergence according to the functional Central Limit Theorem to a mesokurtic bell, a normally distributed random variable, and a corresponding simplification of the forward integro-differential equation to the heat equation. This is the same limit, after accounting for reparameterization, as the Poisson process upon which the random sum is based and the same limit of all diffusions with bases that have components with real and finite mean and variance;

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$$\begin{aligned}\lim_{t\to\infty} S(t) &\sim \lim_{t\to\infty} N\left(x_0 + \mu P(\lambda t),\, \sigma^2 P(\lambda t)\right)\\ &\to N\left(x_0 + \mu N(\lambda t, \lambda t),\, \sigma^2 N(\lambda t, \lambda t)\right)\\ &\sim N\left(x_0 + N(\mu\lambda t, \mu^2\lambda t),\, \sigma^2 N(\lambda t, \lambda t)\right)\\ &\to N\left(x_0 + \mu\lambda t,\, (\mu^2+\sigma^2)\lambda t\right)\end{aligned}$$

which is a Brownian process. The variance of the normal variance, in effect, diminishes in importance relative to the overall variation of the distribution as the Central Limit Theorem takes effect.

It is further shown, from the properties of random sums of normals, that a Brownian motion with a mean and variance that are Brownian motions, with arbitrary values of drift and variance (in this case, the two are equal, a unit mean-variance ratio) is itself a Brownian motion.

The fact that a normal distribution is defined over the entire real line but the variance of a normal cannot be negative indicates that if the variance is a Brownian motion (together with the mean) which allows the possibility of negative values, there may be some unidentified underlying problem with the normal limiting process of the Poisson process generating the summation of the cumulants of the diffusion. If one considers, say, height or weight measurements or graded scores of a class examination of which negative values are impossible, it is a fact that such variables converge to a normal bell by the Central Limit Theorem, the bell admitting possible negative values even although the variables are always non-negative. This poses no logical difficulties, so long as the normal is positioned sufficiently far above the origin and that the mean to variance ratio is sufficiently large. In this case the mean and variance and exact equal and the limiting diffusion summing the basis cumulants is a unit root SNR diffusion. One can observe quite clearly that simulations of the Poisson process clearly rule out the possibility of negative values for larger periods of time even although the process has become a regular Brownian motion diffusion. For small values of time in relation to the scale set by the parameters, one should consider the continuity correction.

The unconditional density function of the process, for large values of t for which it the process reaches the limiting form with 'ordinary' parameter values, is

$$f_{S(t)}(x) = \frac{1}{\sqrt{2\pi(\mu^2+\sigma^2)\lambda t}} \exp\left(-\frac{[x - x_0 - \mu\lambda t]^2}{2(\mu^2+\sigma^2)\lambda t}\right)$$

Proof: that this limiting distribution is correct and that the limits of the third and fourth standardized measures of shape are necessary and sufficient for this convergence

$$M_{S(t)|N(t)}(\theta) = \exp\left(x_0\theta + \mu N(t)\theta + \tfrac{1}{2}\sigma^2 N(t)\theta^2\right) \qquad \text{since} \qquad S(t)\,|\,N(t) \sim N\left(\mu N(t),\, \sigma^2 N(t)\right)$$

so that, by nested conditional expectation,

$$M_{S(t)}(\theta) = E\left(M_{S(t)|N(t)}(\theta)\right) = e^{x_0\theta} M_{N(t)}(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2)$$

which is an expectation with regard to $N(t)$. As t becomes large, for sufficient frequency λ , by the Poisson process limit theorem, the primary distribution becomes a normal diffusion

$$N(t) \sim P(\lambda t) \to N(\lambda t, \lambda t)$$

so that the moment generating function of $N(t)$ is that of a normal, an exponential quadratic, with equal mean and variance which are both proportional to t, i.e., the moment generating function of the limiting Poisson process being $M_{N(t)}(\theta) = \exp(\lambda t\theta + \tfrac{1}{2}\lambda t\theta^2)$

$$M_{S(t)}(\theta) = \exp\left(\lambda t\theta + \tfrac{1}{2}\lambda t\theta^2\right)$$

so that the compound Poisson diffusion becomes a Brownian motion generated by another Brownian motion. This gives, for the moment generating function of the compound process

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$$
\begin{aligned}
M_{S(t)}(\theta) &= e^{x_0\theta} M_{N(t)}(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2) \\
&= \exp\left(x_0\theta + \lambda t(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2) + \tfrac{1}{2}\lambda t(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2)^2\right) \\
&= \exp\left(x_0\theta + \lambda t(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2) + \tfrac{1}{2}\lambda t(\mu^2\theta^2 + \mu\sigma^2\theta^3 + \tfrac{1}{4}\sigma^4\theta^4)\right) \\
&= \exp\left(x_0\theta + \lambda\mu t\theta + \tfrac{1}{2}\lambda(\mu^2 + \sigma^2)t\theta^2 + \tfrac{1}{2}\lambda\mu\sigma^2 t\theta^3 + \tfrac{1}{8}\lambda\sigma^4 t\theta^4)\right)
\end{aligned}
$$

which is the moment generating function of a generalized Gaussian distribution with cumulants to order four

$$S(t) \sim GeN\left(x_0 + \mu\lambda t,\ (\mu^2 + \sigma^2)\lambda t,\ 3\lambda\mu\sigma^2 t,\ 3\lambda\sigma^2 t\right)$$

The mean and variance of $S(t)$ are $E[S(t)] = x_0 + \mu\lambda t$ and $Var[S(t)] = \lambda(\mu^2 + \sigma^2)\, t$. The moment generating function of the standardized variable of $S(t)$,

$$\{S(t) - E[S(t)]\} / \sqrt{Var[S(t)]} = [S(t) - \mu\lambda t] / \sqrt{(\mu^2 + \sigma^2)\lambda t}$$

is

$$
\begin{aligned}
M_{[S(t)-\mu_{S(t)}]/\sigma_{S(t)}}(\theta) &= \exp\left(\frac{x_0 + \mu\lambda t - x_0 - \mu\lambda t}{[(\mu^2 + \sigma^2)\lambda t]^{1/2}}\theta + \tfrac{1}{2}\frac{(\mu^2 + \sigma^2)\lambda t}{(\mu^2 + \sigma^2)\lambda t}\theta^2 + \tfrac{1}{2}\frac{\mu\sigma^2\lambda t}{[(\mu^2 + \sigma^2)\lambda t]^{3/2}}\theta^3 + \tfrac{1}{8}\frac{\sigma^4\lambda t}{[(\mu^2 + \sigma^2)\lambda t]^2}\theta^4\right) \\
&= \exp\left(\tfrac{1}{2}\theta^2 + \tfrac{1}{2}\frac{\mu\sigma^2}{(\mu^2 + \sigma^2)^{3/2}}\frac{1}{\sqrt{\lambda t}}\theta^3 + \tfrac{1}{8}\frac{\sigma^4}{(\mu^2 + \sigma^2)^2}\frac{1}{\lambda t}\theta^4\right)
\end{aligned}
$$

which depends only on moments up to and including four. Looking at the limiting operation in t, as t becomes large, the 3rd and 4th order cumulants diminish in relative importance leaving the moment generating function

$$\lim_{t\to\infty} M_{[S(t)-\mu_{S(t)}]/\sigma_{S(t)}}(\theta) = e^{\frac{1}{2}\theta^2}$$

which is the moment generating function of a unit standard normal,

$$[S(t) - \mu_{S(t)}]/\sigma_{S(t)} \sim N(0,1)$$

Reintroducing the location and scale gives

$$\lim_{t\to\infty} M_Y(\theta) = \exp\left(x_0\theta + \mu\lambda t\theta,\ \tfrac{1}{2}(\mu^2 + \sigma^2)\lambda t\theta^2\right)$$

which is the distribution of the central limit normal convergence, a Brownian motion

$$S(t) \sim N\left(x_0 + \mu\lambda t,\ (\mu^2 + \sigma^2)\lambda t\right)$$

The proof is complete.

Diffusion Limits: the Diffusion Forward Convolution and Backward Deconvolution Equations

(1) Asymptotic Forward Equation

Remembering the forward equation of the Poisson process compound sum

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \lambda \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)}(x - y)dy - \lambda\ f_{S(t)}(x)$$

Expanding the second density function in the convolution integral, the density of the transition distribution of the compound sum, in a univariate power series[50]

[50] Due to the author.

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$$\frac{\partial}{\partial t} f_{S(t)}(x) = \lambda \int_{-\infty}^{\infty} f_X(y) \left(f_{S(t)}(x) - y \frac{\partial}{\partial x} f_{S(t)}(x) + \tfrac{1}{2} y^2 \frac{\partial^2}{\partial x^2} f_{S(t)}(x) - \ldots \right) dy - \lambda f_{S(t)}(x)$$

Truncating all terms in *y* of degree three and higher with the understanding that, by the Central Limit Theorem, these higher order effects (raw moments) of the convolved basis variables become increasingly negligible with the passage of time in relation to the second order effects of the diffusion variable but never actually vanish. For the convergence to be exact, there is the requirement that

(**i**) the Poisson rate intensity is proportional to the inverse of the span variable *h*,
(**ii**) the cumulants of the basis variable each proportional to *h*, and
(**iii**) the limit $h \to 0$ is taken

rather that appealing to relative dominance of the second order effects for increasing cumulative aggreation of the unit root integration and the Central Limit Theorem. Performing the integrations in the dummy variable *y* for the density of the incremental distribution *X* leaves the forward diffusion equation,

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \tfrac{1}{2} \lambda E(X^2) \frac{\partial^2}{\partial x^2} f_{S(t)}(x) - \lambda E(X) \frac{\partial}{\partial x} f_{S(t)}(x)$$

The cumulants of the incremental distribution are μ and σ^2 which imply the raw moments μ and $\sigma^2 + \mu^2$

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \tfrac{1}{2} \lambda (\sigma^2 + \mu^2) \frac{\partial^2}{\partial x^2} f_{S(t)}(x) - \lambda \mu \frac{\partial}{\partial x} f_{S(t)}(x)$$

which is the forward equation of an Brownian motion with infinitesimal parameters $\lambda\mu$ and $\lambda(\sigma^2 + \mu^2)$.

(2) Asymptotic Backward Equation

From the backward equation of the Poisson process compounding sum

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x \mid x_0 + y) dy - \lambda f_{S(t)|S(t_0)}(x \mid x_0)$$

Expanding the transition distribution of the compound sum, the second density in the convolution integral, in a univariate power series,

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0)$$
$$= \lambda \int_{-\infty}^{\infty} f_X(y) \left(f_{S(t)|S(t_0)}(x \mid x_0) + y \frac{\partial}{\partial x_0} f_{S(t)|S(t_0)}(x \mid x_0) + \tfrac{1}{2} y^2 \frac{\partial^2}{\partial x_0^2} f_{S(t)|S(t_0)}(x \mid x_0) + \ldots \right) dy - \lambda f_{S(t)|S(t_0)}(x \mid x_0)$$

and truncating all terms of in the variable *y* of degree three and higher, again with the understanding that these higher order effects of the raw moments of the secondary distribution become increasingly negligible with the passage of time by the Central Limit Theorem but never actually vanish; performing the required statistical integrations,

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2} \lambda E(X^2) \frac{\partial^2}{\partial x_0^2} f_{S(t)|S(t_0)}(x \mid x_0) + \lambda E(X) \frac{\partial}{\partial x_0} f_{S(t)|S(t_0)}(x \mid x_0)$$

or

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2} \lambda (\sigma^2 + \mu^2) \frac{\partial^2}{\partial x_0^2} f_{S(t)|S(t_0)}(x \mid x_0) + \lambda \mu \frac{\partial}{\partial x_0} f_{S(t)|S(t_0)}(x \mid x_0)$$

which is the backward diffusion equation of asymptotic Brownian motion. For these limiting equations to be exact, rather than increasingly nearer approximations by the Central Limit Theorem, to the forward and backward equations of a Brownian motion with an ordinary Gaussian basis, the parameters of the compound process have to be parameterized in a specific way with the introduction of the span variable h. In particularly, (**i**) the Poisson rate intensity must be proportional to the reciprocal of *h*, $\lambda \propto 1 / h$, (**ii**) the cumulants of the distribution

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of the basis variable must each directly proportional to h and (**iii**) the limit $h \to 0$ taken. If the basis variable is normal, then the mean and the variance of each of the time invariant basis components are μh and $\sigma^2 h$, that is,

$$N(t) \sim P(t/h) \text{ and } X \sim N(\mu h, \sigma^2 h)$$

so that

$$S(t) = x_0 + \sum_{j=1}^{N(t)} X$$

which is in distributional terms

$$\begin{aligned} S(t) &\sim x_0 + \sum_{j=1}^{P(t/h)} N(\mu h, \sigma^2 h) \\ &\sim x_0 + \sum_{j=1}^{P(t/h)} N(\mu, \sigma^2/h)h \end{aligned}$$

which is consistent with the basis parameterization of the discrete diffusion processes of PART II of BOOK 3, the Calculus of Diffusion of Arbitrary Order and Option Valuation. With such a parameterization the exact forward equation becomes

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{S(t)|S(t_0)}(x \mid x_0) - \mu \frac{\partial}{\partial x} f_{S(t)|S(t_0)}(x \mid x_0) + \frac{o(h)}{h}$$

and that of the backward equation

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} f_{S(t)|S(t_0)}(x \mid x_0) + \mu \frac{\partial}{\partial x_0} f_{S(t)|S(t_0)}(x \mid x_0) + \frac{o(h)}{h}$$

which become $\frac{\partial f}{\partial t} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x^2} - \mu \frac{\partial f}{\partial x}$ and $-\frac{\partial f}{\partial t_0} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial x_0^2} + \mu \frac{\partial f}{\partial x_0}$ respectively as $h \to 0$ so that Poisson events occur at every infinitesimal instant (occur infinitely often , $\lambda \to \infty$) and the basis components scale down with the span to keep the cumulants in proportion with infinitesimals in order to keep the relevant derivatives finite and non-negligible while those of higher order vanish.

What Advantage is there if the Process is Simply a Brownian Motion in the Limit?

There are several good reasons why there is additional utility in this specification, given the increase in sophistication and loss of parsimony, given that the limiting distribution reverts to that of an ordinary Brownian motion. The most basic of these is that even although the limiting distribution has the same structural form and the same formulae result, such as Black-Scholes type formulae based on limited moments of the log-normal distribution, after reparameterization, there is more freedom to assign values to parameters and more dynamics can be taken into account in their estimation. This is in addition to the modeling of the specific times at which the randomly occurring endpoints of each increment are observed. Secondly, the compound Poisson diffusion gives a closer correspondence to the variables actually generating the observed outcomes than an ordinary diffusion. As explained shortly, the limiting Brownian motion is the result of the fact that the compound Poisson process is itself a unit-root first order diffusion, like Brownian motion, the sole difference being how the diffusion basis components sum - stochastically and discretely rather than continuously - this difference being negligible as the limiting process becomes dominant. If one assumes that prior to convergence of the process to an ordinary Brownian motion, the Poisson process of the generator has, for all material purposes, converged to a Brownian motion, then the resulting process is equivalent to a diffusion with a generalized Gaussian basis with non-zero cumulants to order four. Although the transition distribution does not have an analytical density, the partial differential equation of the dynamics of the transition density is closed form, in the form of a generalized heat equation[51]

[51] Due to the author, and introduced in FOUNDATIONS I and derived, at numerous stages throughout this thesis, for various infinitesimal parameters; in particular, for additive and multiplicative unit root diffusion and for the additive and multiplicative generalization of the Ornstein-Uhlenbeck diffusion. The general calculus of diffusion can be used to obtain the corrections to the infinitesimal parameters by transforming this equation, such as that for transforming the generalized heat equation and its variants, is also due to the author (of this thesis)

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with diffusion terms to order four (skew and excess kurtosis), which can be solved numerically for the transition probability distribution as well as option premia and other related statistics. This equation is given shortly. Given that these higher order effects never actually vanish, since each are proportional to time *t*, but only become relatively less important with the passage of time, it is likely that the increase in modeling accuracy accruing by using the more exact dynamics is worthwhile. Such generalized equations can also be used if one finds for the limiting process that, after taking into account sampling fluctuations, the higher order cumulants are non-negligible. Then one may start out by modeling with a diffusion basis, or secondary distribution, that accounts for the higher order non-zero cumulants. The methods are given in later parts of this thesis, for continuous diffusion, for incorporating additional skewness and excess kurtosis into the transition distribution for the normal compound Poisson process. Such methods can be extended to higher order non-zero cumulants and the resultant generalized heat equation to arbitrary order. The presence of such features in the observed incremental process of microstructure transaction data, and the absence in the incremental process of the same process observed at larger time frames, say daily, is indicating a scale effect. This may be confirmatory for a compound Poisson diffusion at the microstructure level and the converged compound Poisson diffusion, an ordinary Brownian motion, at the larger time frame, whereby the convergence is reached between the two end-points of any increment at the higher order time scale.

With regard to rare events and how a small Poisson frequency λ may effect convergence, a linear addition of another compound Poisson process, one with a small rate intensity λ and small signal to noise ratio of the secondary normals, will still approach a limiting Brownian motion but the speed of this approach will be slowed, possibly dramatically. If the distribution of the secondary distribution contains complex moments, this may invalidate the convergence to a regular Brownian motion altogether. This is especially so for a secondary distribution that has a complex mean or variance, in which case, the process is no longer Gaussian nor is the limiting form.

Compound Poisson Sums of Normals as a Diffusion Process

For micro-structure transaction data, the Poisson occurrence is frequent and the cumulations of secondary normal distributions are relatively small in mean and variance. This implies that only local changes occur but these changes occur frequently. In this regard, when the Poisson process is generating high frequency additions to the compound sum of homogenous components with variance that produces local movements that are of the same order, rather than in exact proportion, to the mean spacing between Poisson additions, then the process is a diffusion of probability that bears a close resemblance to Brownian motion, even prior to reaching the limiting convergence form. That is, a compound Poisson process is a diffusion like the processes considered in FOUNDATIONS I with continuous additions of uncorrelated infinitesimal normals of a Gaussian basis. The set of uncorrelated homogenous secondary distributions is the basis of the diffusion. This basis is not a Hilbert space since there are always a discrete and finite number of distributions convolved, rather than an infinite number of a continuum, onto the discrete aggregate of the diffusion in any time interval. The correspondence between the two becomes very obvious in the analysis of random walks, in particular, ordinary random walks and random walks generated by a Poisson process. This is found in the appendix to BOOK 1, **A Compendium of Random Walks**. A discussion of the transformations of the Poisson diffusion that correspond to the transformations between the processes of FOUNDATIONS I, taking the process discussed in this section as the basic or fundamental process equivalent to Brownian motion, is deferred until BOOK 3 of this dissertation. Such transformations include introducing a moderator between the convolution components in the summations, as per the Ornstein-Uhlenbeck process, and the additive to multiplicative exponential transformation, as well as the generalization to diffusions of higher order. This is easily achieved within the framework of BOOK 3 of this dissertation.

The fact that the limiting form of the process is a Brownian motion, an ordinary first order unit-root diffusion, gives further justification for recognizing the compound Poisson process as a diffusion process and as a direct counterpart to Brownian motion;

- **Brownian motion:** the continuous deterministic (linear) summation of the basis components
- **Compound Poisson diffusion:** the discrete stochastic (linear) summation of the basis components

One familiar with the literature on stochastic processes will perhaps further agree when it is remembered that a compound Poisson process is a Levy process, often used to move beyond Brownian motion, as a process with the (first order) Markov property and the corresponding property of independent increments. The reader is reminded that the (first order) Markov property is a defining feature of first order diffusion.

In the limit, the difference in parameterization between the ordinary Brownian motion and the Brownian motion resulting from the compound Poisson process accumulation of Gaussian variables accounts for

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(1) the additional normal dispersion arising from the Poisson process which adds an additional component to the variance of the limiting normal transition distribution; and
(2) the inclusion of the Poisson parameter for rate intensity, λ, the parameter that adjusts for a stochastically fractal time dimension.

This parameter is the analogue to *h* that the work in this thesis uses to time scale the increments of ordinary diffusion. This parameter, h, in an ordinary diffusion is simply a scaling for time units and is not a distributional parameter as it is in a compound Poisson diffusion.

Compound Poisson Process – a Diffusion Occurring Over a First Order Diffusion Time Variable

To extend the argument and shed further light on the increased sophistication of a compound Poisson unit-root diffusion relative to Brownian motion, one may again consider the fact that the compound Poisson process is a stochastic process with regard to the two separate dimensions, the dimension in the time variable (T) and the dimension in the diffusion variable (X_T). A compound Poisson diffusion is a monotone aggregation of unit steps, generated by a unit-root aggregation of a Poisson basis, on the diffusion variable and a linear unit-root aggregation of an exponential basis on the time variable. The compound Poisson process turns the aggregation of unit steps into an aggregation of uncorrelated unit-root normal convolutions, and with the fact that the Poisson process is itself a diffusion, the compound Poisson process is an amalgam of a Gaussian diffusion on the axis of the dependent variable and an exponential diffusion on the axis of the independent variable, time.

Compound Poisson Diffusion or Regular Diffusion with Exponential Observation Times?

The requirement for a diffusion to be a compound Poisson diffusion, a compound Poisson aggregated sum of normals (or any secondary distribution with finite second moment, due to the Central Limit Theorem) rather than an ordinary diffusion, is that the secondary distributions composing the elements of the basis are time invariant. For example, a long elapsed time since the last transaction or Poisson event does not result in an addition of the next basis component to the cumulative aggregation with a larger mean or variance. If the mean and variance were in exact proportion to the time between transactions, then the process would be an ordinary diffusion with exponential observation times. Determining whether a diffusion is a compound Poisson diffusion can be tested empirically. The times between transactions verse the incremental differences of the diffusion variable being observed at the transaction times should be uncorrelated for a compound Poisson diffusion. A scatter plot of the two variables should show no (significant) linear trend and zero heteroscedasticity. If the process is an ordinary diffusion, and not a compound Poisson diffusion, the scatter plot should show a linear increase for larger waiting times between transactions with linearly increasing variance, i.e., linear heteroscedasticity, around the linear trend.

Maximum Likelihood Estimation

A. Exponentially Sampled Ordinary Unit-Root Diffusion (or Log-Diffusion)

For comparison purposes, say that it is assumed that the transaction prices form a discrete sampling at random points in time of a continuous log-diffusion. For this process, the width of each increment, h, is now variable and determines $h_j = t_j - t_{j-1} = \Delta t_{j-1}$. That is, the sampling of the continuous log-diffusion is at arbitrary times and the increment between sample observations is a variable of arbitrary magnitude which scales the mean and variance of the normal variable that generates the incremental diffusion convolution. Because of the property of independent and scalable increments or log-increments of a Brownian or geometric Brownian motion (ignoring the ito-correction), each increment between transactions is normal with parameters $\mu(t_j - t_{j-1}) = \mu h_j$ and $\sigma^2(t_j - t_{j-1}) = \sigma^2 h_j$,

$$\Delta \log x_j = \delta_j \sim N(\mu h_j, \sigma^2 h_j)$$

If one was modeling an additive process, rather than a multiplicative process, the increment would be the same function without the logarithmic transformation of each data point, i.e., the increment $x_j - x_{j-1} = \delta_j$ and this would be the variable of the likelihood components

$$x_j - x_{j-1} = \delta_j \sim N(\mu h_j, \sigma^2 h_j)$$

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and the procedure follows along the same lines. The likelihood equation is

$$L(\mu,\sigma)=\prod_{j=1}^{n}\ \frac{1}{\sigma\sqrt{h_j}\sqrt{2\pi}}\exp\left(-\frac{[\delta_j-\mu h_j]^2}{2\sigma^2 h_j}\right)=\frac{1}{(\sigma\sqrt{2\pi})^n\prod_{j=1}^{n}\ \sqrt{h_j}}\exp\left(-\sum_{j=1}^{\infty}\ \frac{[\delta_j-\mu h_j]^2}{2\sigma^2 h_j}\right)$$

and the log-likelihood is

$$l(\mu,\sigma)=\log L(\mu,\sigma)=-n\log\sigma\sqrt{2\pi}-\sum_{j=1}^{n}\ \log\sqrt{h_j}-\sum_{j=1}^{n}\ \frac{[\delta_j-\mu h_j]^2}{2\sigma^2 h_j}$$

The components of the gradient log-likelihood surface are

(1) $$l_\mu=\frac{\partial}{\partial\mu}\log L(\mu,\sigma)=\sum_{j=1}^{n}\ \frac{\delta_j-\mu h_j}{\sigma^2}$$

(2) $$l_\sigma=\frac{\partial}{\partial\sigma}\log L(\mu,\sigma)=-\frac{n}{\sigma}+\sum_{j=1}^{n}\ \frac{[\delta_j-\mu h_j]^2}{\sigma^3 h_j}$$

giving the likelihood equations $l_\mu = 0$ and $l_\sigma = 0$. The estimators are then

$$\hat{\mu}=\frac{\bar{\delta}}{\bar{h}}=\frac{\log(x_n/x_0)}{t_n-t_0}$$

which is the average per period log-return, the same as for the estimation over equally spaced observations, and

$$\hat{\sigma}^2=\frac{1}{n}\sum_{j=1}^{n}\ \frac{(\delta_j-\bar{\delta})^2}{h_j}$$

the average squared discrepancies from the mean value, scaled by the width of each increment.

B. Poisson Process Compound Sum of Homogenous Independent Normals (Compound Poisson Unit-Root Diffusion)

Since the Poisson process is independent of each of the normal noise contributions, the parameter estimates are uncorrelated. The likelihood equation, given that $N(t) = n$ Poisson events have occurred over an observation period of length t is then

$$L(\mu,\sigma)=\prod_{j=1}^{n}\ \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{[\delta_j-\mu]^2}{2\sigma^2}\right)=\frac{1}{(\sigma\sqrt{2\pi})^n}\exp\left(-\sum_{j=1}^{\infty}\ \frac{[\delta_j-\mu]^2}{2\sigma^2}\right)$$

and the log-likelihood is

$$l(\mu,\sigma)=\log L(\mu,\sigma)=-n\log\sigma\sqrt{2\pi}-\sum_{j=1}^{n}\ \frac{[\delta_j-\mu]^2}{2\sigma^2}$$

The components of the gradient log-likelihood surface are

(1) $$l_\mu=\frac{\partial}{\partial\mu}\log L(\mu,\sigma)=\sum_{j=1}^{n}\ \frac{\delta_j-\mu}{\sigma^2}$$

(2) $$l_\sigma=\frac{\partial}{\partial\sigma}\log L(\mu,\sigma)=-\frac{n}{\sigma}+\sum_{j=1}^{n}\ \frac{[\delta_j-\mu]^2}{\sigma^3}$$

giving the estimators

$$(\hat{\mu},\hat{\sigma}^2)=\left(\frac{1}{n}\log(x_n/x_0),\frac{1}{n}\sum_{j=1}^{n}\ (\Delta\log x_{j-1}-\hat{\mu})^2\right)$$

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which are the mean and sample variance of the differenced transaction price data. The sampling distributions of the estimators of the basis components are

$$\hat{\mu} \sim \log N(\mu, \sigma^2 / nh) \qquad \text{or} \qquad \log \hat{\mu} \sim N(\mu, \sigma^2 / nh)$$

and

$$\hat{\sigma}^2 \sim \left(\frac{\sigma^2}{n}\right)\chi^2_{n-1} \qquad \text{or, asymptotically,} \quad \hat{\sigma} \sim AN\left(\sigma, \frac{\sigma^2}{2n}\right)$$

The parameter estimates for the parameters for the secondary distributions, and their sampling distributions, are treated in greater depth in the appendices to BOOK 2. The covariance matrix of the Poisson log-diffusion takes a simple form and is simply an additional diagonal element to the covariance matrix, of the secondary normal distributions, with zeroes on the remaining row and column entries[52]. Since the parameters of the normal log-increments are uncorrelated, the covariance matrix is diagonal,

$$\Sigma(\mu, \sigma, \lambda) = \begin{bmatrix} \sigma^2 / nh & 0 & 0 \\ 0 & \sigma^2 / 2n & 0 \\ 0 & 0 & \lambda^2 / n \end{bmatrix}$$

The sampling distributions allows confidence intervals to be constructed, by the method of statistical differentials or delta method, for sampling variation in specified functions of interest, an option premium for example, arising from the sampling variation in the estimated parameters.

Preliminary Diagnostics Regarding the Model Characteristics

Most of the possible deviations between an observed sequence and the defining model characteristics can be remedied by the various generalizations and extensions including transformation, generalizing the Markov property, in either dimension, and generalizing the distribution of the basis components. The majority of such generalizations have been discussed throughout the course of this dissertation.

(1) independence between the Poisson process and the normal basis components – there should be zero statistically significant correlation between the distribution of the times between transactions T_j and the distribution of the changes in the observed price variable that occur between each transaction X_j, i.e., $Corr(T_j, X_j) = 0$ within reasonable bounds of sampling variation.

(2) check the reasonableness of the Poisson process generating the diffusion; i.e., that the basis of the time diffusion is

- **i.** exponential
- **ii.** uncorrelated
- **iii.** homogenous
- **iv.** time invariant

or, alternatively, that the basis of the generator is

- **i.** Poisson
- **ii.** uncorrelated
- **iii.** homogenous
- **iv.** time invariant

(3) check that the basis components are what they are supposed to be

---

[52] the covariance matrix with the variance rather than the standard deviation of the normal secondary distributions, or basis components, is

$$\Sigma(\mu, \sigma, \lambda) = \begin{bmatrix} \sigma^2 / n & 0 & 0 \\ 0 & 2\sigma^4 / n & 0 \\ 0 & 0 & \lambda^2 / n \end{bmatrix}$$

which is associated with two independent normals and one independent chi-square.

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i. normal
ii. uncorrelated
iii. homogenous
iv. time invariant

Further Assessment of the Suitability of the Compound Poisson Diffusion for Observed Transaction Data

The suitability of the process relies upon independence between the Poisson process and the normal components of the basis and exponential transaction times that characterize the Poisson process. Further, features of the exponential distribution that the observed transaction times should have and for which the observed process must not violate, statistically, include a constant hazard rate and mean residual life, and a mode at zero.

(1) constant hazard rate and mean residual life:

The hazard rate of the exponential measures the instantaneous likelihood that a Poisson event, namely a transaction in this context, will occur. For the Poisson process, this is constant and equal to the mean arrival time λ. At any time following the $(j-1)^{th}$ occurrence, the instantaneous likelihood that the j^{th} occurrence will occur immediately, given that it has not yet occurred, is equal to

$$\lambda(t_j) = \frac{f_{D_j}(t)}{S_{D_j}(t)} = \frac{\lambda e^{-\lambda t_j}}{e^{-\lambda t_j}} = \lambda$$

Additionally, the mean residual lifetime of an exponential distribution is also constant and equal to the mean waiting time, the rate of arrivals or transactions, $1/\lambda$. So any time following the $(j-1)^{th}$ Poisson occurrence, the mean waiting time until the j^{th} Poisson occurrence is equal to

$$E(D_j - k \mid D_j > k) = \frac{E(D_j - k \cap D_j > k)}{\Pr(D_j > k)} = \frac{\int_k^\infty (t-k) f_{D_j}(t)dt}{\int_k^\infty f_{D_j}(t)dt} = \frac{[-(t-k)e^{-\lambda t}]_k^\infty + \int_k^\infty e^{-\lambda t}dt}{e^{-\lambda k}}$$

$$= \frac{[-e^{-\lambda t}/\lambda]_k^\infty}{e^{-\lambda k}} = \frac{1}{\lambda}$$

This may be used heuristically to give a 'feel' that the process is correct, or otherwise, may be formally tested statistically with a test or interval regarding λ or $1/\lambda$.

(2) Mode at zero – an empirical histogram of the transaction time differences will show if this is reasonable. This is one manifestation of a proportional density of the exponential distribution and the resulting properties of constant arrival rate and mean residual life.

Option Premia

Since the time variable and the diffusion variable are independent, the parameters can be estimated with regard to the separate dimensions, and since a function of maximum likelihood estimators is the maximum likelihood estimator of the function, the estimates can be used to estimate the option premium as well as further give confidence bands by the method of statistical differentials (the delta method) for sampling variation in the premium due to the estimation of the parameters. Assuming the limiting form of the process, a Brownian motion, will be reached by option maturity, the put and call option formulae are then in the form of a limited normal or log-normal first moment[53]

If it is felt that there is not a sufficient amount of elapsed time for a reasonable convergence, then the premium calculation can proceed by either

(1) summing out the random Poisson parameter or
(2) numerically inverting the moment generating function or characteristic function.

[53] See BOOK 2 and BOOK 3 for the limited moment calculations and the resulting formulae for put and call option premia.

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Summing out the Poisson parameter involves summing the conditional limited normal or log-normal moments weighted by the Poisson probabilities. The summation is infinite but is only required over the values of the count distribution where the probabilities are material in regard to the size of the truncated moments. The sum is best implemented with a do-while loop, the while condition of the recursion being some error threshold past which convergence of the terms of the infinite sum can reasonably be assumed inconsequential. Numerically inverting the moment generating function involves complex integration of the moment generating function along the Bromwhich contour, or alternatively and equivalently, inverting the moment generating function with a complex parameter with an inverse Fourier transform. Summing out the Poisson parameter is preferable to an inversion with complex integration since the inversion has to be implemented numerically[54] and numerical error is harder to control and track when it is on the complex plane.

An important consideration regards how much time must elapse before the limiting stationary distribution is reached, and in the meantime, how the statistical properties of the process are evolving and how the trend towards the limiting distribution can be utilized to make the analysis more intuitive and easy to comprehend. Importantly, one should pay attention to what the convergence process looks like, and in particular, how the moments are changing in relation to one another as the Poisson process diffusion increasingly resembles linear time.

The most relevant question is then "how much time is sufficient for convergence to be reached?" in order for the second order properties to become overridingly dominant and for simplified, closed form, formulae of the limited moments of the normal or log-normal process to produce accurate and reliable results, such that they are not discernibly different from the exact formulae based on summing out the Poisson parameter. The answer to this question depends on a relation between the Poisson rate intensity and the magnitude of the moments of the secondary normals, i.e., the relation between λ, μ, and σ^2. Looking at the lowest four coefficients of the polynomial in the moment generating function parameter in the index of the standardized moment generating function for which the Poisson process has reached convergence to a Brownian motion,

$$M_{(S(t)-\mu_{S(t)})/\sigma_{S(t)}}(\theta) = \exp\left(\tfrac{1}{2}\theta^2 + \tfrac{1}{2}\frac{\mu\sigma^2}{(\mu^2+\sigma^2)^{3/2}}\frac{1}{\sqrt{\lambda t}}\theta^3 + \tfrac{1}{8}\frac{\sigma^4}{(\mu^2+\sigma^2)^2}\frac{1}{\lambda t}\theta^4 \right)$$

It can be seen directly that convergence of the moment generating function to that of a standard normal occurs at a quicker rate the larger the rate at which Poisson events occur, λ, which in this context, is equivalent to speeding up time. Further, this convergence is reached more quickly the larger the mean of the basis components, μ, relative to the variance of the basis components, σ^2. If one considers the fact that both the mean and the variance of the Gaussian components contribute to the mean of the compound sum equally, i.e., a variance of $\kappa_2 = \lambda(\mu^2 + \sigma^2)t$, and that all moments of the compound Poisson sum are proportional to the raw moments of the Gaussian contributions, for which the variance of the components have a greater proportional effect on the rate at which the raw moments grow with (as a function of) order, then convergence which requires the variance of the compound sum to become dominant in relation to the higher order moments is speeded up the smaller the variance relative to the mean of the Gaussian contributions. That is, convergence is reached more quickly the larger the mean to variance ratio, i.e., the larger the signal-to-noise ratio of the secondary Gaussian components of the basis.

With a Poisson process that generates high frequency events in line with what one observes with transactions and required for the compound process to be a diffusion, the limiting Brownian motion or geometric Brownian motion, depending on whether an additive or multiplicative diffusion is required, would generally be reached quickly, and the premia based on the limited moments of the analytic limiting transition distribution would envisaged to be used almost exclusively. Alternatively, the partial differential equation of the process with non-zero skew and excess kurtosis may be more correct if one is willing to numerically solve the partial differential equation or numerically invert the moment generating function.

Convergence Diagnostics

There are a number of ways of proceeding to check whether the limiting transition distribution is suitable for basing the option premia. I will suggest two **techniques for convergence diagnostics.** The first method is to

[54] noting that a model, or process, that requires numerical inversion of an moment generating function or characteristic function is not closed form

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track the convergence with a plot of a rolling Berry-Esseen[55] bound. For a given option expiry, the value of the Berry-Esseen bound can be calculated and used as a basis for deciding whether the skewness bound between the distribution function of the compound Poisson process and the normal limiting distribution is tight enough to reasonably substitute one for the other and allow for an accurate premium calculation. The second is to test for convergence empirically, for some fixed time interval, by calculating the empirical third and fourth order standardized measures of distributional shape, the **coefficient of skewness** $\hat{\zeta}_3$ and **coefficient of excess kurtosis** $\hat{\zeta}_4 - 3$ (on logs if one is modeling a multiplicative diffusion) and decide whether it is reasonable to assume that both the statistics are zero after allowing for sampling variation. One can look at **confidence intervals** for these two statistics or **test** the null hypothesis that the underlying parameters are zero and that the model has converged to a limiting Brownian motion. The alternative hypothesis is that they are not zero. A rejection of the null hypothesis is providing evidence towards either a lack of adequate time for convergence or that there is an error in the model specification. I will leave these for the reader to pursue further.

LINEAR SUPERPOSITION OF A BROWNIAN MOTION TO A COMPOUND POISSON SUM OF GAUSSIANS

The Diffusion Equation

It may be of interest to add an ordinary continuous function $g(t)$ to the compound Poisson process, in which case the diffusion variable conditional on the generator becomes

$$S(t) = s_0 + g(t) + \sum_{j=1}^{N(t)} X_j$$

where $N(t) \sim P(\lambda t)$ and $X \sim N(\mu, \sigma^2)$

Transition Density and MGF of the Compound Poisson Sum

The unconditional transition density function is

$$f_{S(t)}(x \mid s_0) = \sum_{j=1}^{\infty} p_n(t) f_{S(t)|N(t)=n}(x - g(t) \mid s_0)$$

From a signal processing and engineering perspective, this is a signal inside noise where the noise is a compound Poisson diffusion. Since the density of the deterministic function is a Dirac singularity, the mgf of this function is its exponential, $e^{g(t)\theta}$. The mgf of the aggregative diffusion variable becomes

$$M_{S(t)}(\theta) = \exp\left(s_0\theta + g(t)\theta + \lambda t[M_X(\theta) - 1]\right)$$

Kolmogorov Forward Integro-Differential Convolution Equation

Differentiating with respect to time t

$$\frac{\partial}{\partial t} M_{S(t)}(\theta) = \left(g'(t)\theta + \lambda[M_X(\theta) - 1]\right) M_{S(t)}(\theta) \qquad \text{where} \qquad g'(t) = \frac{d}{dt} g(t)$$

and converting to the differential equation in the generating function to the equivalent equation in the transition density by reference to the convolution integral for the product of generating functions

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \lambda \int_{-\infty}^{\infty} f_X(y) f_{S(t)}(x - y) dy - \lambda f_{S(t)}(x) - g'(t) \frac{\partial}{\partial x} f_{S(t)}(x)$$

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[55] Discussed in
i. Feller, W., (1966) An Introduction to Probability Theory and Its Applications, Volume 2. John Wiley & Sons.
ii. Feller, W., (1966) On the Berry-Esseen Theorem, Z. *Wahrscheinlichkeitstheorie verw* Vol. 10, (1968), pp. 261-268.
iii. Pinsky, M. A., (2002) *Introduction to Fourier Analysis and Wavelets.* Series in Advanced Mathematics, Brooks/Cole.

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which is what Feller[56] refers to as the Fokker-Planck equation with the exception that the translation β, here, is made a function of time $\beta = g'(t)$. When β is a constant, this amounts to adding linear drift to the compound Poisson diffusion in an equivalent manner as that for Brownian motion. This may be inclusive of non-zero basis mean of the basis distributions X_j. This equation and the underlying process can be obtained from a linear supposition of a compound Poisson diffusion and a Brownian motion, considered next, by setting the infinitesimal mean μ equal to the derivative of the required function of time $\mu = g'(t)$ (which is the equivalent to the symbol β in Feller's note on the Fokker-Planck equation) and letting the infinitesimal variance approach zero $\sigma^2 \to 0$.

For Gaussian compound Poisson diffusion, which has normal basis components $X \sim N(\mu, \sigma^2)$, this is

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \frac{\lambda}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y-\mu)^2}{2\sigma^2}} f_{S(t)}(x-y)dy - \lambda f_{S(t)}(x) - g'(t)\frac{\partial}{\partial x} f_{S(t)}(x)$$

and for exponential compound Poisson diffusion, which has exponential basis components with scale β $X \sim \Gamma(1, \beta)$, this is

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \frac{\lambda}{\beta} \int_{-\infty}^{\infty} e^{-y/\beta} f_{S(t)}(x-y)dy - \lambda f_{S(t)}(x) - g'(t)\frac{\partial}{\partial x} f_{S(t)}(x)$$

Diffusion Limits

(1) normal basis components $z_t \sim N\left(\mu, \sigma^2/dt\right)$

(2) exponential/gamma basis components $z_t \sim \Gamma\left(\alpha dt,\ \beta/dt\right)$

Limiting arguments of these two diffusion bases results in

(i) the forward equation of a regular Brownian motion and
(ii) the forward equation of a gamma diffusion

superimposed on top of the regular continuous function $g(t)$.

The equation of a gamma diffusion is an example of the generalized heat equation described throughout this thesis, a specific example of the gamma diffusion given in the appendix to BOOK 2.

Example, Insurance Risk

An example, from actuarial risk theory, is modeling the solvency level of a general insurer. This example is indication of the close connection between actuarial risk theory and the mathematics of this thesis and further suggestion of the appropriateness and suitability of mathematicians with actuarial background [and qualifications] to manage the risk aspects of option trading departments and quantitative aspects of organized derivative exchanges as suggested and argued in BOOK 2 regarding a revised philosophy for option markets

The assumptions are the insurance company has **(i)** an initial surplus s_0, which is the backward diffusion variable, **(ii)** incurs claims with frequency lambda λ, **(iii)** has claim cost severity X_j, and **(iv)** premium inflow occurring at rate P which is ordinarily assumed to occur at a constant rate but can, and probably more appropriately given additional information or particulars for a given scenario, made a suitable function of time, say periodic or logisitc, etc or even more realistically, say, a Brownian motion. The financial position of the insurer, over the transition interval, is then

$$S(t) = s_0 + \bar{P} \times t - \sum_{j=1}^{N(t)} X_j$$

whereby the company, from the initial surplus of s_0 has earned $\bar{P} \times t$ from premium income and incurred $N(t)$ claims of X_j. This is a linear supposition of a compound Poisson Process and linear drift, the dynamics of the density for which follows the Fokker-Planc equation. The transition distribution of the company's financial position is then

[56] Feller, W., (1966) An Introduction to Probability Theory and Its Applications, Volume 2. John Wiley & Sons; page 296.

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$S(t) \sim s_0 + \overline{P} \times t - \sum_{j=1}^{P(\lambda t)} N(a, b^2)$ for normally distributed claims or

$S(t) \sim s_0 + \overline{P} \times t - \sum_{j=1}^{P(\lambda t)} \Gamma(a, b)$ for gamma distributed claims.

For premium inflow that is a Brownian motion, a specification considered shortly with linear supposition, the transition distribution is

$$S(t) \sim s_0 + N(\mu t, \sigma^2 t) + \sum_{j=1}^{P(\lambda t)} N(a, b^2)$$

When the basis components are normal, the mean of the basis components of the compound Poisson diffusion is negative.

Differential change in the financial position of the insurer is

$$\frac{d}{dt} S(t)$$

which is the diffusion side of the differential form of the diffusion equation and the financial position at the upper extremum of the transition interval, say T, is $S(T)$.

For probability calculations, most notably to determine the probability that the company is solvent $S(T) > 0$ at some value of the forward variable, it is usual to use the normal approximation regardless of the assumed distribution of the basis components by reference to the Central Limit Theorem. This is equivalent to taking the continuous time limit and convergence to a regular Brownian motion and the limit of the generalized heat equation to the standard heat equation. This approximation is relatively accurate given the right conditions for adequate convergence to a regular diffusion.

A put on ruin would is an interesting academic concept. Considered with discounting and inflation, the basis additions can be modulated as per a discrete Ornstein-Uhlenbeck process summed with exponential increments.

From this example, it is clear that **given certain alterations for the nuances of the stochastic variation**, those trained in the mathematics of risk theory would feel very comfortable with the mathematics of this thesis and, reiterating, have the ability to respond professionally to the suggestions made in BOOK 2 regarding a revised philosophy for option markets and the management of the risk aspects of option trading departments, specifically for listed options traded on exchanges.

Integro-Differential Equation of Put and Call Option Premia for Linear Sup-Position of Compound Poisson Diffusion and Continuous Signal – Option Formulae Form for Fokker-Planc

Further, the equation for an option on a linear supposition of a compound Poisson diffusion and a deterministic function $g(t)$ is

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t;x_0,t_0) = \lambda \int_{-\infty}^{\infty} f_X(y) V(k,t;x_0+y,t_0)dy - \lambda V(k,t;x_0,t_0) - g'(t)\frac{\partial}{\partial x_0} V(k,t;x_0,t_0)$$

which is the Fokker-Planck equation with a sink. For a put with a strike at zero, this equation would describe the dynamics of a premium for a put on ruin of an insurer. Alternatively, this equation could describe the premium for a call on survival of an insurer. The limit of this equation is an ordinary Brownian motion with time dependent infinitesimal drift

$$rV - \frac{\partial V}{\partial t_0} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x_0^2} + \left(\mu + g'(t)\right)\frac{\partial V}{\partial x_0} \quad \text{where} \quad V = V(k,t;x_0,t_0)$$

If one prefers to model with a multiplicative diffusion, for rates of return for example, then the backward integro- differential equation should first be transformed to the multiplicative equivalent.

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LINEAR SUPERPOSITION OF A BROWNIAN MOTION TO A COMPOUND POISSON SUM OF GAUSSIANS

The linear superposition of a Brownian motion or regular diffusion $Y(t)$ to a random compound sum of normals is described by the following relations between random variables, given the initial or starting value s_0 ,

$$S(t) = s_0 + Y(t) + \sum_{j=1}^{N(t)} X_j$$

where $Y(t) \sim N(\mu t, \sigma^2 t)$ $N(t) \sim P(\lambda t)$ $X \sim N(a, b^2)$

which implies that

$$S(t) \sim s_0 + N(\mu t, \sigma^2 t) + \sum_{j=1}^{P(\lambda t)} N(a, b^2)$$

The main result of this section is that this process, the sum of a regular arithmetic Brownian motion diffusion and a compound Poisson process of independent homogenous normals, gives another regular diffusion in the central limit,

$$\lim_{t\to\infty} S(t) \to N\left(s_0 + (\mu + a\lambda)t, (\sigma^2 + a^2\lambda + b^2\lambda)t\right)$$

Conditional on the underlying process, the sum of each of the contributions to movements in state, has the density function

$$f_{S(t)|N(t)}(x \mid s_0) = \frac{1}{\sqrt{2\pi[\sigma^2 t + b^2 N(t)]}} \exp\left(-\frac{[x - s_0 - \mu t - aN(t)]^2}{2[\sigma^2 t + b^2 N(t)]}\right)$$

and for a given value of the count process, n, the density is

$$f_{S(t)|N(t)=n}(x \mid s_0, n) = \frac{1}{\sqrt{2\pi[\sigma^2 t + b^2 n]}} \exp\left(-\frac{[x - s_0 - \mu t - an]^2}{2[\sigma^2 t + b^2 n]}\right)$$

If n was an independent variable that was proportional to time t then this function would be identical, after reparameterization, to the transition density of Brownian motion. When n is held fixed, it is the density of the components of a basis. As it stands, n is the variable of a separate random variable, the forward variable of the Poisson process.

The unconditional transition density is

$$f_{S(t)}(x \mid s_0) = \sum_{n=0}^{\infty} p_n(t) f_{S(t)|N(t)=n}(x \mid s_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \frac{1}{\sqrt{2\pi[\sigma^2 t + b^2 n]}} \exp\left(-\frac{[x - s_0 - \mu t - an]^2}{2[\sigma^2 t + b^2 n]}\right)$$

The moment generating function of the supposition of a regular diffusion $Y(t)$ to a compound Poisson process of identically but unspecified distributions

$$\sum_{j=1}^{N(t)} X_j$$

is

$$M_{S(t)}(\theta) = M_{s_0 + Y(t) + \sum_{j=1}^{N(t)} X_j}(\theta) = e^{s_0\theta} M_{Y(t)}(\theta) M_{\sum_{j=1}^{N(t)} X_j}(\theta)$$

Since the component processes are independent so that

$$\begin{aligned} M_{S(t)}(\theta) &= \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2\right) \exp\left(\lambda t[M_X(\theta) - 1]\right) \\ &= \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 + \lambda t[M_X(\theta) - 1]\right) \end{aligned}$$

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and in the format consistent with the theory of Levy processes, without the complexity since all cumulants are real and introducing complexity is not parsimonious, the moment generating function of the transition distribution can be expressed as

$$M_{S(t)}(\theta) = \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 + \lambda t \int_{-\infty}^{\infty} (e^{\theta x} - 1) f_X(x) dt \right)$$

Since the secondary distributions are normal $X \sim N(a,b^2)$, the original expression becomes the representation

$$M_{S(t)}(\theta) = \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 + \lambda t (e^{a\theta + \frac{1}{2}b^2\theta^2} - 1) \right)$$

Of some interest is the fact that by expanding the exponential term in a power series and evaluating the integral, this function can be expressed in all of the existing moments of the component parts

$$\begin{aligned} M_{S(t)}(\theta) &= \exp\left(s_0\theta + \mu_1(Y) t\theta + \tfrac{1}{2}\sigma_2(Y) t\theta^2 + \lambda t[\theta\mu_1(X) + \tfrac{1}{2!}\theta^2\mu_2(X) + \tfrac{1}{3!}\theta^3\mu_3(X) + ...] \right) \\ &= \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 + \lambda t \sum_{r=1}^{\infty} \tfrac{1}{r!}\theta^r \mu_r(X) \right) \end{aligned}$$

Therefore, a standard forward equation from the expanded integro-differential equation has a diffusion side that is of infinite order, before convergence, which restricts the usefulness of the expanded form prior to reaching the central limit.

The Diffusion Equation

On the infinitesimal scale, the process for some general secondary distribution, or diffusion basis, is specified by the stochastic differential equation which is the same as that for a regular diffusion plus a stochastic term for the Poisson process accumulation of the time homogenous random variable X . This equation is

$$dS(t) = \mu dt + \sigma dB(t) + X dN(t)$$

where $dB(t) \sim N(0,dt)$ is standard Brownian motion and $dN(t) \sim P(\lambda dt)$ is a Poisson process.

Kolmogorov Forward Integro-Differential Convolution Equation

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{S(t)}(x) - \mu \frac{\partial}{\partial x} f_{S(t)}(x) - \lambda f_{S(t)}(x) + \lambda \int_{-\infty}^{\infty} f_X(y) f_{S(t)}(x-y) dy$$

for which the diffusion side is the sum of the diffusion side of the forward equation for a regular diffusion (the first two terms) and the diffusion side of a compound Poisson process diffusion (the second two terms). The time side, which is a single time derivative, establishes the dynamic time dependent process and the diffusion side gives it its evolution properties in the diffusion variable as the statistical properties develop through time.

Proof

Considering the moment generating function of the transition density function,

$$M_{S(t)}(\theta) = \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 + \lambda t[M_X(\theta) - 1] \right)$$

Differentiating partially with respect to time,

$$\begin{aligned} \frac{\partial}{\partial t} M_{S(t)}(\theta) &= \left(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2 + \lambda[M_X(\theta) - 1] \right) M_{S(t)}(\theta) \\ &= \mu\theta M_{S(t)}(\theta) + \tfrac{1}{2}\sigma^2\theta^2 M_{S(t)}(\theta) + \lambda M_X(\theta) M_{S(t)}(\theta) - \lambda M_{S(t)}(\theta) \end{aligned}$$

by the definition of moment generating functions and convolution integral,

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x) dx &= \mu\theta \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x) dx + \tfrac{1}{2}\sigma^2\theta^2 \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x) dx \\ &\quad - \lambda \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x) dx + \lambda \int_{-\infty}^{\infty} e^{\theta x} \int_{-\infty}^{\infty} f_X(x-y) f_{S(t)}(y) dy dx \end{aligned}$$

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By the converse of the result converting the forward equation with the transition density as the dependent variable to that with the moment generating function of as the dependent variable for

$$\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^n}{\partial x^n} f_{S(t)}(x)dx = (-\theta)^n M_{S(t)}(\theta)$$

differentiating with respect to the forward diffusion variable to the respective order, i.e., once and then twice for the first and second order effects

(1) $\theta M_{S(t)}(\theta) = -\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{S(t)}(x)dx$

(2) $\theta^2 M_{S(t)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{S(t)}(x)dx$

and the symmetrical property of the convolution integral, the equation can be expressed as

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} e^{\theta x} f_{S(t)}(x)dx = -\mu \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{S(t)}(x)dx + \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{S(t)}(x)dx$$
$$- \lambda \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x)dx + \lambda \int_{-\infty}^{\infty} e^{\theta x} \int_{-\infty}^{\infty} f_X(y) f_{S(t)}(x-y)dydx$$

provided that the moment generating function parameter is within the radius of convergence, in particular, that the density function of the diffusion and basis components are sub-exponential in the tails, which each are since the process is Gaussian. Inverting the moment generating function on both sides of the equation gives the forward equation in the required form

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{S(t)}(x) - \mu \frac{\partial}{\partial x} f_{S(t)}(x) - \lambda f_{S(t)}(x) + \lambda \int_{-\infty}^{\infty} f_X(y) f_{S(t)}(x-y)dy$$

The proof is complete.

Limiting Diffusion Under Conditions of Regularly Sized Parameter Values

The limiting form of this process becomes a Brownian motion provided that the rate intensities are not too small and the secondary distributions are not too large or extreme and the first and second moments are real and finite. That is, the Poisson compound sum of normal secondary distributions converges to a Wiener process and this process added to another Wiener process produces another Wiener process. Note that for very rare events whereby the Poisson process has a small rate intensity and the basis components being summed have large means and variances, the approach to convergence will be very slow and may well result in a non-convergence on the relevant time scale whereby the higher order effects remain material. If

$$S(t) = s_0 + Y(t) + \sum_{j=1}^{N(t)} X$$

where $Y(t) \sim N(\mu t, \sigma^2 t)$, $N(t) \sim P(\lambda t)$, and $X \sim N(a,b)$

or

$$S(t) \sim s_0 + N(\mu t, \sigma^2 t) + \sum_{j=1}^{N(t)} N(a, b^2)$$

then

$$\lim_{t\to\infty} S(t) \sim N\left(x_0 + (\mu + a\lambda)t, (\sigma^2 + \lambda a^2 + \lambda b^2)\lambda t\right)$$

that is,

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$$\begin{aligned}\lim_{t\to\infty} S(t) &\sim \lim_{t\to\infty} N\left(s_0+\mu t+a\,P(\lambda t),\ \sigma^2 t+b^2 P(\lambda t)\right)\\ &\to N\left(x_0+\mu t+a\,N(\lambda t,\lambda t),\ \sigma^2 t+b^2 N(\lambda t,\lambda t)\right)\\ &\sim N\left(x_0+\mu t+N(a\lambda t,a^2\lambda t),\ \sigma^2 t+N(b^2\lambda t,b^4\lambda t)\right)\\ &\to N\left(x_0+(\mu+a\lambda)t,\ (\sigma^2+\lambda a^2+\lambda b^2)\lambda t\right)\end{aligned}$$

The unconditional transition density of the limiting stationary process is

$$f_{S(t)}(x\mid s_0)=\frac{1}{\sqrt{2\pi(\sigma^2+\lambda a^2+\lambda b^2)t}}\exp\left(-\frac{[x-s_0-\mu t-a\lambda t]^2}{2(\sigma^2+\lambda a^2+\lambda b^2)t}\right)$$

Proof: that this limiting distribution is correct and that the limits of the third and fourth standardized measures of shape are necessary and sufficient for this convergence. From the previous section of linear supposition,

$$S(t)\sim N\left(s_0+\mu t+a\,P(\lambda t),\ \sigma^2 t+b^2 P(\lambda t)\right)$$

the conditional moment generating function of the process given Poisson process is,

$$\begin{aligned}M_{S(t)|N(t)}(\theta)&=\exp\left((s_0+\mu t+aN(t))\theta+\tfrac{1}{2}(\sigma^2 t+b^2N(t))\theta^2\right)\\ &=\exp\left(s_0\theta+\mu t\theta+aN(t)\theta+\tfrac{1}{2}\sigma^2 t\theta^2+\tfrac{1}{2}b^2N(t)\theta^2\right)\\ &=\exp\left(s_0\theta+\mu t\theta+\tfrac{1}{2}\sigma^2 t\theta^2\right)\exp\left((a\theta+\tfrac{1}{2}b^2\theta^2)N(t)\right)\end{aligned}$$

by nested expectations, the unconditional moment generating function of the process is

$$\begin{aligned}M_{S(t)}(\theta)&=E\left(M_{S(t)|N(t)}(\theta)\right)\\ &=\exp\left(s_0\theta+\mu t\theta+\tfrac{1}{2}\sigma^2 t\theta^2\right)M_{N(t)}\left(a\theta+\tfrac{1}{2}b^2\theta^2\right)\end{aligned}$$

which is an expectation with regard to $N(t)$. By the Poisson process limit theorem, the primary distribution becomes a normal diffusion $N(t)\sim P(\lambda t)\to N(\lambda t,\ \lambda t)$ for which the moment generating function is

$$M_{N(t)}(\theta)=\exp\left(\lambda t\theta+\tfrac{1}{2}\lambda t\theta^2\right)$$

This gives, for the moment generating function of the compound process

$$\begin{aligned}M_{S(t)}(\theta)&=\exp\left(s_0\theta+\mu t\theta+\tfrac{1}{2}\sigma^2 t\theta^2\right)\left(\lambda t(a\theta+\tfrac{1}{2}b^2\theta^2)+\tfrac{1}{2}\lambda t(a\theta+\tfrac{1}{2}b^2\theta^2)^2\right)\\ &=\exp\left(s_0\theta+\mu t\theta+\tfrac{1}{2}\sigma^2 t\theta^2\right)\left(\lambda t(a\theta+\tfrac{1}{2}b^2\theta^2)+\tfrac{1}{2}\lambda t(a^2\theta^2+ab^2\theta^3+\tfrac{1}{4}b^4\theta^4)\right)\\ &=\exp\left(s_0\theta+\mu t\theta+\tfrac{1}{2}\sigma^2 t\theta^2\right)\left(\lambda at\theta+\tfrac{1}{2}\lambda(a^2+b^2)t\theta^2+\tfrac{1}{2}\lambda ab^2t\theta^3+\tfrac{1}{8}\lambda b^4t\theta^4)\right)\\ &=\exp\left(s_0\theta+(\mu+\lambda a)t\theta+\tfrac{1}{2}(\sigma^2+\lambda a^2+\lambda b^2)t\theta^2+\tfrac{1}{2}\lambda ab^2t\theta^3+\tfrac{1}{8}\lambda b^4t\theta^4)\right)\end{aligned}$$

From the moment generating function of $S(t)$, the mean and variance of $S(t)$ are

$$E[S(t)]=s_0\theta+(\mu+a\lambda)t \qquad \text{and} \qquad Var[S(t)]=(\lambda a^2+\lambda b^2+\sigma^2)t$$

The moment generating function of the standardized variable of $S(t)$,

$$\{S(t)-E[S(t)]\}/\sqrt{Var[S(t)]}=[S(t)-s_0-(\mu+a\lambda)t]/\sqrt{(\lambda a^2+\lambda b^2+\sigma^2)t}$$

is

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$$M_{(S(t)-\mu_{S(t)})/\sigma_{S(t)}}(\theta)$$
$$= \exp\left(\frac{s_0 + (\mu + a\lambda)t - s_0 - (\mu + a\lambda)t}{[(\lambda a^2 + \lambda b^2 + \sigma^2)t]^{1/2}} \theta + \tfrac{1}{2} \frac{(\lambda a^2 + \lambda b^2 + \sigma^2)t}{(\lambda a^2 + \lambda b^2 + \sigma^2)t} \theta^2 + \tfrac{1}{2} \frac{\lambda a b^2 t}{[(\lambda a^2 + \lambda b^2 + \sigma^2)t]^{3/2}} \theta^3 + \tfrac{1}{8} \frac{\lambda b^4 t}{[(\lambda a^2 + \lambda b^2 + \sigma^2)t]^2} \theta^4 \right)$$
$$= \exp\left(\tfrac{1}{2}\theta^2 + \tfrac{1}{2} \frac{\lambda a b^2}{(\lambda a^2 + \lambda b^2 + \sigma^2)^{3/2}} \frac{1}{\sqrt{t}} \theta^3 + \tfrac{1}{8} \frac{\lambda b^4}{(\lambda a^2 + \lambda b^2 + \sigma^2)^2} \frac{1}{\lambda t} \theta^4 \right)$$

so that he moment generating function of the limiting distribution

$$\lim_{t\to\infty} M_{[S(t)-\mu_{S(t)}]/\sigma_{S(t)}}(\theta) = \exp\left(\tfrac{1}{2}\theta^2\right)$$

is that of a unit standard normal,

$$[S(t) - \mu_{S(t)}]/\sigma_{S(t)} \sim N(0,1)$$

and the process has converged to a mesokurtic normal bell. Reintroducing the location and scale gives

$$\lim_{t\to\infty} M_{S(t)}(\theta) = \exp\left(s_0\theta + (\mu + \lambda a)t\theta + \tfrac{1}{2}(\sigma^2 + \lambda a^2 + \lambda b^2)t\theta^2 \right)$$

which is the moment generating function for a normal variable with parameters

$$(\mu + a\lambda)t \qquad \text{and} \qquad (\lambda a^2 + \lambda b^2 + \sigma^2)t$$

Hence,

$$\lim_{t\to\infty} S(t) \sim N\left(s_0 + (\mu + \lambda a)t,\ (\sigma^2 + \lambda a^2 + \lambda b^2)t \right)$$

The proof is complete.

FURTHER LINEAR SUPERPOSITION

Remembering that compound Poisson process sums of independent and homogenous distributions have independent increments and the fact that a Wiener process has independent increments, it follows that

a linear supposition of linear combinations of Brownian motions(which is itself another Brownian motion) and linear combinations of compound Poisson processes of sums of independent and identically distributed variables within each compound Poisson process has independent increments[57].

This is directly due to the fact that adding Markov processes to other Markov processes produces a Markov process due to the linearity property of derivatives, in this case, time derivatives. The moment generating function of such a process, upon reparameterization, is that given by the moment generating function of the distribution stated in the previous section

$$M_{S(t)}(\theta) = \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta + t\sum\nolimits_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} (e^{\theta x} - 1) f_{X_k}(x) dx \right)$$

Where n is the number of compound Poisson processes in the supposition, c_k is the weighting given to the k^{th} Poisson process in the supposition which sums independent and homogeneous variables with distribution X_k at rate intensity λ_k. The parameters of the Wiener component, or Brownian motion, are linear combinations of the first two cumulants of the individual Wiener processes, since cumulants are summable and scalable over the operation of convolution and weighted convolution. The moment generating function can be expressed more compactly, but less informatively, as a Levy representation in real form

$$M_{S(t)}(\theta) = \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 + \lambda t \int_{-\infty}^{\infty} (e^{\theta x} - 1) dF(x) \right)$$

[57] and consequently also a (first order) Markov process.

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Kolmogorov Forward Integro-Differential Convolution Equation: a Brownian Motion Superimposed on a Compound Poisson Sum

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{S(t)}(x) - \mu \frac{\partial}{\partial t} f_{S(t)}(x) - \left(\sum_{k=1}^{n} c_k \lambda_k \right) f_{S(t)}(x) + \sum_{k=1}^{n} \left(c_k \lambda_k \int_{-\infty}^{\infty} f_{X_k}(y) f_{S(t)}(x-y) dy \right)$$

The derivation is similar to that with only a single compound Poisson sum,

Proof

$$\begin{aligned} M_{S(t)}(\theta) &= E\left(E[M_{S(t)|N(t)}(\theta) \mid N(t)] \right) \\ &= \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 \right) M_{N(t)}\left(\log M_X(\theta) \right) \\ &= \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 \right) \exp\left(\lambda t [M_X(\theta) - 1] \right) \\ &= \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta + \lambda t \int_{-\infty}^{\infty} (e^{\theta x} - 1) dF(x) \right) \\ &= \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta + t \sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} (e^{\theta x} - 1) f_{X_k}(x) dx \right) \end{aligned}$$

Differentiating with respect to time by the product rule

$$\begin{aligned} \frac{\partial}{\partial t} M_{S(t)}(\theta) &= (\mu\theta + \tfrac{1}{2}\sigma^2\theta^2) \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 \right) \exp\left(t \sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} (e^{\theta x} - 1) f_{X_k}(x) dx \right) \\ &\quad + \left(\sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} (e^{\theta x} - 1) f_{X_k}(x) dx \right) \exp\left(s_0\theta + \mu t\theta + \tfrac{1}{2}\sigma^2 t\theta^2 \right) \exp\left(t \sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} (e^{\theta x} - 1) f_{X_k}(x) dx \right) \\ &= (\mu\theta + \tfrac{1}{2}\sigma^2\theta^2) M_{S(t)}(\theta) + \sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} (e^{\theta x} - 1) f_{X_k}(x) dx M_{S(t)}(\theta) \end{aligned}$$

or

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x) dx &= (\mu\theta + \tfrac{1}{2}\sigma^2\theta^2) \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x) dx \\ &\quad + \sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} e^{\theta x} \int_{-\infty}^{\infty} f_X(y) f_{S(t)}(x-y) dy dx - \sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x) dx \end{aligned}$$

From the converse of the results converting the forward equation with the transition density as the dependent variable to that with the moment generating function of as the dependent variable, i.e., differentiating for

(1) $\theta M_{S(t)}(\theta) = -\int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{S(t)}(x) dx$

(2) $\theta^2 M_{S(t)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{S(t)}(x) dx$

the equation becomes

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial t} f_{S(t)}(x) dx &= \tfrac{1}{2}\sigma^2 \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^2}{\partial x^2} f_{S(t)}(x) dx - \mu \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial}{\partial x} f_{S(t)}(x) dx \\ &\quad + \sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} e^{\theta x} \int_{-\infty}^{\infty} f_X(y) f_{S(t)}(x-y) dy dx - \sum_{k=1}^{n} c_k \lambda_k \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x) dx \end{aligned}$$

inverting the moment generating function on both sides gives the required forward integro-differential equation,

$$\frac{\partial}{\partial t} f_{S(t)}(x) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{S(t)}(x) - \mu \frac{\partial}{\partial t} f_{S(t)}(x) - \left(\sum_{k=1}^{n} c_k \lambda_k \right) f_{S(t)}(x) + \sum_{k=1}^{n} \left(c_k \lambda_k \int_{-\infty}^{\infty} f_{X_k}(y) f_{S(t)}(x-y) dy \right)$$

The proof is complete.

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Practical Aspects of Linear Suppositions

A linear supposition of a Brownian motion and a high frequency compound Poisson process results in a process with five distributional parameters, three-parameters from the compound Poisson sum of normals and two-parameters from the Brownian motion. Four of these parameters relate to the mean and variance of two separate diffusion bases, for which only one sequence is observed and this observed sequence is the result of the convolution of a function of the two aggregated bases. This makes it impossible to write out a likelihood equation for the two parts since one cannot obtain the deconvolved components to insert into the separate density functions. It is therefore not possible to estimate the supposition of a Brownian motion and a compound Poisson process from a single observed sequence by the techniques of ordinary maximum likelihood. The reason for this is that the process is over-parameterized with regard to the observed sequence. In effect, there is model redundancy – a redundancy between the random variables generating the process of the model and those generating the sequence that is observed. A numerical maximization may work but one would have to be very cautious of multiple maxima (a non-unique maximum), and any one of the local maxima may be more logically correct than the absolute maximum. Such a situation is equivalent to having free parameters and a possible infinite number of solutions to the optimization. With regard to moment estimators, there are logical inconsistencies with the solutions to the moment matching equations. This has been hinted at by some of the early finance literature[58] which focus on equally spaced observations of the linear supposition rather than transaction data.

By ordinary techniques, it would only be possible to estimate the two processes in the supposition by observing the two separate sequences before they are combined, or convolved, into the single series. If only the combined sequence was observed and information was available on one of the processes, either the Brownian motion or the compound Poisson diffusion, then one could proceed with the conventional techniques since there would then be a single parameter for each distributional aspect (the mean and variance of the Gaussian basis, and the time scaling parameter for the Poisson process). Alternatively, the known process with known parameters, could be deconvolved from the observed process leaving an observed sequence for the unknown process from which the unknown parameters could then be estimated.

Techniques of the literature, such as 'Disentangling Diffusion from Jumps[59]', to separate out the contributions of Brownian motion from Poisson compound sums regard Poisson events with small rate intensity of contributions with large variation and the propensity for large outcomes. Since the compound Poisson processes considered in this thesis are high frequency (large λ) compounds of locally scaled normals (small σ^2/μ), they are also diffusions like regular Brownian motion and very similar in stochastic behavior. The basis components of the two processes are, then, of comparable order of magnitude, namely infinitesimals and near-infinitesimals, so that the techniques and arguments to separate out and obtain estimates of the measure of the Poisson counts and normal additions are not directly applicable. As it stands, it is not possible to obtain the information regarding the two distinct, but like, processes from a single observed sequence.

The moral of this story is that a nice formula or process is like a flashy performance car – whatever its performance or prestige, it isn't any use unless you have fuel (or reasonable estimates of the parameters).

A Time Contiguous Process for Microstructure Transaction Data

From the previous discussion regarding microstructure transaction data, it may be beneficial to estimate a Poisson compound sum of normals for intra-day transaction data in conjunction with an estimation of a separate ordinary Brownian motion from the daily summary statistics[60] for the 'out-of-trading-hours' periods, and then combine the results by appending the consecutive process into a single time contiguous process. From this process, pricing can proceed with the limiting process, which is an appendum of consecutive Brownian motions (or log-diffusion if one is modeling a multiplicative process[61]). From the limiting distribution, the put and call pre-

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[58] Such literature includes
(i) Press, J. S., (1967) A Compound Events Model for Security Prices. *The Journal of Business*, Vol. 40, No. 3 (Jul., 1967), pp. 317-335.
(ii) Beckers, S., (1981) A Note on Estimating the Parameters of the Diffusion-Jump Model of Stock Returns. *Journal of Quantitative Finance*, Vol. 16, No. 1 (Mar., 1981), pp. 127-140.

[59] Ait-Sahalia, Y., (2004) Disentangling Diffusion from Jumps. *Journal of Financial Economics*, Vol. 74, No. 3 (Dec., 2004), pp. 487-528.

[60] last trade and first trade of trading days.

[61] Both processes must be either additive or multiplicative (or square root). It is not possible to model one with an additive process, say, and the other with a multiplicative process and the resulting appendum to remain analytic.

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miums are then, as previously, the limited first moments of a normal or log-normal variable. On logs for example, there would be an ordinary log-diffusion for the out-of-hours price dynamics and a compound Poisson log-diffusion for the intra-day price dynamics, with each then consecutively appended to one another.

Under this specification, it is possible to write down the likelihood components of each process for which each of the distributional parameters have analytic parameter estimators and for which the maximum likelihood estimators also correspond to the method of moment estimators, the mean and variance of the out-of-trading-hours process and the mean, variance, and Poisson rate intensity for the intra-day process. The out-of-trading-hours process is estimated from the differences of the daily close-open statistics or last trade before close and first trade after open, which correspond to overnight and weekend differences that are free of micro-structure effects. The within-trading-hours process is estimated from the differences of the observed transaction prices and differences of the observed transaction times.

Again, if it is not felt that there has been a sufficient amount of time for convergence to be reached in the compound Poisson component and one wants the exact price, then there are two ways to proceed. The option premium limited moment can be obtained by summing out the single Poisson parameter in the weighted conditional option premium limited moments for the intra-day process, with a do-while loop, over the values where the Poisson probabilities are material in relation to the conditional option premium. Alternatively, the moment generating function of the compound sum can be inverted with an inverse Fourier transform applied to the moment generating function with a complex parameter, from which the option premia can be obtained by the usual limited first moment integration.

The amount of time required for an adequate convergence, so that the simplified formulae for the intra-day components are accurate and reliable, again depends on the relation between the mean rate of Poisson occurrences and the magnitude of the moments of the secondary Gaussians; i.e., the size and relation between the parameters of the primary and secondary distribution, λ to μ and σ^2. In particular, the larger the Poisson rate intensity λ and the larger the signal to noise ratio of the diffusion basis components, μ / σ^2, the quicker the convergence occurs. For a relatively liquid traded security, the convergence could reasonably assumed to be reached within a relatively short time. Since these are being applied to successive entire trading days and evenings and weekend periods, it is only required that the compound Poisson sum reaches a Brownian motion by the end of each trading day. If convergence does not occur by the end of trade in the afternoon, then there may be a more appropriate model for the security dynamics or the process has to be modified to capture some additional feature of the dynamics.

THE MULTIPLICATIVE COMPOUND POISSON PROCESS

Recurrence Dynamics, the Diffusion Equation, and its Resolution

The **forward recurrence** of a multiplicative first order compound Poisson diffusion is

$$S(t+dt) = S(t)^{dN(t)} Y_t^{\,hdN(t)} S(t)^{1-dN(t)} e^{o(dt)}$$

Expressing this in terms of the scaled Poisson basis function $z_t\,dt$

$$S(t+dt) = S(t)^{z_t dt} Y_t^{\,hz_t dt} S(t)^{1-z_t dt} e^{o(dt)}$$

and taking logarithms of both sides

$$\log S(t+dt) = [\log S(t) + \log Y_t\ h] z_t dt + \log S(t)[1 - z_t dt] + o(dt)$$

Canceling $\log S(t)\ dN(t)$, subtracting $\log S(t)$ from both sides

$$\log S(t+dt) - \log S(t) = (\log Y_t\ ) h\, z_t dt + o(dt)$$

and dividing through by dt

$$\frac{\log S(t+dt) - \log S(t)}{dt} = (\log Y_t\ ) h\, z_t + \frac{o(dt)}{dt}$$

Letting $dt \to 0$ so that $o(dt) / dt \to 0$ results in the **differential form of the diffusion equation** for a multiplicative first order compound Poisson diffusion

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$$\frac{d}{dt}\log S(t) = (\log Y_t\ )h\,z_t$$

expressed alternatively as

$$\frac{d}{h\,z_t dt}\log S(t) = \log Y_t \qquad\qquad \frac{d}{h\,dN(t)}\log S(t) = \log Y_t$$

Conversion from $d\log S(t) = (\log Y_t h)dN(t)$ to the variable of the original scale, rather than logarithms, for any non-infinitesimal h, which is $dS(t) = S(t)\,(e^{\log Y_t h} - 1)\,dN(t)$, requires the total differential

$$\frac{dg(X)}{dt} = \frac{\partial g(X)}{\partial X}\frac{dX}{dt} + \tfrac{1}{2}\frac{\partial^2 g(X)}{\partial X^2}\frac{(dX)^2}{dt} + \tfrac{1}{3!}\frac{\partial^3 g(X)}{\partial X^3}\frac{(dX)^3}{dt} + \ldots$$

Applying the exponential transform to the logarithm of the diffusion variable

$$g[\log S(t)] = \exp[\log S(t)]$$

which has identical partial derivatives of all orders

$$\frac{\partial^n g[\log S(t)]}{\partial \log S(t)^n} = \exp[\log S(t)] = S(t)$$

This gives an expansion of the total differential of

$$\frac{dS(t)}{dt} = S(t)\frac{[d\log S(t)]}{dt} + \tfrac{1}{2}S(t)\frac{[d\log S(t)]^2}{dt} + \tfrac{1}{3!}S(t)\frac{[d\log S(t)]^3}{dt} + \tfrac{1}{4!}S(t)\frac{[d\log S(t)]^4}{dt} + \ldots$$

$$\frac{dS(t)}{dt} = S(t)\frac{[\log Y_t h\,z_t dt]}{dt} + \tfrac{1}{2}S(t)\frac{[\log Y_t h\,z_t dt]^2}{dt} + \tfrac{1}{3!}S(t)\frac{[\log Y_t h\,z_t dt]^3}{dt} + \tfrac{1}{4!}S(t)\frac{[\log Y_t h\,z_t dt]^4}{dt} + \ldots$$

$$\frac{dS(t)}{dt} = S(t)\left((\log Y_t h)\frac{dN(t)}{dt} + \tfrac{1}{2}(\log Y_t h)^2\frac{dN(t)^2}{dt} + \tfrac{1}{3!}(\log Y_t h)^3\frac{dN(t)^3}{dt} + \tfrac{1}{4!}(\log Y_t h)^4\frac{dN(t)^4}{dt} + \ldots\right)$$

or

$$dS(t) = S(t)\left((\log Y_t h)dN(t) + \tfrac{1}{2}(\log Y_t h)^2 dN(t)^2 + \tfrac{1}{3!}(\log Y_t h)^3 dN(t)^3 + \tfrac{1}{4!}(\log Y_t h)^4 dN(t)^4 + \ldots\ \right)$$

From the assertion that $(dN_t)^n = dN_t + o(dt)$ and consequently that $(dN_t)^n \to dN_t$ as dt becomes negligible

$$\begin{aligned} dS(t) &= S(t)\left((\log Y_t h)dN(t) + \tfrac{1}{2}(\log Y_t h)^2 dN(t) + \tfrac{1}{3!}(\log Y_t h)^3 dN(t) + \tfrac{1}{4!}(\log Y_t h)^4 dN(t) + \ldots\ \right) + o(dt) \\ &= S(t)\left((\log Y_t h) + \tfrac{1}{2}(\log Y_t h)^2 + \tfrac{1}{3!}(\log Y_t h)^3 + \tfrac{1}{4!}(\log Y_t h)^4 + \ldots\ \right)dN(t) + o(dt) \\ &= S(t)\left(e^{\log Y_t\ h} - 1\right)dN(t) + o(dt) \end{aligned}$$

Dividing by dt

$$\frac{dS(t)}{dt} = S(t)\left(e^{\log Y_t\ h} - 1\right)\frac{dN(t)}{dt} + \frac{o(dt)}{dt}$$

and this formula, as dt becomes negligible so that $o(dt)\,/\,dt \to 0$, results in the differential form of the diffusion equation for a multiplicative compound Poisson diffusion

$$\frac{dS(t)}{dN(t)} = S(t)\left(e^{\log Y_t\ h} - 1\right) \qquad\qquad \frac{d}{dt}S(t) = S(t)\left(e^{\log Y_t\ h} - 1\right)z_t$$

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Limiting Form of the Differential Form of Diffusion Equation as h Becomes Infinitesimal, $h \to 0$

Putting the differential form of the diffusion equation into expanded form

$$\frac{dS(t)}{dN(t)} = S(t)\left((\log Y_t h) + \tfrac{1}{2}(\log Y_t h)^2 + \tfrac{1}{3!}(\log Y_t h)^3 + \tfrac{1}{4!}(\log Y_t h)^4 + \ldots \right)$$

and dividing both sides by h

$$\frac{dS(t)}{h\,dN(t)} = S(t)\left(\frac{(\log Y_t h) + \tfrac{1}{2}(\log Y_t h)^2 + \tfrac{1}{3!}(\log Y_t h)^3 + \tfrac{1}{4!}(\log Y_t h)^4 + \ldots}{h} \right)$$

If $\log Y_t \sim N(\mu, \sigma^2 / h)$ then $(\log Y_t h)^2 = \sigma^2 h + o(h)$ and $(\log Y_t h)^r = o(h)$ for $r > 2$, hence

$$\frac{dS(t)}{h\,dN(t)} = S(t)\left(\log Y_t + \tfrac{1}{2}\sigma^2 + \frac{o(h)}{h} \right)$$

Letting h become infinitesimal so that $o(dt) / dt \to 0$ and $\lim_{h\to 0} h\,dN(t) \sim \lim_{h\to 0} h\,P(dt/h) \to dt$

$$\frac{dS(t)}{dt} = S(t)\left(\log Y_t + \tfrac{1}{2}\sigma^2 \right)$$

which is the diffusion equation of geometric Brownian motion. Noting that $(\log Y_t)dt \sim N(\mu dt, \sigma^2 dt)$ so that $\log Y_t\,dt = \mu dt + \sigma dB_t$ where $dB_t \sim N(0, dt)$ is differential standard Brownian motion, results in the more familiar notational form

$$dS(t) = \left(\mu + \tfrac{1}{2}\sigma^2 \right) S(t)dt + \sigma S(t) dB_t$$

Resolving the Diffusion Equation

Integrating the differential form of the logarithm of the compound Poisson diffusion variable, over the transition interval from time 0 to time t

$$\log S(t) - \log S(0) = \int_0^t (\log Y_y) h\, z_y dy$$

and moving the logarithm of the backward diffusion variable to the basis side

$$\log S(t) = \log S(0) + \int_0^t (\log Y_y) h\, z_y dy$$

Exponentiating results in one form of the integral form of the diffusion equation for the multiplicative compound Poisson diffusion

$$S(t) = S(0) \exp\left(\int_0^t (\log Y_y) h\, z_y dy \right)$$

Converting to the integral to a discrete sum from the indicators of the Poisson basis function $z_y\,dy$ juxtaposed within the integrand, as per the ordinary additive process; if the times at which the Poisson events of the generator basis z_y occur are t_j then the scaled Poisson basis is one at the times t_j and zero everywhere else

$$z_y dy = \begin{cases} 1 & \text{if } y = t_j \\ 0 & \text{if } y \neq t_j \end{cases} \qquad \text{for } j = 1,\ 2,\ ,3,\ \ldots,\ N(t) \text{ and } 0 \le y \le t$$

This means that

$$(\log Y_y) z_y dy = \begin{cases} 1 \times (\log Y_y) & \text{if } y = t_j \\ 0 \times (\log Y_y) & \text{if } y \neq t_j \end{cases} = \begin{cases} (\log Y_y) & \text{if } y = t_j \\ 0 & \text{if } y \neq t_j \end{cases} \quad \text{for } j = 1,\ 2,\ ,3,\ \ldots,\ N(t) \text{ and } 0 \le y \le t$$

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and that there is a total of $N(t)$ times t_j at which the Poisson generator produces ones and results in multiplicative convolutions of diffusion basis variables Y onto the diffusion variable, i.e., where Y is multiplicatively convolved onto the diffusion variable S. The resolved diffusion equation can be written as an exponential of a discrete sum in log Y

$$S(t) = S(0)\exp\left(\sum_{j=1}^{N(t)} (\log Y_{t_j})h\right)$$

This, together with the fact that the Y are time homogenous so that t_j is simply an ordering variable for Y, means that $\log Y_{t_j} = \log Y_{jh}$ and the resolved diffusion equation in discrete sum form can alternatively be written with a linear dependence for the diffusion basis components

$$S(t) = S(0)\exp\left(\sum_{j=1}^{N(t)} (\log Y_{jh})h\right)$$

in sum form or

$$S(t) = S(0)\prod_{j=1}^{N(t)} Y_{jh}{}^{h}$$

in product form. The unconditional resolved diffusion equation is that with the Poisson generator summed out,

$$S(t) = S(0)\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\exp\left(\sum_{j=1}^{N(t)} (\log Y_{jh})h\right)$$

or

$$S(t) = S(0)\sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\prod_{j=1}^{n} Y_{jh}{}^{h}$$

Limiting Form of Resolved Diffusion Variable as h Becomes Infinitesimal, h → 0

Again utilizing the limit of the scaled Poisson process, which is linear time

$$\lim_{h\to 0} hN(t) \sim \lim_{h\to 0} hP(t/h) \to t$$

Since $j = 1, 2, 3, \ldots, N(t)$ in the discrete sum in the exponential index of the resolved diffusion equation

$$\sum_{j=1}^{N(t)} (\log Y_{jh})h$$

the time indexing argument of the basis function X is $jh = h, 2h, 3h, \ldots, h\,N(t)$. Letting the Poisson parameter be equal to the inverse of $\lambda = 1/h$, and taking limits $h \to dt \to 0$ so that $jh \to y$ and becomes a time continuum from 0 to $h\,N(t)$, and noting that $h\,N(t/h) \sim h\,P(t/h) \to t$ as $h \to 0$

$$S(t) = S(0)\exp\left(\int_0^t (\log Y_y)dy\right)$$

which is the resolved diffusion variable of first order unit root log-diffusion, geometric Brownian motion. If

$$\log Y_t = \mu + \sigma\frac{dB_t}{dt}$$

is a Brownian motion and hence an ordinary differential Gaussian basis

$$\log Y_t \sim N(\mu, \sigma^2/dt)$$

then the resolved diffusion equation can be written in terms of standard Brownian motion

$$S(t) = S(0)\exp\left(\mu t + \sigma dB_t\right)$$

which is the resolved diffusion variable of geometric Brownian motion.

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Transition Density of a Multiplicative Compound Poisson Diffusion

From the resolved diffusion equation, it is seen that $N(t)$ counts the number of random multiplications, rather than additions, of independent copies of time invariant distribution Y onto the random compound product. The transition variable of the multiplicative compound Poisson process, expressed as the resolved diffusion equation, is

$$S(t) = S(0)\, Y_1 Y_2 .. Y_{N(t)} = \prod_{j=0}^{N(t)} Y_j$$

where $S(0) = x_0$ is the backward diffusion variable. The diffusion transition density and distribution function of the multiplicative unit root compound Poisson process are $N(t)$ fold multiplicative convolutions of the secondary density and distribution functions

$$f_{S(t)|S(0)} = f_{x_0 X_1 X_2 ... X_{N(t)}} \qquad \text{and} \qquad F_{S(t)|S(0)} = F_{x_0 X_1 X_2 ... X_{N(t)}}$$

Taking logarithms of $S(t)$ results in an additive compound Poisson process as per taking logs of geometric Brownian motion

$$\log S(t) = \log S(0) + \log Y_1 + \log Y_2 + \ldots + \log Y_{N(t)} = \log S(0) + \sum_{j=1}^{N(t)} \log Y_j$$

Modeling with a multiplicative diffusion in financial applications is usually to model compound returns.

Unconditionally Transition Density of the Compound Poisson Product

Summing out the Poisson variable gives the unconditional transition density

$$\begin{aligned} f_{S(t)|S(0)}(x \mid x_0) &= \sum_{n=0}^{\infty} p_n(t)\, f_{\prod_{j=0}^{n} Y_n}(x \mid x_0) \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!}\, f_{\prod_{j=0}^{n} Y_n}(x \mid x_0) \end{aligned}$$

and transition distribution function

$$\begin{aligned} F_{S(t)|S(0)}(x \mid x_0) &= \sum_{n=0}^{\infty} p_n(t) F_{\prod_{j=0}^{n} Y_n}(x \mid x_0) \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F_{\prod_{j=0}^{n} Y_n}(x \mid x_0) \end{aligned}$$

where $f_{X^{*0}}(x) = f_{S(0)|S(0)}(x \mid x_0) = \delta_{x_0}(x)$ is a Dirac delta function. When the scaled basis components are log-normal, with log-mean μh and log-variance $\sigma^2 h$

$$Y_j \sim \log N(\mu h, \sigma^2 h)$$

The distribution of the transition interval conditional on the Poisson process is log-normal

$$\prod_{j=0}^{N(t)} Y_j \sim \log N\left(x_0 + \mu N(t),\ \ \sigma^2 N(t) \right)$$

so that the transition distribution has density

$$f_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \frac{1}{x\sigma\sqrt{n}\sqrt{2\pi}} \exp\left(-\frac{[\log x - x_0 - \mu n]^2}{2\sigma^2 n} \right)$$

and transition distribution function

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$$F_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \int_{-\infty}^{x} \frac{1}{y\sigma\sqrt{hn}\sqrt{2\pi}} \exp\left(-\frac{[\log y - x_0 - \mu h n]^2}{2\sigma^2 h n} \right) dy$$
$$= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \Phi\left(\frac{\log y - x_0 - \mu h n}{\sigma\sqrt{hn}} \right)$$

where $\Phi(x)$ is the cumulative distribution function of a standard normal $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy$.

Transition MGF

From the definition of a moment generating function

$$M_{S(t)}(\theta) = E(e^{\theta S(t)})$$
$$= E\left(1 + \theta S(t) + \tfrac{1}{2}\theta^2 S(t)^2 + \tfrac{1}{3!}\theta^3 S(t)^3 + \ldots\right)$$
$$= E\left(1 + \theta e^{\log S(t)} + \tfrac{1}{2}\theta^2 e^{2\log S(t)} + \tfrac{1}{3!}\theta^3 e^{3\log S(t)} + \ldots\right)$$
$$= 1 + \theta M_{\log S(t)}(1) + \tfrac{1}{2}\theta^2 M_{\log S(t)}(2) + \tfrac{1}{3!}\theta^3 M_{\log S(t)}(3) + \ldots$$

which is the general formula for the moment generating function of an exponentiated process $S(t)$ in terms of the underlying additive process $\log S(t)$,

$$M_{S(t)}(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j!} M_{\log S(t)}(j)$$

Hence, with logarithms of the multiplicative basis components result in additive basis components $\log Y = X$

$$M_{S(t)}(\theta) = 1 + \theta e^{S_0} M_{N(t)}\left(\log M_{\log Y}(1)\right) + \tfrac{1}{2}(\theta e^{x_0})^2 M_{N(t)}\left(\log M_{\log Y}(2)\right) + \tfrac{1}{3!}(\theta e^{x_0})^3 M_{N(t)}\left(\log M_{\log Y}(3)\right) + \ldots$$
$$= \sum_{j=0}^{\infty} \tfrac{1}{j!}(\theta e^{S_0})^j M_{N(t)}\left(\log M_{\log Y}(j)\right)$$

For example, if Y are log-normal $Y \sim \log N(\mu h, \sigma^2 h)$ so that $X = \log Y \sim N(\mu h, \sigma^2 h)$ are normal with mgf

$$M_{\log Y}(\theta) = \exp(\mu h\theta + \tfrac{1}{2}\sigma^2 h\theta^2)$$

then the mgf of the multiplicative diffusion variable is

$$M_{S(t)}(\theta) = \sum_{j=0}^{\infty} \tfrac{1}{j!}(\theta e^{S_0})^j M_{N(t)}\left(\mu h j + \tfrac{1}{2}\sigma^2 h j^2\right)$$

and for a Poisson generator

$$M_{S(t)}(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j!} \exp\left(j \log S_0 + \lambda t (e^{\mu h j + \frac{1}{2}\sigma^2 h j^2} - 1) \right)$$

The moment generating function is used primarily to derive and deduce results and, therefore, the fact that it is an infinite series is not material.

Mathematical Preliminary: Multiplication and Division of Random Variables and the Multiplicative Convolution and Deconvolution of Density Functions

When the diffusion basis components sum multiplicatively rather than additively, such as for geometric Brownian motion, the convolution integrals that convolve the successive basis components onto the diffusion variable alter so that each basis component is convolved multiplicatively into expanding products rather than sums. The relevant multiplicative convolution and deconvolution formulae for the transition density and cumulative density are now stated;

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Convolution formula: the multiplicative convolution integral

$$\begin{aligned} F_{S\times X}(y) &= \Pr(S\times X \le y) \\ &= \int_{-\infty}^{\infty} \Pr(X = x)\Pr(S\times x \le y \mid X = x)dx \\ &= \int_{-\infty}^{\infty} f_X(x)\Pr(S \le y/x \mid X = x)dx \\ &= \int_{-\infty}^{\infty} f_X(x)F_{S|X}(y/x \mid x)dx \end{aligned}$$

which is the multiplicative convolution formula for the distribution function. That for the density is obtained by differentiating $DF_{S\times X}(y)$

$$f_{S\times X}(y) = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x) f_{S|X}(y/x \mid x)dx$$

Noting that the variables in the multiplicative convolution are commutative, the symmetric relation with a change of variable is

$$f_{S\times X}(y) = \int_{-\infty}^{\infty} \frac{1}{u} f_X(y/u) f_{S|X}(u \mid y/u)du$$

Deconvolution formula: the multiplicative deconvolution integral

$$\begin{aligned} F_{S/X}(y) &= \Pr(S/X \le y) \\ &= \int_{-\infty}^{\infty} \Pr(X = x)\Pr(S/x \le y \mid X = x)dx \\ &= \int_{-\infty}^{\infty} f_X(x)\Pr(S \le y\times x \mid X = x)dx \\ &= \int_{-\infty}^{\infty} f_X(x)F_{S|X}(y\times x \mid x)dx \end{aligned}$$

which is the multiplicative deconvolution formula for the distribution function. That for the density is obtained by differentiating $DF_{S/X}(y)$

$$f_{S/X}(y) = \int_{-\infty}^{\infty} x\, f_X(x) f_{S|X}(y\times x \mid x)dx$$

and noting that the variables in the multiplicative deconvolution do not commute, the alternative multiplicative convolution formula which does not share symmetry with the above, is with a change of variable

$$f_{S/X}(y) = \int_{-\infty}^{\infty} \frac{u}{y^2} f_X(u/y) f_{S|X}(u \mid u/y)du$$

It is the former set that are required, the latter are stated simply for completeness. When the convolution or deconvolution takes place over the backward variables, the Jacobian factors resulting from differentiating the distribution functions are absent in the convolution and deconvolution formulae for the density function

$$f_{S|S_0\times X}(y) = \int_{-\infty}^{\infty} f_X(x) f_S(y \mid y_0 \div x)dx \qquad \text{for backward convolution}$$

and

$$f_{S|S_0\div X}(y \mid y_0) = \int_{-\infty}^{\infty} f_X(x) f_S(y \mid y_0 \times x)dx \qquad \text{for backward deconvolution}$$

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Proof: for the backward convolution formula,

$$\begin{aligned} F_{S|X\times Y}(w\,|\,u) &= \Pr(S \le w\,|\,XY = u) \\ &= \int_{-\infty}^{\infty} \Pr(Y = y)\Pr(S \le w\,|\,Xy = u)dy \\ &= \int_{-\infty}^{\infty} f_Y(y)\Pr(S \le w\,|\,X = u/y)dy \\ &= \int_{-\infty}^{\infty} f_Y(y)F_{S|X}(w\,|\,u/y)dy \end{aligned}$$

and hence differentiating with respect to *w* for the density results in formulae free from a Jacobian

$$f_{S|X\times Y}(w\,|\,u) = \int_{-\infty}^{\infty} f_Y(y)\,f_{S|X}(w\,|\,u/y)dy$$

and for the backward deconvolution formula,

$$\begin{aligned} F_{S|X\div Y}(w\,|\,u) &= \Pr(S \le w\,|\,X \div Y = u) \\ &= \int_{-\infty}^{\infty} \Pr(Y = y)\Pr(S \le w\,|\,X \div y = u)dy \\ &= \int_{-\infty}^{\infty} f_Y(y)\Pr(S \le w\,|\,X = u \times y)dy \\ &= \int_{-\infty}^{\infty} f_Y(y)F_{S|X}(w\,|\,u \times y)dy \end{aligned}$$

and differentiating with respect to *w* similarly results in formulae free from a Jacobian

$$f_{S|X\div Y}(w\,|\,u) = \int_{-\infty}^{\infty} f_Y(y)\,f_{S|X}(w\,|\,u \times y)dy$$

Not requiring the Jacobian factors in these formulae for the backward variables is consistent with **transformations of the backward equations similarly not requiring Jacobian scale factors**. It Is clear, here, why this thesis sometimes refers to the **(1)** forward and **(2)** backward equations more specifically as **(1)** the forward convolution and deconvolution equations (convoluation and deconvolution of basis variables at the forward diffusion and time varaibles) and **(2)** the backward convolution and deconvolution equations (convoluation and deconvolution of basis variables at the backward diffusion and time varaibles).

Kolmogorov Forward (Convolution) Integro-Differential Equation of a Multiplicative Compound Poisson Process

The intuitive approach is taken since the technique of moment generating functions is awkward due to the process being multiplicative. This approach is based upon the forward recurrence relation of the Poisson process generating the successive incremental convolutions. On any differential increment in time a Poisson event occurs or it does not occur, with the probabilities λdt and $1 - \lambda dt$. If **(i)** a Poisson event occurs, an additional secondary distribution is multiplied onto the compound sum while if **(ii)** no Poisson event occurs, the number of secondary distributions in the compound product stays fixed and the transition distribution is unchanged. Mathematically, this is stated as

$$f_{S(t+dt)|S(t_0)}(x\,|\,x_0) = \lambda dt \times f_{S(t)\times X|S(t_0)}(x\,|\,x_0) + (1 - \lambda dt) \times f_{S(t)|S(t_0)}(x\,|\,x_0) + o(dt)$$

and in terms of a convolution integral,

$$f_{S(t+dt)|S(t_0)}(x\,|\,x_0) = \lambda dt \int_{-\infty}^{\infty} \frac{1}{y} f_X(y)\,f_{S(t)|S(t_0)}(x/y\,|\,x_0)dy + (1 - \lambda dt)\,f_{S(t)|S(t_0)}(x\,|\,x_0) + o(dt)$$

Subtracting $f_{S(t)|S(t_0)}(x\,|\,x_0)$ from both sides

$$f_{S(t+dt)|S(t_0)}(x\,|\,x_0) - f_{S(t)|S(t_0)}(x\,|\,x_0) = \lambda dt \int_{-\infty}^{\infty} \frac{1}{y} f_X(y)\,f_{S(t)|S(t_0)}(x/y\,|\,x_0)dy - \lambda f_{S(t)|S(t_0)}(x\,|\,x_0)dt + o(dt)$$

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and dividing through by *dt*

$$\frac{f_{S(t+dt)|S(t_0)}(x\,|\,x_0)-f_{S(t)|S(t_0)}(x\,|\,x_0)}{dt}=\lambda\int_{-\infty}^{\infty}\frac{1}{y}f_X(y)\,f_{S(t)|S(t_0)}(x/y\,|\,x_0)dy-\lambda f_{S(t)|S(t_0)}(x\,|\,x_0)+\frac{o(dt)}{dt}$$

Letting *dt* approach zero so that $o(dt)/dt\to 0$, results in the forward equation

$$\frac{\partial}{\partial t}f_{S(t)|S(t_0)}(x\,|\,x_0)=\lambda\int_{-\infty}^{\infty}\frac{1}{y}f_X(y)\,f_{S(t)|S(t_0)}(x/y\,|\,x_0)dy-\lambda f_{S(t)|S(t_0)}(x\,|\,x_0)$$

Kolmogorov Backward (Deconvolution) Integro-Differential Equation of a Multiplicative Compound Poisson Process

On any backward differential increment in time a Poisson event has either occurred or not occurred, and the probabilities of each of these two outcomes is again λdt and $1-\lambda dt$. **(i)** If no Poisson events occurred, then the convolution product will have the same number of components and the transition distribution is unchanged. **(ii)** If a Poisson event did occur, then the convolution product would have had one less component in the aggregate product and therefore needs to be multiplicatively deconvolved of one variable from the back end of the product, i.e., the first multiplicative increment of the compound product is removed;

$$f_{S(t)|S(t_0-dt)}(x\,|\,x_0)=\lambda dt\times f_{S(t)|S(t_0)\div X}(x\,|\,x_0)+(1-\lambda dt)\times f_{S(t)|S(t_0)}(x\,|\,x_0)+o(dt)$$

From the deconvolution formula,

$$f_{S(t)|S(t_0-dt)}(x\,|\,x_0)=\lambda dt\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)|S(t_0)}(x\,|\,x_0\times y)dy+(1-\lambda dt)\,f_{S(t)|S(t_0)}(x\,|\,x_0)+o(dt)$$

Subtracting $f_{S(t)|S(t_0)}(x\,|\,x_0)$ from both sides

$$f_{S(t)|S(t_0-dt)}(x\,|\,x_0)-f_{S(t)|S(t_0)}(x\,|\,x_0)=\lambda dt\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)|S(t_0)}(x\,|\,x_0\times y)dy-\lambda\,f_{S(t)|S(t_0)}(x\,|\,x_0)dt+o(dt)$$

and dividing through by *dt*

$$\frac{f_{S(t)|S(t_0-dt)}(x\,|\,x_0)-f_{S(t)|S(t_0)}(x\,|\,x_0)}{dt}=\lambda\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)|S(t_0)}(x\,|\,x_0\times y)dy-\lambda\,f_{S(t)|S(t_0)}(x\,|\,x_0)+\frac{o(dt)}{dt}$$

Letting *dt* approach zero so that $o(dt)/dt\to 0$, gives the backward equation,

$$-\frac{\partial}{\partial t_0}f_{S(t)|S(t_0)}(x\,|\,x_0)=\lambda\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)|S(t_0)}(x\,|\,x_0\times y)dy-\lambda\,f_{S(t)|S(t_0)}(x\,|\,x_0)$$

For backward convolution, rather than deconvolution, the variable of the time derivative changes to forward running time and the integral becomes that of backward convolution

$$\frac{\partial}{\partial t_0}f_{S(t)|S(t_0)}(x\,|\,x_0)=\lambda\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)|S(t_0)}(x\,|\,x_0\div y)dy-\lambda\,f_{S(t)|S(t_0)}(x\,|\,x_0)$$

which can be obtained by the same line of reasoning with a forward increment in the backward time variable and a convolution of the backward diffusion variable

$$f_{S(t)|S(t_0+dt)}(x\,|\,x_0)=\lambda dt\times f_{S(t)|S(t_0)\times X}(x\,|\,x_0)+(1-\lambda dt)\times f_{S(t)|S(t_0)}(x\,|\,x_0)+o(dt)$$

Limiting Form of Forward Integro-Differential Equation of a Multiplicative Compound Poisson Process

Considering the forward integro-differential equation of a multiplicative compound Poisson diffusion

$$\frac{\partial}{\partial t}f_{S(t)|S(t_0)}(x\,|\,x_0)=\lambda\int_{-\infty}^{\infty}\frac{1}{y}f_X(y)\,f_{S(t)|S(t_0)}(x\div y\,|\,x_0)dy-\lambda\,f_{S(t)|S(t_0)}(x\,|\,x_0)$$

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The expansion of the forward argument of the transition density function within the convolution integral requires a Jacobian term. This is obtained by transforming the diffusion variable. Letting $Y(t) = \log S(t)$ .The forward variable and its transform are $y = \log x$ and the Jacobian is

$$\left|\frac{dy}{dx}\right| = \frac{1}{x}$$

The transition density is then related to the equivalent transformed density as

$$f_{S(t)|S(t_0)}(x \mid x_0) = f_{Y(t)|Y(t_0)}(y \mid y_0)\left|\frac{dy}{dx}\right| = \frac{1}{x} f_{Y(t)|Y(t_0)}(y \mid y_0)$$

that is,

$$f_{S(t)|S(t_0)}(x \mid x_0) = \frac{1}{x} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0)$$

so that the integro-differential forward convolution equation becomes

$$\frac{\partial}{\partial t}\frac{1}{x} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0)$$
$$= \lambda \int_{-\infty}^{\infty} \frac{1}{y} f_X(y) \frac{1}{(x/y)} f_{\log S(t)|\log S(t_0)}(\log x - \log y \mid \log x_0) dy - \lambda \frac{1}{x} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0)$$

i.e.,

$$\frac{1}{x}\frac{\partial}{\partial t} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0)$$
$$= \frac{\lambda}{x} \int_{-\infty}^{\infty} f_X(y) f_{\log S(t)|\log S(t_0)}(\log x - \log y \mid \log x_0) dy - \frac{\lambda}{x} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0)$$

Expanding the transition density within the convolution integral in a Taylor series

$$\frac{1}{x}\frac{\partial}{\partial t} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0)$$
$$= \frac{\lambda}{x} \int_{-\infty}^{\infty} f_X(y) \left\{ 1 - (\log y)\frac{\partial}{\partial \log x} + \tfrac{1}{2}(\log y)^2 \frac{\partial^2}{(\partial \log x)^2} \right\} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0) dy$$
$$- \frac{\lambda}{x} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0)$$

and noting that $X \sim N(\mu h, \sigma^2 h)$

(1) $\int_{-\infty}^{\infty} f_X(y) dy = 1$

(2) $\int_{-\infty}^{\infty} (\log y) f_X(y) dy = \mu h$

(3) $\int_{-\infty}^{\infty} (\log y)^2 f_X(y) dy = \sigma^2 h + \mu^2 h^2$

resulting in

$$\frac{1}{x}\frac{\partial}{\partial t} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0) = \frac{\lambda}{x} \left\{ -\mu h \frac{\partial}{\partial \log x} + \tfrac{1}{2}(\sigma^2 h + \mu^2 h^2) \frac{\partial^2}{(\partial \log x)^2} + \ldots \right\} f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0) dy$$

Converting back to the original diffusion variable with $f_{\log S(t)|\log S(t_0)}(\log x \mid \log x_0) = x f_{S(t)|S(t_0)}(x \mid x_0)$

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$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \frac{\lambda}{x}\left\{ -\mu h \frac{\partial}{\partial \log x} x\, f_{S(t)|S(t_0)}(x \mid x_0) + \tfrac{1}{2}(\sigma^2 h + \mu^2 h^2) \frac{\partial^2}{(\partial \log x)^2} x\, f_{S(t)|S(t_0)}(x \mid x_0) + o(h) \right\}$$

and from[62]

$$\begin{aligned}
\frac{\partial}{\partial \log x} x\, f_{S(t)|S(t_0)}(x \mid x_0) &= \frac{\partial x}{\partial \log x}\frac{\partial}{\partial x} x\, f_{S(t)|S(t_0)}(x \mid x_0) \\
&= x\frac{\partial}{\partial x} x\, f_{S(t)|S(t_0)}(x \mid x_0) \\
\frac{\partial^2}{(\partial \log x)^2} x\, f_{S(t)|S(t_0)}(x \mid x_0) &= \frac{\partial}{\partial \log x}\left(\frac{\partial}{\partial \log x} x\, f_{S(t)|S(t_0)}(x \mid x_0) \right) \\
&= \frac{\partial x}{\partial \log x}\frac{\partial}{\partial x}\left(\frac{\partial x}{\partial \log x}\frac{\partial}{\partial x} x\, f_{S(t)|S(t_0)}(x \mid x_0) \right) \\
&= x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} x\, f_{S(t)|S(t_0)}(x \mid x_0) \right) \\
&= x\frac{\partial^2}{\partial x^2} x^2\, f_{S(t)|S(t_0)}(x \mid x_0) - x\frac{\partial}{\partial x} x\, f_{S(t)|S(t_0)}(x \mid x_0)
\end{aligned}$$

with the Poisson parameter $\lambda = 1 / h$ this expression becomes

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x^2\ f_{S(t)|S(t_0)}(x \mid x_0) \right) - (\mu + \tfrac{1}{2}\sigma^2)\frac{\partial}{\partial x}\left(x\ f_{S(t)|S(t_0)}(x \mid x_0) \right) + \frac{o(h)}{h}$$

Letting $h \to 0$ so that the $o(h) / h$ term vanishes results in

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}\left(x^2\ f_{S(t)|S(t_0)}(x \mid x_0) \right) - (\mu + \tfrac{1}{2}\sigma^2)\frac{\partial}{\partial x}\left(x\ f_{S(t)|S(t_0)}(x \mid x_0) \right)$$

which is the forward equation of a proportional Brownian motion.

Limiting Form of Backward Integro-Differential Equation of a Multiplicative Compound Poisson Process

This is almost identical to that of the option premium considered shortly. The result is therefore simply stated,

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{\partial x_0^2} f_{S(t)|S(t_0)}(x \mid x_0) + (\mu + \tfrac{1}{2}\sigma^2) x_0 \frac{\partial}{\partial x_0} f_{S(t)|S(t_0)}(x \mid x_0)$$

Backward Integro-Differential Equation of an Option on a Multiplicative Compound Poisson Process

Considering the backward integro-differential equation for multiplicative compound Poisson diffusion

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[62] the latter derivative is very important to the mathematics of this thesis,

$$\frac{\partial^2}{\partial x^2} x^2 f(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} x[x\, f(x)] \right) = \frac{\partial}{\partial x}\left(x\, f(x) + x\frac{\partial}{\partial x} x\, f(x) \right) = \frac{\partial}{\partial x} xf(x) + \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} x\, f(x) \right)$$

and consequently,

$$\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} x\, f(x) \right) = \frac{\partial^2}{\partial x^2} x^2 f(x) - \frac{\partial}{\partial x} x\, f(x)$$

which is seen again in Compendium of Random Walks and BOOK 2, Replicating Portfolios and a Revised Philosophy for Option Markets.

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$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x \mid x_0 \times y) dy - \lambda f_{S(t)|S(t_0)}(x \mid x_0)$$

Defining the put and call option premia as discounted expected values of the maturity intrinsic value at the forward time variable

$$V(k,t;x_0,t_0) = \begin{cases} e^{-r(t-t_0)} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx & \text{for a call} \\ e^{-r(t-t_0)} \int_{-\infty}^{k} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx & \text{for a put} \end{cases}$$

and applying the same method as that of the additive compound Poisson diffusion, namely, multiplying both sides of the equation by the options intrinsic value, integrating over the moneyness,

$$-\int_k^{\infty} (x-k) \frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) dx = \lambda \int_k^{\infty} (x-k) \int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x \mid x_0 \times y) dy dx - \lambda \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$

taking the integrations in the forward variables inside the time derivative and deconvolution integral in the backward variables,

$$-\frac{\partial}{\partial t_0} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$
$$= \lambda \int_{-\infty}^{\infty} f_X(y) \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0 \times y) dx dy - \lambda \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$

and adding a sink by discounting to the backward time variable at the risk free rate $e^{-r(t-t_0)}$

$$-e^{-r(t-t_0)} \frac{\partial}{\partial t_0} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$
$$= \lambda e^{-r(t-t_0)} \int_{-\infty}^{\infty} f_X(y) \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0 \times y) dx dy - \lambda e^{-r(t-t_0)} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$

gives the transformed equation in the option premium

$$re^{-r(t-t_0)} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx - \frac{\partial}{\partial t_0} e^{-r(t-t_0)} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$
$$= \lambda \int_{-\infty}^{\infty} f_X(y) e^{-r(t-t_0)} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0 \times y) dx dy - \lambda e^{-r(t-t_0)} \int_k^{\infty} (x-k) f_{S(t)|S(t_0)}(x \mid x_0) dx$$

i.e.,

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t;x_0,t_0) = \lambda \int_{-\infty}^{\infty} f_X(y) V(k,t;x_0 \times y,t_0) dy - \lambda V(k,t;x_0,t_0)$$

For log-normal basis components $X \sim N(\mu,\sigma^2)$ which has density

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\log x - \mu)^2/\sigma^2}$$

the equation is

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t;x_0,t_0) = \frac{\lambda}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{y} e^{-\frac{1}{2}(\log y - \mu)^2/\sigma^2} V(k,t;x_0 \times y,t_0) dy - \lambda V(k,t;x_0,t_0)$$

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Limiting Form of Integro-Differential Equation of Put and Call Option Premia for Multiplicative Compound Poisson Diffusion

From the multiplicative form of a Taylor series expansion

$$\begin{aligned} f(xy) &= f(e^{\log x+\log y}) \\ &= f(e^{\log x}) + (\log y)\frac{\partial}{\partial \log x} f(e^{\log x}) + \tfrac{1}{2!}(\log y)^2 \frac{\partial^2}{(\partial \log x)^2} f(e^{\log x}) + \tfrac{1}{3!}(\log y)^3 \frac{\partial^3}{(\partial \log x)^3} f(e^{\log x}) + \dots \\ &= f(x) + (\log y)\frac{\partial}{\partial \log x} f(x) + \tfrac{1}{2!}(\log y)^2 \frac{\partial^2}{(\partial \log x)^2} f(x) + \tfrac{1}{3!}(\log y)^3 \frac{\partial^3}{(\partial \log x)^3} f(x) + \dots \end{aligned}$$

since the backward equation does not require a Jacobian scale factor. Expanding the option premium in the convolution integral of the equation

$$\begin{aligned} & rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t;x_0,t_0) \\ &= \lambda \int_{-\infty}^{\infty} f_X(y) \left\{ 1 + (\log y)\frac{\partial}{\partial \log x_0} + \tfrac{1}{2}(\log y)^2 \frac{\partial^2}{(\partial \log x_0)^2} + \dots \right\} V(k,t;x_0,t_0)dy - \lambda V(k,t;x_0,t_0) \end{aligned}$$

say that $X \sim \log N(\mu h, \sigma^2 h)$ so that $\log X \sim N(\mu h, \sigma^2 h)$ and

$$\int_{-\infty}^{\infty} (\log y) f_X(y)dy = \mu h \qquad \text{and} \qquad \int_{-\infty}^{\infty} (\log y)^2 f_X(y)dy = \sigma^2 h + \mu^2 h^2$$

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t;x_0,t_0) = \lambda \left\{ \mu h \frac{\partial}{\partial \log x_0} + \tfrac{1}{2}(\sigma^2 h + \mu^2 h^2) \frac{\partial^2}{(\partial \log x_0)^2} + o(h) \right\} V(k,t;x_0,t_0)$$

Letting $\lambda = 1 / h$

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t;x_0,t_0) = \left\{ \mu \frac{\partial}{\partial \log x_0} + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{(\partial \log x_0)^2} + \frac{o(h)}{h} \right\} V(k,t;x_0,t_0)$$

and noting that

(1) $\dfrac{\partial}{\partial \log x_0} = x_0 \dfrac{\partial}{\partial x_0}$ and

(2) $\dfrac{\partial^2}{(\partial \log x_0)^2} = \dfrac{\partial}{\partial \log x_0}\left(\dfrac{\partial}{\partial \log x_0}\right) = x_0 \dfrac{\partial}{\partial x_0}\left(x_0 \dfrac{\partial}{\partial x_0}\right) = x_0^2 \dfrac{\partial^2}{\partial x_0^2} + x_0 \dfrac{\partial}{\partial x_0}$

gives

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t;x_0,t_0) = \left\{ \tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{(\partial x_0)^2} + (\mu + \tfrac{1}{2}\sigma^2) x_0 \frac{\partial}{\partial x_0} + \frac{o(h)}{h} \right\} V(k,t;x_0,t_0)$$

letting $h \to 0$ so that the Poisson events occur infinitely often and the basis cumulants scale in proportion to λ results in the unbiased equivalent to the Black-Scholes-Merton pde

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t;x_0,t_0) = \tfrac{1}{2}\sigma^2 x_0^2 \frac{\partial^2}{(\partial x_0)^2} V(k,t;x_0,t_0) + (\mu + \tfrac{1}{2}\sigma^2) x_0 \frac{\partial}{\partial x_0} V(k,t;x_0,t_0)$$

or $\quad rV - \dfrac{\partial V}{\partial t_0} = \tfrac{1}{2}\sigma^2 x_0^2 \dfrac{\partial^2 V}{\partial x_0^2} + (\mu + \tfrac{1}{2}\sigma^2) x_0 \dfrac{\partial V}{\partial x_0}$ where $\quad V = V(k,t;x_0,t_0)$

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The subsidized option premium of return-neutral misevaluation results when the log-drift is replaced by the risk free rate $\mu + \frac{1}{2}\sigma^2 = r$.

Integro-Differential Equation of Put and Call Option Premia for Linear Superposition of Brownian Motion and Compound Poisson Diffusion

Converting the backward integro-differential equation, that which describes the successive deconvolutions of the basis components with the time-decay as maturity nears, for the linear supposition to that of the option premium[63] gives

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0}V(k,t;x_0,t_0)$$
$$= \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial x_0^2}V(k,t;x_0,t_0) + \mu\frac{\partial}{\partial x_0}V(k,t;x_0,t_0) + \int_{-\infty}^{\infty} V(k,t;x_0+y,t_0)\,f_X(y)dy - \lambda V(k,t;x_0,t_0)$$

This could conceivably be a model for actuarial risk theory whereby premium income is a Brownian motion (with a sign change in the drift) rather than linear inflow. Ignoring ito-corrections, the integro-differential equation for the equivalent multiplicative diffusion is

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0}V(k,t;x_0,t_0)$$
$$= \tfrac{1}{2}\sigma^2 x_0^2\frac{\partial^2}{\partial x_0^2}V(k,t;x_0,t_0) + \mu x_0\frac{\partial}{\partial x_0}V(k,t;x_0,t_0) + \int_{-\infty}^{\infty} V(k,t;x_0\times y,t_0)\,f_X(y)dy - \lambda V(k,t;x_0,t_0)$$

where the diffusion terms are transformed by the techniques of FOUNDATIONS I of this book and BOOK 2 and the integral term relating to the compound Poisson component relates. If the secondary components are normal $X \sim N(a,b^2)$ or log-normal $X \sim \log N(a,b^2)$, letting $\lambda = 1/h$ and letting the Poisson events occur every instant $h \to 0$, this equation by the same limiting arguments results in a reparameterized version of a pde for an option on Brownian motion

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0}V(k,t;x_0,t_0) = \tfrac{1}{2}(\sigma^2+b^2)\frac{\partial^2}{\partial x_0^2}V(k,t;x_0,t_0) + (\mu+a)\frac{\partial}{\partial x_0}V(k,t;x_0,t_0)$$

for additive components and

$$rV(k,t;x_0,t_0) - \frac{\partial}{\partial t_0}V(k,t;x_0,t_0) = \tfrac{1}{2}(\sigma^2+b^2)x_0^2\frac{\partial^2}{\partial x_0^2}V(k,t;x_0,t_0) + (\mu+a-\tfrac{1}{2}b^2)x_0\frac{\partial}{\partial x_0}V(k,t;x_0,t_0)$$

for multiplicative components.

THE MODULATED COMPOUND POISSON PROCESS

Recurrence Dynamics, the Diffusion Equation, and its Resolution

To modulate the diffusion aggregation, there are two possible approaches. The diffusion could be modulated at every instantaneous increment whether or not a Poisson increment occurs. Alternatively, the diffusion could be modulated only when a Poisson event occurs and an additional basis variable is convolved onto the diffusion variable. One possible interpretation of this is the revision of old news plus the addition of new news, revisions only occurring when new news has become available. The **forward recurrence** differential dynamics of the first approach is

$$S(t+dt) = [aS(t) + X\ h]dN(t) + aS(t)[1-dN(t)] + o(dt)$$

while that of the second approach is

[63] see BOOK 2.

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$$S(t+dt) = [aS(t) + X\ h]dN(t) + S(t)[1 - dN(t)] + o(dt)$$

This thesis takes the second approach as the resulting diffusion equations correspond exactly with those of BOOK 3. By this approach, the compound Poisson diffusion either modulates by a factor of a and increases with a sultus of a convolved scaled basis component $X\,h$ or remains unchanged and this depends on whether the Poisson process increments or does not increment, i.e., whether the differential increment in the generator, $dN(t)$, is one or zero. With the alternative approach that is not pursued by this thesis, the compound Poisson diffusion variable modulates independently of the Poisson process. Expressing the forward differential dynamics of the approach of interest, that whereby the diffusion variable modulates only when the Poisson generator increments, in terms of the differential form of the diffusion equation of the Poisson process, $dN(t) = z_t\,dt$,

$$S(t+dt) = z_t dt[aS(t) + X\,h] + (1 - z_t dt)S(t) + o(dt)$$

Rearranging terms gives

$$S(t+dt) - S(t) = (a-1)z_t S(t)dt + z_t X\,h dt + o(dt)$$

Letting $a = 1 - \alpha h$ and dividing by dt

$$\frac{S(t+dt) - S(t)}{dt} = -\alpha h z_t S(t) + z_t X\,h + \frac{o(dt)}{dt}$$

and letting $dt \to 0$ so that $o(dt) / dt \to 0$ results in the **differential form of the diffusion equation** for a modulated (non-unit root) compound Poisson diffusion

$$\frac{dS(t)}{dt} + \alpha h z_t S(t) = z_t X\,h$$

where, like the ordinary compound Poisson diffusion, h is proportional to the inverse of the Poisson intensity, $h \propto 1/\lambda$. Whenever $z_t\,dt$ is held fixed at one, $z_t\,dt = 1$, for all values of t and the span variable h is proportional to dt , $h \propto dt$, the differential diffusion equation and all of the derived equations simplify to those of an Ornstein-Uhlenbeck process $dS(t) \propto -\alpha S(t)dt + X_t\,dt$.

Dividing both sides of the diffusion equation of the compound Poisson sum by z_t

$$\frac{dS(t)}{z_t dt} + \alpha h S(t) = X\ h$$

and from the fact that $dN(t) = z_t\,dt$, the differential form can alternatively be expressed as

$$\frac{dS(t)}{dN(t)} + \alpha h S(t) = X\ h \qquad \text{or} \qquad \frac{1}{h}\frac{dS(t)}{dN(t)} + \alpha S(t) = X$$

which is the diffusion equation of an Ornstein-Uhlenbeck process which changes differentially with respect to the Poisson basis, i.e., the differential change in the Poisson process generator. In terms of the discrete characteristic root a, this is

$$\frac{dS(t)}{dN(t)} + (1-a)S(t) = X\ h$$

Resolving the Diffusion Equation:

In order to resolve the modulated compound Poisson diffusion equation, the differential form has to be modified to incorporate the fact that $N(t)$ is a discrete variable. This modification is the additional Poisson indicator $a^{dN(t)-1}$ appended to the non-differential diffusion term $S(t)$

$$\frac{dS(t)}{dN(t)} + a^{dN(t)-1}(1-a)S(t) = X\ h$$

This, then, allows the equation to be differentially factorized incorporating the discrete stochastic calculus of the Poisson variable. Rearranging

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$$\frac{dS(t)}{dt} + a^{N(t+dt)-N(t)}(a^{-1}-1)S(t)\frac{dN(t)}{dt} = X\ h\frac{dN(t)}{dt}$$

or

$$a^{-N(t+dt)}\frac{dS(t)}{dt} + a^{-N(t)}(a^{-1}-1)S(t)\frac{dN(t)}{dt} = a^{-N(t+dt)}X\ h\frac{dN(t)}{dt}$$

which can be differentially factorized into the form

$$\frac{d}{dt}a^{-N(t)}S(t) = a^{-N(t+dt)}X\ h\frac{dN(t)}{dt}$$

Because the forward recurrence and differential expression is expressed in terms of forward derivatives, the exact timing of the convolution of the diffusion basis component X is at $t + dt$ (rather than t) so that $X = X(t + dt)$. The equation is written more specifically as

$$\frac{d}{dt}a^{-N(t)}S(t) = a^{-N(t+dt)}X(t+dt)\ h\frac{dN(t)}{dt}$$

Integrating over time from time zero to the forward time variable

$$\int_0^T \frac{d}{dt}a^{-N(t)}S(t)dt = \int_0^T a^{-N(t+dt)}X(t+dt)\ hdN(t)$$

or

$$a^{-N(T)}S(T) - a^{-N(0)}S(0) = \int_0^T a^{-N(t+dt)}X(t+dt)\ hdN(t)$$

results in one form of the **integral form of the diffusion equation** of the modulated compound Poisson diffusion variable[64]

$$S(T) = a^{N(T)}S(0) + a^{N(T)}\int_0^T a^{-N(t+dt)}X(t+dt)\ hdN(t)$$

Because the forward derivative differs from the backward derivative for Poisson variables because such variables are discrete, the argument to convert the integration to a discrete sum differs slightly from the much simpler case where the Poisson additions of basis variables onto S are not modulated. Since $dN(t)$ is a forward time differential, $dN(t) = N(t + dt) - N(t)$, whenever $z_t\,dt$ is one indicating a Poisson event has occurred, the exact timing of this is at $t + dt$ where $t + dt \to t$ as $dt \to 0$. Only one Poisson event can occur on the infinitesimal interval from t to $t + dt$. Therefore, if $dN(t) = 1$ generates the j^{th} Poisson event, the timing of this event t_j occurs at $t + dt$. **[BOOK 3 will use backward derivatives so this complication will not occur]**

If the Poisson events of the generator basis z_y occur at t_j, as for the ordinary compound Poisson diffusion, then the scaled Poisson basis is one at the t_j and zero everywhere else

$$z_y dy = \begin{cases} 1 & \text{if } y = t_j \\ 0 & \text{if } y \neq t_j \end{cases} \qquad \text{for } j = 1,\ 2,\ ,3,\ \ldots$$

---

[64] if the alternative approach is taken, that with forward relation

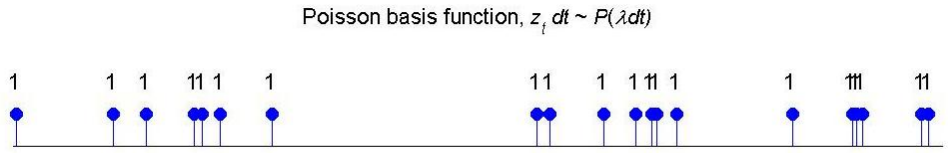
$$S(t+dt) = [aS(t) + X_t h]dN(t) + aS(t)[1 - dN(t)] + o(dt)$$

then the resolved diffusion variable would have the form

$$S(T) = e^{-\alpha T}S(0) + e^{-\alpha T}\int_0^T e^{\alpha t}z_t X_t dt$$

which is not of the form that is most advantageous for the goals of this thesis. This is the method that, for example, describes the continuous revision of old news whether or not new information has become available and requires modeling of the timing of the Poisson basis increments, i.e., the damped diffusion basis variable is not independent of the timing of the Poisson basis increments.

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so that

$$a^{-N(t+dt)}X_{hN(t+dt)}dN(t)=\begin{cases}1\times a^{-N(t+dt)}X_{hN(t+dt)} & \text{if } t+dt=t_j\\ 0\times a^{-N(t+dt)}X_{hN(t+dt)} & \text{if } t+dt\neq t_j\end{cases}$$

$$=\begin{cases}a^{-N(t_j)}X_{hN(t_j)} & \text{if } t+dt=t_j\\ 0 & \text{if } t+dt\neq t_j\end{cases}\qquad \text{for } j=1,\ 2,\ ,3,\ \ldots,\ N(y)$$

There is a total of $N(t)$ times t_j at which the Poisson generator produces ones and resulting convolutions of diffusion basis variables since with any point in time at which a Poisson event occurs and $dN(t)$ being t_j where $j = 1, 2, 3, \ldots, N(t)$ and $0 \le y \le t$,

$$N(y)=\int_0^y z_x dx=j \quad \text{at any} \quad y=t_j \qquad \text{for } j=1,\ 2,\ ,3,\ \ldots,\ N(y) \qquad \text{and } 0\le y\le t$$

so that

$$a^{-N(t+dt)}X_{hN(t+dt)}dN(t)=\begin{cases}a^{-j}X_{jh} & \text{if } t+dt=t_j\\ 0 & \text{if } t+dt\neq t_j\end{cases}\qquad \text{for } j=1,\ 2,\ ,3,\ \ldots,\ N(y)$$

Therefore, exponential factors of the resolved diffusion equation of the modulated compound Poisson process can be expressed simply as a^{-j} which are those for discrete diffusion. The integral part of can alternatively be written as a discrete sum

$$\int_0^T a^{-N(t+dt)}X_{hN(t+dt)}hdN(t)=\sum_{j=1}^{N(T)}a^{-j}X_{jh}h$$

giving an alternative form of the resolved variable $S(t)$

$$S(T)=a^{N(T)}S(0)+a^{N(T)}\sum_{j=1}^{N(T)}a^{-j}X_{jh}h$$

which is exactly the form of the integral expression of discrete first order non-unit root diffusion from PART II of BOOK 3 with the forward time index n replaced by a Poisson process $N(T)$. The diffusion variable unconditional on the Poisson generator $N(t)$ is then

$$S(T)=\sum_{n=0}^{\infty}e^{-\lambda T}\frac{(\lambda T)^n}{n!}\times\left(a^nS(0)+a^n\sum_{j=1}^{n}a^{-j}X_{jh}h\right)$$

where, for convenience, it is assumed that a summation from 1 to zero, $\sum_1^0$ is a null sum, i.e.,

$$S(T)=S(0)\sum_{n=0}^{\infty}e^{-\lambda T}\frac{(a\lambda T)^n}{n!}+\sum_{n=1}^{\infty}e^{-\lambda T}\frac{(\lambda T)^n}{n!}\times\left(a^n\sum_{j=1}^{n}a^{-j}X_{jh}h\right)$$

which, noting that $e^{-\lambda t}\sum_{n=0}^{\infty}(a\lambda T)^n/n!=e^{\lambda T(a-1)}=e^{-\alpha h\lambda T}$, can be expressed as

$$S(T)=e^{\lambda T(a-1)}S(0)+\sum_{n=1}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\times\left(a^n\sum_{j=1}^{n}a^{-j}X_{jh}h\right)$$

Limiting Form as h Becomes Infinitesimal, h → 0

Recalling that the limit of the scaled Poisson process, proved by moment generating functions, is linear time

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$$\lim_{h\to 0} hN(t) \sim \lim_{h\to 0} hP(t/h) \to t$$

Since $j = 1, 2, 3, \ldots, N(t)$ in the discrete sum of

$$\sum_{j=1}^{N(T)} a^{-j} X_{jh} h$$

the time indexing argument of the basis function X_{jh} is $jh = h, 2h, 3h, \ldots, h\,N(t)$. The infinitesimal limit of the indices of the exponential factors and the upper limit of integration of the secondary basis variables in the integral solution are consequently proportional to the forward time variable t; with $\lambda = 1/h$, taking the limits $h \to dt \to 0$, subject to $hj \to t$, and utilizing the limit $h\,N(t) \sim h\,P(t/h) \to t$ as $h \to 0$ so that hj becomes a time continuum from 0 to t, results in the resolved diffusion variable of first order non-unit root diffusion, the Ornstein-Uhlenbeck process (a damped Brownian motion),

$$S(t) = e^{-\alpha t} S(0) + e^{-\alpha t} \int_{y=0}^{t} e^{\alpha y} X_y dy$$

This theory is relevant and complementary to **BOOK 3**, **the Diffusion Calculus of Arbitrary Order**, for diffusion with modulated roots to model high frequency transaction data.

Transition density of a modulated compound Poisson diffusion

From the resolved diffusion equation, it is clear that $N(t)$ counts the number of random additions of independent copies of the time invariant basis distribution X onto the modulated compound sum, and at the time of every addition of another convolved basis component X, the diffusion variable is modulated by a factor of a. The transition variable of the diffusion, from the integral form of the diffusion equation is

$$S(t) = a^{N(t)} x_0 + a^{N(t)-1} X_1 + a^{N(t)-2} X_2 + \ldots + a X_{N(t)-1} + X_{N(t)} = \sum_{j=0}^{N(t)} a^{N(t)-j} X_j$$

where $a = 1 - \alpha h$, which was derived from the difference/differential form

$$S(t+dt) = aS(t) + X_{N(t+dt)} h \qquad \text{or} \qquad S(t) = aS(t-dt) + X_{N(t)} h \text{ in backward differentials}$$

if a Poisson event occurs and

$$S(t+dt) = S(t) \qquad \text{or} \qquad S(t) = S(t-dt) \text{ in backward differentials}$$

if a Poisson event does not occur, where $a = 1 - \alpha h$. Since S(t) only changes at t_j this can be expressed more concisely as $S(t_j) = aS(t_{j-1}) + X_{t_{j-1}} h$ where the j^{th} Poisson event occurs at time t_j. The density and distribution function of the process are consequently $N(t)$-fold modulated convolutions of the basis density and distribution functions

$$f_{S(t)|S(0)} = f_{a^{N(t)}x_0 + a^{N(t)-1}X_1 + a^{N(t)-2}X_2 + \ldots + aX_{N(t)-1} + X_{N(t)}} \quad \text{and} \quad F_{S(t)|S(0)} = F_{a^{N(t)}x_0 + a^{N(t)-1}X_1 + a^{N(t)-2}X_2 + \ldots + aX_{N(t)-1} + X_{N(t)}}$$

or

$$f_{S(t)|S(0)} = f_{\sum_{j=0}^{N(t)} a^{N(t)-j} X_j} \qquad \text{and} \qquad F_{S(t)|S(0)} = F_{\sum_{j=0}^{N(t)} a^{N(t)-j} X_j}$$

The conditional transition cumulants are those of a discrete Ornstein-Uhlenbeck process, a process which is looked at in depth in BOOK 3.

$$f_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} p_n(t) f_{\sum_{j=0}^{n} a^{n-j} X_j}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f_{\sum_{j=0}^{n} a^{n-j} X_j}(x \mid x_0)$$

where $f_{X^{*0}}(x) = \delta_{x_0}(x)$ is a Dirac delta function. When the scaled basis components are normal $X \sim N(\mu h, \sigma^2 h)$, the distribution of the transition interval conditional on the Poisson process is log-normal

$$\sum_{j=0}^{N(t)} X_j \sim N\left(x_0 a^n + \mu h \frac{1-a^n}{1-a},\ \sigma^2 h \frac{1-a^{2n}}{1-a^2} \right)$$

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so that

$$f_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \frac{1}{\sigma\sqrt{h(1-a^{2n})/(1-a^2)}\sqrt{2\pi}} \exp\left(-\frac{[x - x_0 a^n - \mu h(1-a^n)/(1-a)]^2}{2\sigma^2 h(1-a^{2n})/(1-a^2)} \right)$$

This gives mathematical credibility, and the foundation for, letting the forward time index of discrete diffusion of PART II of BOOK 3 be modeled as a Poisson process, i.e., a unit root integrated Poisson basis. This is the suggested mathematical process, of this thesis, to model high frequency transaction data.

Transition MGF

The transition moment generating function for a fixed Poisson generator $N(t) = n$ is

$$M_{S(t)|S(0),N(t)=n}(\theta) = \exp\left(x_0 a^n \theta + \mu h \frac{1-a^n}{1-a}\theta + \tfrac{1}{2}\sigma^2 h \frac{1-a^{2n}}{1-a^2}\theta^2 \right)$$

The mgf of the unconditional transition variable, for a fixed transition period, is then

$$f_{S(t)|S(0)}(x \mid x_0) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \exp\left(x_0 a^n \theta + \mu h \frac{1-a^n}{1-a}\theta + \tfrac{1}{2}\sigma^2 h \frac{1-a^{2n}}{1-a^2}\theta^2 \right)$$

which does not (?) sum into a convenient analytic form unlike the case of a unit root whereby the cumulants sum proportional to $N(t)$.

First Two Transition Moments

The first two derivatives of the conditional transition moment generating function are

$$\frac{d}{d\theta} M_{S(t)|S(0),N(t)=n}(\theta) = \left(x_0 a^n + \mu h \frac{1-a^n}{1-a} + \sigma^2 h \frac{1-a^{2n}}{1-a^2}\theta \right) M_{S(t)|S(0),N(t)=n}(\theta)$$

$$\frac{d^2}{d\theta^2} M_{S(t)|S(0),N(t)=n}(\theta) = \left\{ \left(x_0 a^n + \mu h \frac{1-a^n}{1-a} + \sigma^2 h \frac{1-a^{2n}}{1-a^2}\theta \right)^2 + \left(\sigma^2 h \frac{1-a^{2n}}{1-a^2} \right) \right\} M_{S(t)|S(0),N(t)=n}(\theta)$$

so that the first two raw moments are

$$E\left(S(t) \mid S(0), N(t) = n \right) = x_0 a^n + \mu h \frac{1-a^n}{1-a}$$

$$E\left(S(t)^2 \mid S(0), N(t) = n \right) = \left(x_0 a^n + \mu h \frac{1-a^n}{1-a} \right)^2 + \left(\sigma^2 h \frac{1-a^{2n}}{1-a^2} \right)$$

resulting in a conditional transition variance of

$$Var\left(S(t) \mid S(0), N(t) = n \right) = \left(x_0 a^n + \mu h \frac{1-a^n}{1-a} \right)^2 + \left(\sigma^2 h \frac{1-a^{2n}}{1-a^2} \right) - \left(x_0 a^n + \mu h \frac{1-a^n}{1-a} \right)^2 = \sigma^2 h \frac{1-a^{2n}}{1-a^2}$$

The unconditional transition mean, from $E(S) = E[E(S \mid N)]$ is

$$E[S(t) \mid S(0)] = E\left(E[S(t) \mid S(0)] \mid \quad N(t) = n \right)$$

which evaluates to

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$$
\begin{aligned}
E\big(S(t)\,|\,S(0)=x_0\big) &= \sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\times\left(x_0a^n+\mu h\frac{1-a^n}{1-a}\right)\\
&= x_0\sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda at)^n}{n!}+\mu h\frac{e^{-\lambda t}}{1-a}\sum_{n=0}^{\infty}\ \frac{1}{n!}\Big((\lambda t)^n-(\lambda at)^n\Big)\\
&= x_0e^{-\lambda t}e^{\lambda at}+\mu he^{-\lambda t}\,\frac{e^{\lambda t}-e^{\lambda at}}{1-a}\\
&= x_0e^{-\lambda(1-a)t}+\mu h\frac{1-e^{-\lambda(1-a)t}}{1-a}
\end{aligned}
$$

From either $Var(S) = E(S^2) - E(S)^2 = E[E(S^2|\,N)] - E[E(S\,|\,N)]^2$ or $Var(S) = Var[E(S\,|\,N)] + E[Var(S\,|\,N)]$, the transition variance is

$$
\begin{aligned}
&Var\big(S(t)\,|\,S(0)=x_0\big)\\
&=\sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\times\left\{\left(x_0a^n+\mu h\frac{1-a^n}{1-a}\right)^2+\left(\sigma^2h\frac{1-a^{2n}}{1-a^2}\right)\right\}-\left\{\sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\times\left(x_0a^n+\mu h\frac{1-a^n}{1-a}\right)\right\}^2\\
&=\sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\times\left\{{x_0}^2a^{2n}+2x_0a^n\left(\mu h\frac{1-a^n}{1-a}\right)+\left(\mu h\frac{1-a^n}{1-a}\right)^2+\left(\sigma^2h\frac{1-a^{2n}}{1-a^2}\right)\right\}\\
&\qquad-\left\{\sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\times\left(x_0a^n+\mu h\frac{1-a^n}{1-a}\right)\right\}^2\\
&=\sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\times\left\{{x_0}^2a^{2n}+2x_0\mu h\frac{a^n-a^{2n}}{1-a}+\mu^2h^2\frac{1-2a^n+a^{2n}}{(1-a)^2}+\left(\sigma^2h\frac{1-a^{2n}}{1-a^2}\right)\right\}\\
&\qquad-\left\{\sum_{n=0}^{\infty}\ e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\times\left(x_0a^n+\mu h\frac{1-a^n}{1-a}\right)\right\}^2
\end{aligned}
$$

so that

$$
\begin{aligned}
&Var\big(S(t)\,|\,S(0)=x_0\big)\\
&={x_0}^2e^{-\lambda(1-a^2)t}+2x_0\left(\mu h\frac{e^{-\lambda(1-a)t}-e^{-\lambda(1-a^2)t}}{1-a}\right)+\mu^2h^2\,\frac{e^{-\lambda t}-2e^{-\lambda(1-a)t}+e^{-\lambda(1-a^2)t}}{(1-a)^2}+\left(\sigma^2h\frac{1-e^{-\lambda(1-a^2)t}}{1-a^2}\right)\\
&\qquad-\left(x_0e^{-\lambda(1-a)t}+\mu h\frac{1-e^{-\lambda(1-a)t}}{1-a}\right)^2\\
&=\left\{{x_0}^2e^{-\lambda(1-a^2)t}+2x_0\mu h\frac{e^{-\lambda(1-a)t}-e^{-\lambda(1-a^2)t}}{1-a}+\mu^2h^2\,\frac{e^{-\lambda t}-2e^{-\lambda(1-a)t}+e^{-\lambda(1-a^2)t}}{(1-a)^2}+\sigma^2h\frac{1-e^{-\lambda(1-a^2)t}}{1-a^2}\right\}\\
&\qquad-\left({x_0}^2e^{-2\lambda(1-a)t}+2x_0\mu h\frac{e^{-\lambda(1-a)t}-e^{-2\lambda(1-a)t}}{1-a}+\mu^2h^2\,\frac{1-2e^{-\lambda(1-a)t}+e^{-2\lambda(1-a)t}}{(1-a)^2}\right)\\
&=\sigma^2h\frac{1-e^{-\lambda(1-a^2)t}}{1-a^2}+{x_0}^2(e^{-\lambda(1-a^2)t}-e^{-2\lambda(1-a)t})-2x_0\mu h\frac{e^{-\lambda(1-a^2)t}-e^{-2\lambda(1-a)t}}{1-a}+\mu^2h^2\,\frac{e^{-\lambda(1-a^2)t}-e^{-2\lambda(1-a)t}}{(1-a)^2}
\end{aligned}
$$

Because of the Poisson process generating the modulated convolved aggregation, the transition distribution has non-zero higher order cumulants despite the basis components only having cumulants to order two. These, like the variance, are overly complicated and are omitted as the evaluations are not material. Letting $\lambda = 1/h$ and anticipating results from BOOK 3, $a = 1 - \alpha h$, the transition mean and variance (and higher order cumulants) become those of a continuous Ornstein-Uhlenbeck discussed following Brownian motion in FOUNDATIONS I as $h \to 0$.

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Limits of Moments

Taking limits for the transition mean

$$\lim_{\substack{h\to 0\\ \lambda\to 1/h}} E\big(S(t)\,|\,S(0)=x_0\big)=x_0e^{-(1/h)(\alpha h)t}+\mu h\frac{1-e^{-(1/h)(\alpha h)t}}{\alpha h}=x_0e^{-\alpha t}+\mu\frac{1-e^{-\alpha t}}{\alpha}$$

and the transition variance, from

$$\begin{aligned}e^{-\lambda(1-a^2)t}-e^{-2\lambda(1-a)t}&=e^{-\lambda[1-(1-\alpha h)^2]t}-e^{-2\lambda[1-(1-\alpha h)]t}\\&=e^{-\lambda(2\alpha h-\alpha^2h^2)t}-e^{-2\lambda(\alpha h)t}\\&=e^{-2\alpha h\lambda t+\lambda\alpha^2h^2t}-e^{-2\alpha\lambda ht}\\&=e^{-2\alpha h\lambda t+o(h)}-e^{-2\alpha\lambda ht}\\&=e^{-2\alpha\lambda ht}(e^{o(h)}-1)\end{aligned}$$

and

$$\lim_{\substack{h\to 0\\ \lambda\to 1/h}}\left(e^{-\lambda(1-a^2)t}-e^{-2\lambda(1-a)t}\right)=\lim_{\substack{h\to 0\\ \lambda\to 1/h}}e^{-2\alpha\lambda ht}(e^{o(h)}-1)=\lim_{h\to 0}e^{-2\alpha t}(e^{o(h)}-1)=e^{-2\alpha t}(1-1)=0$$

the limiting variance as $h\to 0$ is

$$\lim_{\substack{h\to 0\\ \lambda\to 1/h}} Var\big(S(t)\,|\,S(0)=x_0\big)$$

$$=\lim_{\substack{h\to 0\\ \lambda\to 1/h}}\left\{\sigma^2h\frac{1-e^{-\lambda(1-a^2)t}}{1-a^2}+{x_0}^2(e^{-\lambda(1-a^2)t}-e^{-2\lambda(1-a)t})-2x_0\mu h\frac{e^{-\lambda(1-a^2)t}-e^{-2\lambda(1-a)t}}{1-a}+\mu^2h^2\frac{e^{-\lambda(1-a^2)t}-e^{-2\lambda(1-a)t}}{(1-a)^2}\right\}$$

$$=\sigma^2\frac{1-e^{-2\alpha t}}{2\alpha}$$

The higher order moments are such that all cumulants above second are zero, as per ordinary continuous time diffusion. Hence

$$\lim_{\substack{h\to 0\\ \lambda\to 1/h}} S(t)\sim N\left(x_0e^{-\alpha t}+\mu\frac{1-e^{-\alpha t}}{\alpha},\ \sigma^2\frac{1-e^{-2\alpha t}}{2\alpha}\right)$$

which is the transition distribution of an Ornstein-Uhlenbeck process.

Mathematical Preliminary: Modulated Addition and Subtraction of Random Variables and the Modulated Convolution and Deconvolution of Density Functions

Convolution formula: the modulated additive convolution integral

$$\begin{aligned}F_{aS+X}(y)&=\Pr(aS+X\le y)\\&=\int_{-\infty}^{\infty}\Pr(X=x)\Pr(aS+x\le y\,|\,X=x)dx\\&=\int_{-\infty}^{\infty}f_X(x)\Pr(S\le(y-x)/a\,|\,X=x)dx\\&=\int_{-\infty}^{\infty}f_X(x)F_{S|X}\big((y-x)/a\,|\,x\big)dx\end{aligned}$$

which is the modulated additive convolution formula for the distribution function. That for the density is obtained by differentiating $DF_{aS+X}(y)$

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$$f_{aS+X}(y)=\frac{1}{a}\int_{-\infty}^{\infty} f_X(x)\,f_{S|X}\left((y-x)/a\,|\,x\right)dx$$

By change of variable, this formula is alternatively stated as

$$f_{aS+X}(y)=\int_{-\infty}^{\infty} f_X(y-au)\,f_{S|X}\left(u\,|\,y-au\right)du$$

noting the non-symmetry.

Deconvolution formula: the modulated additive deconvolution integral

$$\begin{aligned}F_{S/a-X}(y)&=\Pr(S/a-X\le y)\\&=\int_{-\infty}^{\infty}\Pr(X=x)\Pr(S/a-x\le y\,|\,X=x)dx\\&=\int_{-\infty}^{\infty}f_X(x)\Pr(S\le a(y+x)\,|\,X=x)dx\\&=\int_{-\infty}^{\infty}f_X(x)F_{S|X}\left(a(y+x)\,|\,x\right)dx\end{aligned}$$

which is the modulated additive deconvolution formula for the distribution function. That for the density is obtained by differentiating $DF_{S/a-X}(y)$

$$f_{S/a-X}(y)=a\int_{-\infty}^{\infty} f_X(x)\,f_{S|X}\left(a(y+x)\,|\,x\right)dx$$

By change of variable, this formula is alternatively stated with a change of variable as

$$f_{S/a-X}(y)=\int_{-\infty}^{\infty} f_X(u/a-y)\,f_{S|X}\left(u\,|\,u/a-y\right)du$$

When the basis component is also demodulated, since the basis component is deconvolved before the diffusion variable is demodulated, the maths for the distribution function can be written as

$$\begin{aligned}F_{(S-X)/a}(y)&=\Pr\left((S-X)/a\le y\right)\\&=\int_{-\infty}^{\infty}\Pr(X=x)\Pr\left((S-x)/a\le y\,|\,X=x\right)dx\\&=\int_{-\infty}^{\infty}f_X(x)\Pr(S\le ay+x\,|\,X=x)dx\\&=\int_{-\infty}^{\infty}f_X(x)F_{S|X}(ay+x\,|\,x)dx\end{aligned}$$

giving that for the density, upon differentiation $DF_{(S-X)/a}(y)$,

$$f_{(S-X)/a}(y)=a\int_{-\infty}^{\infty} f_X(x)\,f_{S|X}(ay+x\,|\,x)dx$$

or with a change of variable

$$f_{(S-X)/a}(y)=a\int_{-\infty}^{\infty} f_X(u-ay)\,f_{S|X}(u\,|\,u-ay)du$$

Kolmogorov Forward (Convolution) Integro-Differential Equation of a Modulated Additive Compound Poisson Process

From the forward recurrence relation of the Poisson process generating the successive incremental convolutions onto the modulated diffusion variable, on any differential increment in time a Poisson event either occurs or does not occur, and these events have the probabilities λdt and $1-\lambda dt$. If **(i)** a Poisson event occurs, the diffusion variable $S(t)$ is modulated by a factor of a and an additional basis variable X is convolved onto the modulated diffusion variable $a\,S(t)$ while if **(ii)** no Poisson event occurs, the diffusion variable $S(t)$ does not change and no

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additional basis variable is convolved onto the diffusion aggregation, that is, the distribution of the diffusion variable stays fixed and the transition distribution is unchanged. Mathematically, this is stated in terms of the transition density function as

$$f_{S(t+dt)|S(t_0)}(x \mid x_0) = \lambda dt \times f_{aS(t)+X|S(t_0)}(x \mid x_0) + (1 - \lambda dt) \times f_{S(t)|S(t_0)}(x \mid x_0) + o(dt)$$

and in terms of a convolution integral,

$$f_{S(t+dt)|S(t_0)}(x \mid x_0) = \frac{\lambda}{a}\int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}((x-y)/a \mid x_0)dydt + (1 - \lambda dt) \times f_{S(t)|S(t_0)}(x \mid x_0) + o(dt)$$

Subtracting $f_{S(t)|S(t_0)}(x \mid x_0)$ from both sides

$$f_{S(t+dt)|S(t_0)}(x \mid x_0) - f_{S(t)|S(t_0)}(x \mid x_0) = \frac{\lambda}{a}\int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}((x-y)/a \mid x_0)dydt - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0)dt + o(dt)$$

and dividing through by dt

$$\frac{f_{S(t+dt)|S(t_0)}(x \mid x_0) - f_{S(t)|S(t_0)}(x \mid x_0)}{dt} = \frac{\lambda}{a}\int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}((x-y)/a \mid x_0)dy - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0) + \frac{o(dt)}{dt}$$

Letting dt approach zero so that $o(dt)/dt \to 0$, gives the forward equation

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \frac{\lambda}{a}\int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}((x-y)/a \mid x_0)dy - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0)$$

Kolmogorov Backward (Deconvolution) Integro-Differential Equation of a Modulated Additive Compound Poisson Process

From the backward recurrence relation of the Poisson process generating the successive incremental convolutions onto the modulated diffusion variable, on any backward differential increment in time a Poisson event has either occurred or not occurred, and the probabilities of each of these two outcomes is again λdt and $1 - \lambda dt$. **(i)** If no Poisson events occurred, then the modulated convolution sum has the same number of components and the transition distribution $S(t) \mid S(t_0)$ is unchanged. **(ii)** If a Poisson event did occur, then the modulated convolution sum has one less basis component X in the aggregate sum and the diffusion variable (following being deconvolved of one X) will be demodulated by the factor a; the backward diffusion variable therefore needs to be **i** deconvolved of one basis variable (the first basis increment of the transition interval) and demodulated by a, that is $S(t) \mid [S(t_0) - X]/a$. The relation in the transition density function is

$$f_{S(t)|S(t_0-dt)}(x \mid x_0) = \lambda dt \times f_{S(t)|[S(t_0)-X]/a}(x \mid x_0) + (1 - \lambda dt) \times f_{S(t)|S(t_0)}(x \mid x_0) + o(dt)$$

From the deconvolution formula,

$$f_{S(t)|S(t_0-dt)}(x \mid x_0) = \lambda dt \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x \mid ax_0 + y)dy + (1 - \lambda dt)\, f_{S(t)|S(t_0)}(x \mid x_0) + o(dt)$$

Subtracting $f_{S(t)|S(t_0)}(x \mid x_0)$ from both sides

$$f_{S(t)|S(t_0-dt)}(x \mid x_0) - f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x \mid ax_0 + y)dydt - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0)dt + o(dt)$$

and dividing through by dt

$$\frac{f_{S(t)|S(t_0-dt)}(x \mid x_0) - f_{S(t)|S(t_0)}(x \mid x_0)}{dt} = \lambda \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x \mid ax_0 + y)dy - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0) + \frac{o(dt)}{dt}$$

Letting dt approach zero so that $o(dt)/dt \to 0$, gives the backward equation,

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)|S(t_0)}(x \mid ax_0 + y)dy - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0)$$

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Limiting Form of Forward Integro-Differential Equation of a Modulated Additive Compound Poisson Process

From the forward integro-differential equation of an additive modulated compound Poisson process

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \frac{\lambda}{a}\int_{-\infty}^{\infty} f_z(y)\, f_{S(t)|S(t_0)}((x-y)/a \mid x_0)dy - \lambda\ f_{S(t)|S(t_0)}(x \mid x_0)$$

making the substitution $u = x - (x-y)/a$ with differential $dy = du/a$ and inverse function $y = (1-a)x + a\,u$ gives

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda\int_{-\infty}^{\infty} f_z(x - ax + au)\, f_{S(t)|S(t_0)}(x-u \mid x_0)du - \lambda\ f_{S(t)|S(t_0)}(x \mid x_0)$$

Expanding the forward argument of the transition density within the convolution integral

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda\int_{-\infty}^{\infty} f_z(x - ax + au)\left\{1 - u\frac{\partial}{\partial x} + \tfrac{1}{2}u^2\frac{\partial^2}{\partial x^2} + \ldots\right\} f_{S(t)|S(t_0)}(x \mid x_0)du - \lambda\ f_{S(t)|S(t_0)}(x \mid x_0)$$

and noting the distribution of the basis variable with a translation and scaling

$$\begin{aligned}\frac{a-1}{a}x + \frac{1}{a}Z &\sim \frac{a-1}{a}x + \frac{1}{a}N(\mu h, \sigma^2 h)\\ &\sim \frac{a-1}{a}x + N\left(\frac{\mu h}{a}, \frac{\sigma^2 h}{a^2}\right)\\ &\sim N\left(\frac{a-1}{a}x + \frac{\mu h}{a}, \frac{\sigma^2 h}{a^2}\right)\end{aligned}$$

which result in the following integrations

$$\textbf{(1)}\ \int_{-\infty}^{\infty} f_z(x - ax + au)du = \frac{1}{a}$$

$$\textbf{(2)}\ \int_{-\infty}^{\infty} u\ f_z(x - ax + au)du = \frac{1}{a}\int_{-\infty}^{\infty} (x - (x-y)/a)\ f_z(y)dy = \frac{a-1}{a^2}x + \frac{\mu h}{a^2}$$

$$\textbf{(3)}\ \int_{-\infty}^{\infty} u^2 f_z(x - ax + au)du = \frac{1}{a}\int_{-\infty}^{\infty} (x - (x-y)/a)^2 f_z(y)dy = \frac{\sigma^2 h}{a^3} + \frac{1}{a}\left(\frac{a-1}{a}x + \frac{\mu h}{a}\right)^2$$

The equation becomes

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda\left\{\frac{1}{a} - \left(\frac{a-1}{a^2}x + \frac{\mu h}{a^2}\right)\frac{\partial}{\partial x} + \tfrac{1}{2}\left\{\frac{\sigma^2 h}{a^3} + \frac{1}{a}\left(\frac{a-1}{a}x + \frac{\mu h}{a}\right)^2\right\}\frac{\partial^2}{\partial x^2} + \ldots\right\} f_{S(t)|S(t_0)}(x \mid x_0) - \lambda\ f_{S(t)|S(t_0)}(x \mid x_0)$$

letting $a = 1 - \alpha h$ and $\lambda = 1/h$

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \frac{1}{h}\left\{\frac{1}{1-\alpha h} - \left(\frac{\mu h - \alpha x h}{1-\alpha h^2}\right)\frac{\partial}{\partial x} + \tfrac{1}{2}\left(\frac{\sigma^2 h + (\mu h - \alpha x h)^2}{(1-\alpha h)^3}\right)\frac{\partial^2}{\partial x^2} + o(h)\right\} f_{S(t)|S(t_0)}(x \mid x_0) - \frac{1}{h} f_{S(t)|S(t_0)}(x \mid x_0)$$

and collecting terms and canceling the factors in

$$\begin{aligned}&\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0)\\ &\quad= \frac{\alpha}{1-\alpha h} f_{S(t)|S(t_0)}(x \mid x_0) - \left(\frac{\mu - \alpha x}{1-\alpha h^2}\right)\frac{\partial}{\partial x} f_{S(t)|S(t_0)}(x \mid x_0) + \tfrac{1}{2}\left(\frac{\sigma^2 + h(\mu - \alpha x)^2}{(1-\alpha h)^3}\right)\frac{\partial^2}{\partial x^2} f_{S(t)|S(t_0)}(x \mid x_0) + \frac{o(h)}{h}\end{aligned}$$

As $h \to 0$, $o(h)/h$ vanishes leaving

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$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \alpha f_{S(t)|S(t_0)}(x \mid x_0) - (\mu - \alpha x) \frac{\partial}{\partial x} f_{S(t)|S(t_0)}(x \mid x_0) + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{S(t)|S(t_0)}(x \mid x_0)$$

Factorizing the first order diffusion differential

$$\left(-\alpha + (\mu - \alpha x) \frac{\partial}{\partial x} \right) f_{S(t)|S(t_0)}(x \mid x_0) = \frac{\partial}{\partial x} \left((\mu - \alpha x) \quad f_{S(t)|S(t_0)}(x \mid x_0) \right)$$

results in the forward equation of the Ornstein-Uhlenbeck process in the conventional format

$$\frac{\partial}{\partial t} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} f_{S(t)|S(t_0)}(x \mid x_0) - \frac{\partial}{\partial x} \left((\mu - \alpha x) \quad f_{S(t)|S(t_0)}(x \mid x_0) \right)$$

Limiting Form of Backward Integro-Differential Equation of a Modulated Additive Compound Poisson Process

From the backward integro-differential equation of a modulated additive compound Poisson process

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \int_{-\infty}^{\infty} f_z(y) f_{S(t)|S(t_0)}(x \mid a x_0 + y) dy - \lambda \; f_{S(t)|S(t_0)}(x \mid x_0)$$

making the change of variable $x_0 + u = a\,x_0 + y$ to put the expression in a form that can be expanded easily, with change of variable differential $du = dy$ and inverse function $y = u + (1 - a)\,x_0$

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \int_{-\infty}^{\infty} f_z(u + x_0(1-a)) f_{S(t)|S(t_0)}(x \mid x_0 + u) du - \lambda \; f_{S(t)|S(t_0)}(x \mid x_0)$$

Expanding the backward argument of the transition density within the convolution integral in a Taylor series

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \int_{-\infty}^{\infty} f_z(u + x_0(1-a)) \left\{ 1 + u \frac{\partial}{\partial x_0} + \tfrac{1}{2} u^2 \frac{\partial^2}{\partial x_0^2} + \ldots \right\} f_{S(t)|S(t_0)}(x \mid x_0) du - \lambda \; f_{S(t)|S(t_0)}(x \mid x_0)$$

Noting the distribution of the scaled and shifted basis variable is $Z \sim N(\mu h, \sigma^2 h)$, the variable of the integrated basis function is shifted downwards by $(1 - a)\,x_0$

$$\begin{aligned} Z - x_0(1-a) &\sim N(\mu h, \sigma^2 h) - (1-a)x_0 \\ &\sim N\left(\mu h - (1-a)x_0, \;\; \sigma^2 h\right) \end{aligned}$$

and gives the following integral evaluations

(1) $\int_{-\infty}^{\infty} f_z\left(u + x_0(1-a)\right) du = 1$

(2) $\int_{-\infty}^{\infty} u \;\; f_z\left(u + x_0(1-a)\right) du = \int_{-\infty}^{\infty} \left(y - (1-a)x_0\right) f_z(y) dy = \mu h - (1-a)x_0$

(3) $\int_{-\infty}^{\infty} u^2 f_z\left(u + x_0(1-a)\right) du = \int_{-\infty}^{\infty} \left(y - (1-a)x_0\right)^2 f_z(y) dy = \sigma^2 h + [\mu h - (1-a)x_0]^2$

so that the terms $\lambda f_{S(t)|S(t_0)}(x \mid x_0)$ cancel and the equation becomes

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \left\{ \{\mu h - (1-a)x_0\} \frac{\partial}{\partial x_0} + \tfrac{1}{2}[\sigma^2 h + \{\mu h - (1-a)x_0]^2\} \frac{\partial^2}{\partial x_0^2} + \ldots \right\} f_{S(t)|S(t_0)}(x \mid x_0)$$

Letting $a = 1 - \alpha h$

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda \left\{ \{\mu h - \alpha h x_0\} \frac{\partial}{\partial x_0} + \tfrac{1}{2}[\sigma^2 h + \{\mu h - \alpha h x_0]^2\} \frac{\partial^2}{\partial x_0^2} + \ldots \right\} f_{S(t)|S(t_0)}(x \mid x_0)$$

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and the Poisson intensity proportional to the inverse of the span length $\lambda = 1 / h$

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \frac{1}{h}\left\{\{\mu h - \alpha h x_0\}\frac{\partial}{\partial x_0} + \tfrac{1}{2}[\sigma^2 h + \{\mu h - \alpha h x_0]^2\}\frac{\partial^2}{\partial x_0^2} + \ldots\right\} f_{S(t)|S(t_0)}(x \mid x_0)$$

gives

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \left\{(\mu - \alpha x_0)\frac{\partial}{\partial x_0} + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2}\right\} f_{S(t)|S(t_0)}(x \mid x_0) + \frac{o(h)}{h}$$

Letting $h \to 0$ so that the $o(h) / h \to 0$ term vanishes gives the backward equation of the Ornstein-Uhlenbeck process

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} f_{S(t)|S(t_0)}(x \mid x_0) + (\mu - \alpha x_0)\frac{\partial}{\partial x_0} f_{S(t)|S(t_0)}(x \mid x_0)$$

Backward Integro-Differential Equation of an Option on a Modulated Additive Compound Poisson Process

Transforming the backward equation of the integro-differential equation of the transition density for a modulated additive compound Poisson process to that of the option premium results in

$$rV(k,t \mid x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t \mid x_0,t_0) = \lambda \int_{-\infty}^{\infty} f_{zh}(y) V(k,t \mid a x_0 + y, t_0) dy - \lambda\, V(k,t \mid x_0,t_0)$$

Limiting Form of Integro-Differential Equation of Put and Call Option Premia for Modulated Additive Compound Poisson Diffusion

From the limiting form of the backward equation, that in the option premium becomes

$$rV(k,t \mid x_0,t_0) - \frac{\partial}{\partial t_0} V(k,t \mid x_0,t_0) = (\mu - \alpha x_0)\frac{\partial}{\partial x_0} V(k,t \mid x_0,t_0) + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_0^2} V(k,t \mid x_0,t_0)$$

or

$$rV - \frac{\partial V}{\partial t_0} = \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x_0^2} + (\mu - \alpha x_0)\frac{\partial V}{\partial x_0} \qquad \text{where } V = V(k,t \mid x_0,t_0)$$

as $h \to 0$.

Linear suppositions follow from the same theory as that for ordinary unit root compound Poisson diffusion.

Summary Tables of Main Equations of FOUNDATIONS II of BOOK 1

forward (differential) recurrence in diffusion variable

backward (differential) recurrence in diffusion variable

differential form of diffusion equation

integral form of diffusion equation

diffusion transition distribution and density

forward (differential) recurrence in transition density

backward (differential) recurrence in transition density

forward equation

backward equation

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Table 19: Forward differential recurrence in diffusion variable

Form of compound Poisson diffusion	Forward differential recurrence in diffusion variable	
additive	$S(t+dt)=[S(t)+X_t h]dN(t)+S(t)[1-dN(t)]+o(dt)$	
multiplicative	$S(t+dt)=S(t)^{dN(t)}Y_t^{\,hdN(t)}S(t)^{1-dN(t)}e^{o(dt)}$	$S(t+dt)=S(t)Y_t^{\,hdN(t)}e^{o(dt)}$
additive, modulated	$S(t+dt)=[aS(t)+X_t h]dN(t)+S(t)[1-dN(t)]+o(dt)$	
multiplicative, modulated	$S(t+dt)=S(t)^{adN(t)}Y_t^{\,hdN(t)}S(t)^{1-dN(t)}e^{o(dt)}$	$S(t+dt)=S(t)^{a}Y_t^{\,hdN(t)}e^{o(dt)}$

$$S(t+dt)=[S(t)+X(t+dt)h]dN(t)+S(t)[1-dN(t)]+o(dt)$$
$$S(t+dt)=S(t)^{dN(t)}Y(t+dt)^{hdN(t)}S(t)^{1-dN(t)}e^{o(dt)}$$
$$S(t+dt)=[aS(t)+X(t+dt)h]dN(t)+S(t)[1-dN(t)]+o(dt)$$
$$S(t+dt)=S(t)^{adN(t)}Y(t+dt)^{hdN(t)}S(t)^{1-dN(t)}e^{o(dt)}$$

where X has the same definition here as the basis function z of PART II of BOOK 3 and Y is related to X by the transformation $X_t=\log Y_t$, for example, if $X_t=\log Y_t$ is normal then $Y_t=\exp(X_t)$ is log-normal (an exponential Gaussian)

$$X_t h=\log Y_t^{\,h}\sim N(\mu h,\sigma^2 h) \qquad Y_t^{\,h}=\exp(X_t h)\sim \log N(\mu h,\sigma^2 h)$$

The forward differential recurrence formulae can alternatively be stated, net of o(dt), as

Table 20: Forward differential recurrence in diffusion variable

Form of compound Poisson diffusion	Forward differential recurrence in diffusion variable
additive	$S(t+dt)=\begin{cases}S(t)+X(t+dt)h & \text{if} \quad z_t dt=1\\ S(t) & \text{if} \quad z_t dt=0\end{cases}$
multiplicative	$S(t+dt)=\begin{cases}S(t)Y(t+dt)^h & \text{if} \quad z_t dt=1\\ S(t) & \text{if} \quad z_t dt=0\end{cases}$
additive, modulated	$S(t+dt)=\begin{cases}aS(t)+X(t+dt)h & \text{if} \quad z_t dt=1\\ S(t) & \text{if} \quad z_t dt=0\end{cases}$
multiplicative, modulated	$S(t+dt)=\begin{cases}S(t)^a Y(t+dt)^h & \text{if} \quad z_t dt=1\\ S(t) & \text{if} \quad z_t dt=0\end{cases}$

Table 21: Backward differential recurrence in diffusion variable

Form of compound Poisson diffusion	Backward differential recurrence in diffusion variable
additive	$S(t)=\Big(S(t+dt)-X(t)h\Big)dN(t)+S(t+dt)[1-dN(t)]+o(dt)$
multiplicative	$S(t)=S(t+dt)^{dN(t)}Y(t)^{-hdN(t)}S(t+dt)^{1-dN(t)}e^{o(dt)}$
additive, modulated	$S(t)=\left(\dfrac{S(t+dt)-X(t)h}{a}\right)dN(t)+S(t+dt)[1-dN(t)]+o(dt)$
multiplicative, modulated	$S(t)=\sqrt[a]{S(t+dt)^{dN(t)}Y(t)^{hdN(t)}}\,S(t+dt)^{1-dN(t)}e^{o(dt)}$

Table 22: resolved diffusion equation conditional on the Poisson generator

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Form of compound Poisson diffusion	Resolved diffusion equation conditional on the Poisson generator
additive	$S(t) = S(0) + \sum_{j=1}^{N(t)} X_{jh} h$
multiplicative	$S(t) = S(0) \prod_{j=1}^{N(t)} Y_{jh}{}^{h} \qquad S(t) = S(0) \exp\left(\sum_{j=1}^{N(t)} (\log Y_{jh}) h \right)$
additive, modulated	$S(t) = a^{N(t)} S(0) + \sum_{j=1}^{N(t)} a^{N(t)-j} X_{jh} h$
multiplicative, modulated	$S(t) = S(0)^{a^{N(t)}} \prod_{j=1}^{N(t)} Y_{jh}{}^{h a^{N(t)-j}} \qquad S(t) = \exp\left(a^{N(t)} \log S(0) + \sum_{j=1}^{N(t)} a^{N(t)-j} (\log Y_{jh}) h \right)$

where $S(0) = S(t_0) \quad t_0 \le 0 < t_1$ (0 being the value of the backward time variable)

Table 23: resolved diffusion equation conditional on the Poisson generator

Form of compound Poisson diffusion	Resolved diffusion equation conditional on the Poisson generator	
additive	$S(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(S(0) + \sum_{j=1}^{n} X_{jh} h \right)$	
multiplicative	$S(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(S(0) \prod_{j=1}^{n} Y_{jh}{}^{h} \right)$	**in product form**
	$S(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \exp\left(\log S(0) + \sum_{j=1}^{n} (\log Y_{jh}) h \right)$	**in sum form**
additive, modulated	$S(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(a^n S(0) + \sum_{j=1}^{n} a^{n-j} X_{jh} h \right)$	
multiplicative, modulated	$S(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(S(0)^{a^n} \prod_{j=1}^{n} Y_{jh}{}^{h a^{n-j}} \right)$	**in product form**
	$S(t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \exp\left(a^n \log S(0) + \sum_{j=1}^{n} a^{n-j} (\log Y_{jh}) h \right)$	**in sum form**

where $S(0) = S(t_0) \quad t_0 \le 0 < t_1$ (0 being the value of the backward time variable)

Table 24: Forward integro differential equation of compound Poisson processes

Form of compound Poisson diffusion	Forward differential recurrence in diffusion variable
additive	$f_{S(t+dt)\|S(t_0)}(x \mid x_0) = \lambda dt \times f_{S(t)+X\|S(t_0)}(x \mid x_0) + (1 - \lambda dt) \times f_{S(t)\|S(t_0)}(x \mid x_0) + o(dt)$
multiplicative	$f_{S(t+dt)\|S(t_0)}(x \mid x_0) = \lambda dt \times f_{S(t)\times X\|S(t_0)}(x \mid x_0) + (1 - \lambda dt) \times f_{S(t)\|S(t_0)}(x \mid x_0) + o(dt)$
additive, modulated	$f_{S(t+dt)\|S(t_0)}(x \mid x_0) = \lambda dt \times f_{aS(t)+X\|S(t_0)}(x \mid x_0) + (1 - \lambda dt) \times f_{S(t)\|S(t_0)}(x \mid x_0) + o(dt)$
multiplicative, modulated	$f_{S(t+dt)\|S(t_0)}(x \mid x_0) = \lambda dt \times f_{S(t)^a \times X\|S(t_0)}(x \mid x_0) + (1 - \lambda dt) \times f_{S(t)\|S(t_0)}(x \mid x_0) + o(dt)$

Table 25: Backward integro differential equation of compound Poisson processes

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Form of compound Poisson diffusion	Backward differential recurrence in transition density
additive	$f_{S(t)\|S(t_0+dt)}(x\mid x_0)=\lambda dt\times f_{S(t)\|S(t_0)-X}(x\mid x_0)+(1-\lambda dt)\times f_{S(t)\|S(t_0)}(x\mid x_0)+o(dt)$
multiplicative	$f_{S(t)\|S(t_0+dt)}(x\mid x_0)=\lambda dt\times f_{S(t)\|S(t_0)\div X}(x\mid x_0)+(1-\lambda dt)\times f_{S(t)\|S(t_0)}(x\mid x_0)+o(dt)$
additive, modulated	$f_{S(t)\|S(t_0+dt)}(x\mid x_0)=\lambda dt\times f_{S(t)\|[S(t_0)-X]\div a}(x\mid x_0)+(1-\lambda dt)\times f_{S(t)\|S(t_0)}(x\mid x_0)+o(dt)$
multiplicative, modulated	$f_{S(t)\|S(t_0+dt)}(x\mid x_0)=\lambda dt\times f_{S(t)\|S(t_0\div X)^{1/a}}(x\mid x_0)+(1-\lambda dt)\times f_{S(t)\|S(t_0)}(x\mid x_0)+o(dt)$

Table 26: Forward integro differential equation of compound Poisson processes

Form of compound Poisson diffusion	Forward integro-differential equation
additive	$\frac{\partial}{\partial t}f_{S(t)\|S(t_0)}(x\mid x_0)=\lambda\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)\|S(t_0)}(x-y\mid x_0)dy-\lambda\,f_{S(t)\|S(t_0)}(x\mid x_0)$
multiplicative	$\frac{\partial}{\partial t}f_{S(t)\|S(t_0)}(x\mid x_0)=\lambda\int_{-\infty}^{\infty}\frac{1}{y}f_X(y)\,f_{S(t)\|S(t_0)}(x\div y\mid x_0)dy-\lambda\,f_{S(t)\|S(t_0)}(x\mid x_0)$
additive, modulated	$\frac{\partial}{\partial t}f_{S(t)\|S(t_0)}(x\mid x_0)=\frac{\lambda}{a}\int_{-\infty}^{\infty}f_z(y)\,f_{S(t)\|S(t_0)}\big((x-y)/a\mid x_0\big)dy-\lambda\,f_{S(t)\|S(t_0)}(x\mid x_0)$
multiplicative, modulated	$\frac{\partial}{\partial t}f_{S(t)\|S(t_0)}(x\mid x_0)=\frac{\lambda}{a}\int_{-\infty}^{\infty}\frac{1}{x}(x\div y)^{1/a}\,f_X(y)\,f_{S(t)\|S(t_0)}\big((x\div y)^{1/a}\mid x_0\big)dy-\lambda\,f_{S(t)\|S(t_0)}(x\mid x_0)$

Table 27: Backward integro differential equation of compound Poisson processes

Form of compound Poisson diffusion	Backward integro-differential equation
additive	$-\frac{\partial}{\partial t_0}f_{S(t)\|S(t_0)}(x\mid x_0)=\lambda\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)\|S(t_0)}(x\mid x_0+y)dy-\lambda\,f_{S(t)\|S(t_0)}(x\mid x_0)$
multiplicative	$-\frac{\partial}{\partial t_0}f_{S(t)\|S(t_0)}(x\mid x_0)=\lambda\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)\|S(t_0)}(x\mid x_0\times y)dy-\lambda\,f_{S(t)\|S(t_0)}(x\mid x_0)$
additive, modulated	$-\frac{\partial}{\partial t_0}f_{S(t)\|S(t_0)}(x\mid x_0)=\lambda\int_{-\infty}^{\infty}f_z(y)\,f_{S(t)\|S(t_0)}\big(x\mid a\,x_0+y\big)dy-\lambda\,f_{S(t)\|S(t_0)}(x\mid x_0)$
multiplicative, modulated	$-\frac{\partial}{\partial t_0}f_{S(t)\|S(t_0)}(x\mid x_0)=\lambda\int_{-\infty}^{\infty}f_X(y)\,f_{S(t)\|S(t_0)}\big(x\mid {x_0}^{a}\times y\big)dy-\lambda\,f_{S(t)\|S(t_0)}(x\mid x_0)$

Table 28: Backward integro differential equation of option premium on compound Poisson processes

Form of compound Poisson diffusion	Backward integro-differential equation
additive	$rV(k,t\mid x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t\mid x_0,t_0)=\lambda\int_{-\infty}^{\infty}f_X(y)V(k,t\mid x_0+y,t_0)dy-\lambda V(k,t\mid x_0,t_0)$
multiplicative	$rV(k,t\mid x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t\mid x_0,t_0)=\lambda\int_{-\infty}^{\infty}f_X(y)V(k,t\mid x_0\times y,t_0)dy-\lambda V(k,t\mid x_0,t_0)$
additive, modulated	$rV(k,t\mid x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t\mid x_0,t_0)=\lambda\int_{-\infty}^{\infty}f_X(y)V(k,t\mid ax_0+y,t_0)dy-\lambda\,V(k,t\mid x_0,t_0)$
multiplicative, modulated	$rV(k,t\mid x_0,t_0)-\frac{\partial}{\partial t_0}V(k,t\mid x_0,t_0)=\lambda\int_{-\infty}^{\infty}f_X(y)V(k,t\mid {x_0}^{a}\times y,t_0)dy-\lambda V(k,t\mid x_0,t_0)$

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RELATION BETWEEN COMPOUNDING SUMS, CONTINUOUS DIFFUSIONS, AND STOCHASTIC VOLATILITY PROCESSES

There is a strong linkage between random walks, diffusions, and the random sum processes. I have classed each of these forms of process under the general term first order diffusion due to the fact that each is the cumulative sum of component probabilities whereby the cumulative sum of these components results in a dispersion of probability over time. Hence, each are first order diffusions and, synonymously, Markov processes. Further, there is a strong linkage to, and between, the high frequency compound sums and stochastic volatility processes. It was shown that adding independent random variables is the same as summing the cumulants of these random variables. As such, the summation of a sequence of the independent random variables in a diffusion basis is alternatively described by the summation of the basis cumulants, which is essentially the logic behind what the forward equations describe and how they were solved. The basis components of the compound Poisson process, the secondary distribution, are normal random variables for which only the first two cumulants, the mean and the variance, are non-zero and the conventional parameterization of these normal components is such that the two-parameters coincide with the two cumulants.

The random sum processes considered have had Poisson discrete and stochastic additions[65], $N(t) \sim P(\lambda t)$, of normal diffusion basis components, $X \sim N(\mu,\sigma^2)$, and are alternatively described as models with stochastic parameters such that the cumulants of the process are all summing together stochastically by discrete counts at the same rate (with perfect correlation)

$$S(t) \sim N\left(\int_0^t \mu P(\lambda)dy,\ \int_0^t \sigma^2 P(\lambda)dy\right) \sim N\left(\mu P(\lambda t),\ \sigma^2 P(\lambda t)\right)$$

ignoring initial values (which is simply a constant $S(0) = s_0$ added to the mean). Because the cumulants are summing stochastically at the same rate, the ratios between the various cumulants are held fixed and are therefore time invariant. Since the basis components are normal with only two conditional non zero cumulants, this characteristic is saying that ratio of the mean to the variance, the signal to noise ratio (SNR), is held fixed

$$\frac{\mu N(t)}{\sigma^2 N(t)} = \frac{\mu}{\sigma^2}$$

The fixed mean to variance ratio μ / σ^2 is also a feature of an ordinary constant parameter Brownian motion (ignoring initial values). The significance of this is most pertinent when it is remembered that the definition of a Brownian motion is a Gaussian process whereby the cumulants sum linearly with time

$$S(t) \sim N\left(\int_0^t \mu dy, \int_0^t \sigma^2 dy\right) \sim N(\mu t, \sigma^2 t)$$

that is, the mean and variance are both proportional to t, and for which the SNR is similarly

$$\frac{\mu t}{\sigma^2 t} = \frac{\mu}{\sigma^2}$$

In effect, for compound Poisson diffusions, the time variable over which the diffusion occurs is essentially replaced with a Poisson process. This is alternatively equivalent to replacing the two infinitesimal parameters of the Brownian motion with two perfectly correlated scaled Poisson distributed random variables, i.e., with a Poisson basis function. The infinitesimal parameters of the Brownian motion μ and σ^2 are replaced with $\mu P(\lambda)$ and $\sigma^2 P(\lambda)$ where $P(\lambda)$ is a time invariant Poisson random variable and a component of a Poisson basis.

In the case of a high frequency Poisson process whereby the discrete additions occur almost instantaneously, and the Poisson spectrum converges and becomes flat and continuous, the Poisson process converges to a Brownian

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[65] The moment generating function of the Poisson distribution is $M_{dN(t)}(\theta) = e^{\lambda(e^{\theta}-1)dt}$ and the moment generating function of the continuous t-fold convolution is then

$$M_{\int_0^t dN(y)}(\theta) = M_{N(t)}(\theta) = e^{\lambda t(e^{\theta}-1)}$$

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motion, resulting in the **discrete** and stochastic additions becoming indistinguishable from **continuous** and stochastic additions of the normal basis components, a Gaussian Hilbert space.

Therefore, as the Poisson process converges to a Brownian motion, the normals begin to sum at a rate determined by this Brownian motion. This is alternatively described by the summation of the normal cumulants at this rate,

$$S(t) \sim N\left(\int_0^t \mu N(\lambda,\lambda)dy, \int_0^t \sigma^2 N(\lambda,\lambda)dy\right) \sim N\left(\mu N(\lambda t,\lambda t),\ \sigma^2 N(\lambda t,\lambda t)\right)$$

the rate determined by the Brownian motion. By the same arguments, this is essentially a Brownian motion for which the time variable over which the diffusion occurs is replaced with another Brownian motion. This is alternatively equivalent to replacing the infinitesimal parameters of the Brownian motion with perfectly correlated unit signal to noise ratio ($\lambda t / \lambda t = 1$) normally distributed random variables. The infinitesimal parameters of the Brownian motion μ and σ^2 are replaced with $\mu N(\lambda,\lambda)$ and $\sigma^2 N(\lambda,\lambda)$ where $N(\lambda,\lambda)$ is a time-invariant unit-SNR normal random variable.

Stochastic volatility models, on the other hand, usually only model the stochastic sum of the second cumulant, while leaving the mean (and other cumulants if they are non-zero) as a deterministic linear function of time, such as the specification

$$S(t) \sim N\left(\int_0^t \mu dy, \int_0^t \sigma^2 N(a,b)dy\right) \sim N\left(\mu t,\ \sigma^2 N(at,bt)\right)$$

or the variance gamma process[66]

$$S(t) \sim N\left(\int_0^t \mu dy, \int_0^t \sigma^2 N^2(0,b)dy\right) \sim N\left(\mu t,\ \sigma^2 b^2 \chi_t^2\right)$$

where χ^2_t is a chi-square random variable on t degrees of freedom. Since

$$\chi_t^2 \sim \Gamma(t/2, 2)$$

and by the Central Limit Theorem $\chi^2_t \sim \Gamma(t/2, 2) \to N(t, 2t)$, that is, the chi-square or gamma variance quickly becomes normal and as soon as it does, the above two models become equivalent i.e., the variance-gamma becomes a sub-model of that with the Brownian variance where, before scaling, the variance of the variance is always twice the mean of the variance;

$$S(t) \sim N\left(\int_0^t \mu dy, \int_0^t \sigma^2 N^2(0,b)dy\right) \sim N\left(\mu t,\ \sigma^2 b^2 \chi_t^2\right) \to N\left(\mu t,\ \sigma^2 b^2 N(t,2t)\right)$$

or

$$S(t) \sim N\left(\mu t,\ N(\sigma^2 b^2 t, 2\sigma^4 b^4 t)\right)$$

both of these processes further converge to ordinary Brownian motions. Stochastic volatility models, often defined by the moment generating function or characteristic function, seldom have a pre-convergent analytic density function, or equivalently, seldom have a closed form inversion of the moment generating function or characteristic function and only do so when it is the only the mean that is stochastic such as

$$S(t) \sim N\left(\int_0^t \mu N(a,b)dy, \int_0^t \sigma^2 dy\right) \sim N\left(\mu N(at,bt),\ \sigma^2 t\right) \sim N\left(\mu at, (\sigma^2 + \mu^2 b)t\right)$$

does, as does the diffusion

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[66] Madan, D. B. and Seneta, E., (1990) The Variance Gamma (V.G.) Model for Share Market Returns. *The Journal of Business*, Vol. 63, No. 4 (Oct., 1990), pp. 511-524.

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$$S(t) \sim N\left(s_0 + e^{-\alpha t}\int_0^t \mu e^{\alpha y} N(a,b)dy, \int_0^t \sigma^2 dy \right) \sim N\left(s_0 + \mu N\left(a\frac{1-e^{-\alpha t}}{\alpha}, b\frac{1-e^{-2\alpha t}}{2\alpha} \right), \sigma^2 t \right)$$

i.e.,

$$S(t) \sim N\left(s_0 + \mu a\frac{1-e^{-\alpha t}}{\alpha}, \left(\sigma^2 t + \mu^2 b\frac{1-e^{-2\alpha t}}{2\alpha} \right) \right)$$

which is proved by moment generating functions and follows from the fact that normals are additive.

It was shown that the Poisson process that stochastically sums the basis components converges to a Brownian motion, which is a continuous process so the components essentially sum continuously. Because exponential and gamma variables do not convolve for a heterogeneous scale parameter (or equivalently, Poisson variables with heterogeneous scaling) into a distribution that is easy to work with, it is difficult to introduce a correlation between the exponential times between Poisson events and generalize the unit-root exponential diffusion[67] for the time variable (which is, equivalently, a unit-root Poisson diffusion for the space variable). Such a generalization of the diffusion in the time variable includes introducing a non-unit-root as well as increasing the order of the diffusion. To bypass this difficulty with not being able to convolve for heterogeneous scale, and assuming an almost instantaneous convergence due to an assumed high rate intensity, as a very reasonable approximation one may consider starting out with the desired limiting form of the primary process for the rate at which the cumulants of the basis sum. This would allow the processes to model the cumulative sum of distribution cumulants to include processes from FOUNDATIONS I in addition to the discrete count diffusions like the Poisson process,

$$S(t) \sim N\left(\int_0^t \mu dX_y, \int_0^t \sigma^2 dX_y \right) \sim N\left(\mu X_t, \sigma^2 X_t \right)$$

where X_t is the integral solution to the differential process dX_t for any of the diffusions from FOUNDATIONS I , in particular, any of the diffusions in the table of ito-transform and convolution ito-transform pairs. Such generalizations include generalization of a Brownian motion for which the cumulants of the diffusion basis components sum according to an Ornstein-Uhlenbeck process or a square-root process. The option premia would then be obtained by integrating out the generating primary process over the probability weighted limited moments of the conditional option premia.

A desirable feature of such models, consistent with some observations of the financial markets[68], is that they have the potential to effectively account for the phenomena of **clustering** and **persistency.** A diffusion whereby the basis components each sum together according to a stochastic process, such as the compound Poisson diffusion, by definition, means that the cumulants sum randomly at different rates at different times but with the important feature that the cumulants at each order each change synchronously with one another. For a compound Poisson diffusion, the Poisson process that sums the cumulants is a unit-root diffusion (a linear aggregation of a Poisson basis) and therefore trends and tends to make long Brownian excursions above and below the mean drift line. Therefore, the rate at which the cumulants of the basis components sum is heavier over some periods of time (when the Poisson generator is drifting above the mean level) and lighter over others (when the Poisson generator is drifting below the mean level). The times over which there is a heavy rate of aggregation of the cumulants, implying a high level of clustering, is persistent to the extent of the excursive properties of the Poisson process and unit-root diffusion.

If one extends the generating process that sums the cumulants to a non-unit-root diffusion, say an Ornstein-Uhlenbeck or square-root process for example, the degree of the trend propensity in the generator and the resulting degree of persistency and clustering in the aggregate diffusion variable can be modulated.

Such diffusions could further be generalized to, say, a non-unit-root diffusion that sums over time with regard to one of these diffusions. An example would be an Ornstein-Uhlenbeck process for which the basis components sum proportionately with regard to another Ornstein-Uhlenbeck process, the diffusion for the time variable. An in depth analysis of such a process is omitted but the general theme is consistent with the discussion regarding the compound Poisson diffusions and generalizing how the basis components of such a diffusion sum, such as the calculus of diffusion of arbitrary order contained in the last book of this thesis, BOOK 3 .

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[67] which becomes normal quickly due to the central limit theorem for exponential and gamma random variables.
[68] Todorov, V., (2007) *Jump Processes in Finance: Modeling, Simulations, Inference and Pricing*. PhD Dissertation, Department of Economics Duke University.

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DISCRETE NON-POISSON COMPOUNDING SUMS

Motivation

A requirement for modeling the transaction times themselves is that the distribution remains discrete. While an independent increments assumption about the discrete generator may be found to be reasonable and an assumption that should be retained in the modeling, it might be found that the mean-variance relationship of the counting distribution is out of proportion for the Poisson process to be suitable, that is, an observed sequence of transaction times that imply counts whereby the mean-variance ratio is statistically different from one. The alternatives for the basis of the generator are the binomial and negative binomial distributions. Either the mean to variance ratio is less than one and the mean is greater than the variance or it is more than one and the variance is greater than the mean.

Negative Binomial Generator – The Negative Binomial Process

It is usually the case that the variance is greater than the mean. This is referred to as excess dispersion. In the case of excess dispersion, the basis of the discrete stochastic generator can be defined as a type 1 negative binomial variable, $NB(m, q)$, the probability function for which is[69]

$$p_n = \frac{(m)_n}{n!}(1-q)^n q^m \quad n = 0,1,2,\ldots$$

where $(m)_n = m(m+1)(m+2)\ldots(m+n-1) = \Gamma(m+n)/\Gamma(m)$ is Pochammer's function which is a rising factorial (of the integer m) limited to n terms. The type I distribution has the intuitive definition of the number of failures, n, in order to obtain m successes where the probability from trial to trial remains fixed. The moment generating function of this distribution is

$$M_{dN(t)}(\theta) = q^m\left(1-(1-q)e^{\theta}\right)^{-m}$$

and the moment generating function of the continuous t-fold convolution $M_{\int_0^t dN(y)}(\theta)$ being

$$M_{N(t)}(\theta) = q^{mt}\left(1-(1-q)e^{\theta}\right)^{-mt}$$

which is a type I negative binomial distribution with the parameters mt and q, $NB(mt, q)$.

The random sum process, which has this discrete (type I) negative binomial monotone diffusion as the compound sum generator, has negative binomial discrete and stochastic additions $N(t) \sim NB(mt, q)$ of normal diffusion basis components $X \sim N(\mu, \sigma^2)$. The transition distribution is

$$S(t) \sim N\left(s_0 + \int_0^t \mu\, NB(m,q)dy,\ \int_0^t \sigma^2 NB(m,q)dy\right) \sim N\left(s_0 + \mu\, NB(mt,q),\ \sigma^2 NB(mt,q)\right)$$

This is the transition distribution of a Brownian motion for which the time variable over which the diffusion occurs is replaced with the (type I) negative binomial process, the integration of a negative binomial basis for the generator and is equivalent to replacing the two infinitesimal parameters of the Brownian motion with two perfectly correlated scaled (type I) negative binomial distributed random variables, i.e., with a negative binomial basis function. The infinitesimal parameters of the Brownian motion μ and σ^2 are essentially replaced with $\mu NB(m,q)$ and $\sigma^2 NB(m,q)$ where $NB(m,q)$ is a time invariant type I negative binomial random variable and a component of a negative binomial basis function. The mean of the generator, the individual negative binomial distribution, is m times the odds ratio of success at any trail, $E[dN(t)] = mq/(1-q)$ and the variance is $1/(1-q) > 1$ of this value, $Var[dN(t)] = mq/(1-q)^2$ so that the mean and the variance of the t-fold convolution are $E[N(t)] = mtq/(1-q)$ and $Var[N(t)] = mtq/(1-q)^2$, both of which are proportional to time t. Noting importantly that all higher order cumulants are non-zero and also proportional to time t. The negative binomial process $N(t) \sim NB(mt, q)$ converges to a Brownian motion $N(t) \to N(mtq/(1-q), mtq/(1-q)^2)$ and the transition distribution of the compound sum then converges to a Brownian motion in the limit

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[69] This is an equivalent to $p_n = \binom{n+m-1}{n}(1-q)^n q^m \quad n = 0,1,2,\ldots$ where $\binom{n+m-1}{n} = (n+m-1)!/\left(n!(m-1)!\right)$.

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$$\lim_{t\to\infty} S(t) \sim N\left(s_0 + \mu N\left(mtq/(1-q), mtq/(1-q)^2\right),\ \sigma^2 N\left(mtq/(1-q), mtq/(1-q)^2\right)\right)$$
$$\to N\left(s_0 + \mu mtq/(1-q),\ \mu^2 mtq/(1-q)^2 + \sigma^2 mtq/(1-q)\right)$$

Binomial Generator – The Binomial Process

When the variance is smaller than the mean, the discrete generator of the random compound sum that models the transaction times can be modeled with a binomial distribution $B(m,q)$ which has probability function[70]

$$p_n = \frac{m^{(n)}}{n!}(1-q)^n q^{m-n} \quad n = 0,1,2,\ldots,m$$

where $m^{(n)} = m(m-1)(m-2)...(m-n+1) = \Gamma(m+1)/\Gamma(m-n+1) = m!/(m-n)!$ is a descending factorial (of the integer m) limited to n terms. This is the permutation of m items taken n at a time. The binomial distribution has the intuitive definition of the number of successes obtained from a fixed number of trials, m, where the probability from trial to trial remains fixed. The moment generating function of this distribution is

$$M_{dN(t)}(\theta) = (1-q+qe^{\theta})^m$$

and the moment generating function of the continuous t-fold convolution $M_{\int_0^t dN(y)}(\theta)$ being

$$M_{N(t)}(\theta) = (1-q+qe^{\theta})^{mt}$$

which is a binomial distribution with the parameters mt and q, $B(mt,q)$.

The random sum process, which has a discrete binomial monotone diffusion as the compound sum generator, has binomial discrete and stochastic additions $N(t) \sim B(mt,q)$ of normal diffusion basis components $X \sim N(\mu, \sigma^2)$ the transition distribution for which is

$$S(t) \sim N\left(s_0 + \int_0^t \mu B(m,q)dy,\ \int_0^t \sigma^2 B(m,q)dy\right) \sim N\left(s_0 + \mu B(mt,q),\ \sigma^2 B(mt,q)\right)$$

This is the transition distribution of a Brownian motion for which the time variable over which the diffusion occurs is replaced with the binomial process, the integration of a binomial basis for the generator and is equivalent to replacing the two infinitesimal parameters of the Brownian motion with two perfectly correlated binomial distributed random variables, i.e., with a binomial basis function. The infinitesimal parameters of the Brownian motion μ and σ^2 are, like that of Poisson and negative binomial generated diffusion, essentially replaced with $\mu B(m,q)$ and $\sigma^2 B(m,q)$ where $B(m,q)$ is a time invariant binomial random variable and a component of a binomial basis function. When $m = 1$ the diffusion has a discrete Bernoulli generator. When m is large and q is small, the Binomial process resembles a Poisson process for which rate intensity is the number of binomial trials times the probability of success at each trail, $\lambda = mq$, i.e., the mean of the right skew binomial. The compound Binomial sum, in this case, becomes equivalent to a compound Poisson sum. For an arbitrary the binomial compound sum, the mean of the basis of the generator, the individual binomial distribution, is m times the probability of success at any trail, $E[dN(t)] = mq$ and the variance is the mean times the probability of failure $1 - q < 1$, $Var[dN(t)] = mq(1-q)$ so that the mean and the variance of the generator resulting from the t fold convolution are $E[dN(t)] = mtq$ and $Var[dN(t)] = mtq(1-q)$, both of which are proportional to time t, as are all of the higher order cumulants. The binomial process $N(t) \sim B(mt,q)$ converges to a Brownian motion $N(t) \to N(mtq, mtq(1-q))$ and the transition distribution of the compound sum then converges to a Brownian motion in the limit

[70] this is an equivalent to $p_n = \binom{m}{n}(1-q)^n q^{m-n} \quad n = 0,1,2,\ldots,m$ where $\binom{m}{n} = m!/\left(n!(m-n)!\right)$

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$$\lim_{t\to\infty} S(t) \sim N\left(s_0 + \mu N\big(mtq, mtq(1-q)\big), \sigma^2 N\big(mtq, mtq(1-q)\big)\right)$$
$$\to N\left(s_0 + \mu mtq\,,\ \mu^2 mtq(1-q) + \sigma^2 mtq\right)$$

Further Theory on the Discrete Generator of a Compound Sum – Binomial, Poisson, and Negative Binomial

Each of these discrete processes, by the Central Limit Theorem of first order unit-root diffusion, have the same domain of attraction, a Gaussian distribution on the standardized variable, and hence each converge to Brownian motion. A Gaussian generator of a real and finite variance basis then further converges by the Central Limit Theorem to Brownian motion. Hence, as was shown with the transition distributions, each of the compound sums converge, whereby each of the higher order cumulants above the second become relatively less important (but never actually vanish entirely), leaving something 'near' Brownian motion. The limit theorem for the Poisson process

$$N(t) \sim P(\lambda t) \to N\big(\lambda t, \lambda t\big)$$

was given earlier in this part, the limit theorem for the Binomial distribution and the binomial process

$$N(t) \sim B(mt, q) \to N\big(mtq, mtq(1-q)\big)$$

is found in the second book on option replicating portfolios and the limit theorem for the negative binomial process

$$N(t) \sim NB(mt, q) \to N\big(mtq/(1-q), mtq/(1-q)^2\big)$$

is similar. The limit theorem of a Brownian motion compound sum of a normal basis whereby the variance of the variance vanishes

$$S(t) \sim N\left(s_0 + \mu N(at, bt), \sigma^2 N(at, bt)\right)$$
$$\sim GenN\left(s_0 + \mu at, (\sigma^2 a + \mu^2 b)t, 3\mu\sigma^2 bt, 3b\sigma^4 t\right)$$
$$\to GenN\left(s_0 + a\mu t, (\mu^2 b + \sigma^2 a)t, 0, 0\right)$$

where the second and third lines regard the random variable of the generalized Gaussian distribution with cumulants to order four. This was also given in FOUNDATIONS II following the Poisson process limit theorem.

As soon as the discrete generator converges to a Brownian motion, with either

i. a generator with infinitesimal mean greater than the infinitesimal variance (binomial),
ii. a generator with equal infinitesimal mean and infinitesimal variance (Poisson), or
iii. a generator with infinitesimal variance greater than the infinitesimal mean (negative binomial),

then one can model the process as a continuous Brownian motion with a generalized Gaussian diffusion basis - one with basis components that have non-zero higher order cumulants. The density and related option premia and other option statistics can each be obtained numerically by solving the backward equation or transformed backward equation subject to the relevant boundary conditions. Alternatively, one can assume further convergence is reached whereby the higher order diffusion effects diminish in importance until the diffusion becomes an ordinary normal Brownian motion, one with a Gaussian basis. The more sophisticated method of incorporating the higher order effects in the partially converged process of the generator

$$S(t) \sim N\left(s_0 + \mu N(at, bt), \sigma^2 N(at, bt)\right)$$
$$\sim N\left(s_0 + \mu at, (\sigma^2 a + \mu^2 b)t, 3\mu\sigma^2 bt, 3b\sigma^4 t\right)$$

is presumably more accurate. The derivation of the backward equation is found at the end of this topic.

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The difficulty in generalizing the Poisson process to a binomial or negative binomial process is that these do not have first passage times distributions with a convenient known statistical distribution, unlike the Poisson process for which it is a gamma variable composed of individual exponentials. Seeing that there is no closed form first passage time distribution of the discrete monotone diffusions for the generator of the binomial and negative binomial alternatives, it is not possible to deconvolve the transition distribution of the generator for the time between each renewal, transaction in this context, for a convenient distribution and density for which the likelihood of the transaction times can be written down and maximized with respect to the parameter set. On the other hand, one may believe that one is analytically fortunate that the Poisson process has exponential times between Poisson occurrences and see this as nothing more than a luxury. When it is not possible to model the transaction times directly due to a lack of an analytic waiting time distribution, the discrete counts themselves have to be modeled. This involves partitioning the times over which transactions occur into equally spaced intervals, of length h say, and obtaining the frequencies by counting how many transactions occur on each interval. The likelihood can then be written down in terms of the relevant count distribution for the time increment, binomial, Poisson, or negative binomial, which can then be maximized analytically. Each parameter then has an analytic maximum likelihood estimator apart from one parameter of the negative binomial process[71]. For the time increment of the j^{th} partition with extremities t and $t + h$, the relevant count distributions are based on the integrated basis of the generator over that partition,

(1) $S(t+h) - S(t) \sim N\left(\int_t^{t+h} \mu\, B(m,q)dy,\ \int_t^{t+h} \sigma^2 B(m,q)dy\right) \sim N\left(\mu\, B(mh,q),\ \sigma^2 B(mh,q)\right)$

so that the generator is $\int_t^{t+h} B(m,q)dy \sim B(mh,q)$ which has density function $p_n = \frac{(mh)^{(n)}}{n!}(1-q)^n q^{mh-n}$

(2) $S(t+h) - S(t) \sim N\left(\int_t^{t+h} \mu P(\lambda)dy,\ \int_t^{t+h} \sigma^2 P(\lambda)dy\right) \sim N\left(\mu P(\lambda h),\ \sigma^2 P(\lambda h)\right)$

so that the generator is $\int_t^{t+h} P(\lambda)dy \sim P(\lambda h)$ which has density function $p_n = e^{-\lambda h}\frac{(\lambda h)^n}{n!}$

(3) $S(t+h) - S(t) \sim N\left(\int_t^{t+h} \mu\, NB(m,q)dy,\ \int_t^{t+h} \sigma^2 NB(m,q)dy\right) \sim N\left(\mu\, NB(mh,q),\ \sigma^2 NB(mh,q)\right)$

so that the generator is $\int_t^{t+h} NB(m,q)dy \sim NB(mh,q)$ which has density function $p_n = \frac{(mh)_n}{n!}(1-q)^n q^{mh}$

where the rising and descending limited factorial functions are not here restricted to the integers but are defined for all positive reals. Such a method and process assumes time homogeneity and this thesis does not make any suggestions with regard to how to best choose the number, or the length, of the time partitions. If time homogeneity holds, it shouldn't matter, although there is a trade-off between obtaining an adequate number of parts to the frequency distribution and obtaining a suitable bin size for the counts. The compound sum processes for each of the three types of count process can be estimated and the likelihoods compared, by the chi-square likelihood ratio test, to choose the process that is most suited to an observed series. Since the modeling of exponential transaction times is exact and there is not the problem of having to bin the time variable, count occurrences, and then further bin the counts to create an empirical frequency distribution, the Poisson process can alternatively be estimated by this method based on counts and then compared with the results based upon the exact maximum likelihood procedure. The results should be very similar otherwise there is a problem with the method of binning the time variable or with the time homogeneity assumption. If the results are similar, then the method of binning of the time variable has produced reasonable results, comparable to modeling the renewals directly, and this suggests that the results regarding the binomial and negative binomial processes are also technically correct.

It is envisaged that

1) The binomial process is most suited to illiquid securities with sparse and infrequent trading since the distribution has a bounded support and low variation;

[71] Klugman, S. A., Panjer, H. H. and Willmot, G. E., (1998) *Loss Models*. John Wiley & Sons.

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2) the Poisson process should serve as a baseline process to model ordinary securities with ordinary or moderate liquidity since it is easy to apply with exact methods and is flexible to incorporate moderate departures of the level of dispersion from that of the mean;
3) the Negative Binomial process is most suitable for moderate to highly liquid securities since the distribution models excess dispersion and has the extra degree of freedom in the parameter set to adjust the distributional shape for higher moments while having an unbounded upper support.

Experience will show what the best approach is. It has been found, in modeling insurance data, that the possible discrete distributions of the basis of the counting process, in this context, the binomial, Poisson, and negative binomial distributions, sometimes do not fit the zeroes adequately. If this is found to be the case, the basis distribution of the generator can be generalized by modifying the zeroes[72]. The estimation proceeds by maximum likelihood and the calculation of option premia and related statistics proceed by summing out the relevant parameter in the ordinary formulae by some recursive method such as a do-while loop. Alternatively, the moment generating function of the compound sum can be inverted along the Bromwhich contour or by the inverse Fourier transform.

Variations in Rate Intensity

The other important generalization to modeling random compound sums with Poisson generators is to model variations in the rate intensity. This relates to incorporating time dependencies and the possible modeling of diurnal variations in intensity, for example, over trading hours where the propensity for trade varies throughout the day with regular periodic peaks and troughs. The most obvious periods of consideration for peaks are just after opening, just before and just after lunch, and just before close. The most important thing to get right, if there are found to be cycles or periodicities, is that they are in phase, i.e., that the function of the rate intensity correctly matches the intensity of the rate of transactions. One may alternatively consider bathtub functions (for open and close, if there are no mid-day effects).

In the context of binomial and negative binomial processes, rather than the rate intensity λ itself, it is the parameter m that is made to be a time dependent function. The transition distribution and the distribution of the increments are then expressed as time integrals of this function;

(1) The Poisson Process

For any differential or non-differential time increment,

$$\int_t^{t+h} P\big(\lambda(y)\big)\,dy \sim P\left(\int_t^{t+h} \lambda(y)dy\right) \sim P\big(\Lambda(t,t+h)\big) \qquad \text{where } \Lambda(t,t+h) = \int_t^{t+h} \lambda(y)dy$$

is the generator, as an integrated generator basis function, for the time interval $[t, t+h)$. $\Lambda(t, t+h)$ is the integrated Poisson intensity over that time interval. The probability function, and likelihood contribution, for the increment in the generator is

$$p_n = e^{-\Lambda(t,t+h)}\,\frac{\Lambda(t,t+h)^n}{n!}$$

The transition distribution for the transition interval $[0, t)$ is

$$S(t) \sim N\left(s_0 + \int_0^t \mu P\big(\lambda(y)\big)\,dy,\ \int_0^t \sigma^2 P\big(\lambda(y)\big)\,dy\right) \sim N\left(s_0 + \mu P\big(\Lambda(0,t)\big),\ \ \sigma^2 P\big(\Lambda(0,t)\big)\right)$$

and the distribution of a time increment $[t, t+h)$ is

$$S(t+h) - S(t) \sim N\left(\int_t^{t+h} \mu P\big(\lambda(y)\big)\,dy,\ \int_t^{t+h} \sigma^2 P\big(\lambda(y)\big)\,dy\right) \sim N\left(\mu P\big(\Lambda(t,t+h)\big),\ \ \sigma^2 P\big(\Lambda(t,t+h)\big)\right)$$

(2) The Binomial Process

For any differential or non-differential time increment,

[72] Klugman, S. A., Panjer, H. H. and Willmot, G. E., (1998) *Loss Models*. John Wiley & Sons.

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$$\int_t^{t+h} B(m(y),q)dy \sim B\left(\int_t^{t+h} m(y)dy, q\right) \sim B(M(t,t+h),q) \qquad \text{where } M(t,t+h)=\int_t^{t+h} m(y)dy$$

is the generator, as an integrated generator basis function, for the time interval $[t, t+h)$. $M(t, t+h)$ is the integrated binomial index over that time interval. The intensity, or instantaneous propensity to generate events, is $q \times m(t)$ and the integrated intensity is $q \times M(t, t+h)$ for the time interval $[t, t+h)$. The probability function, and likelihood contribution, for the increment in the generator is

$$p_n = \frac{(M(t,t+h))^{(n)}}{n!}(1-q)^n q^{M(t,t+h)-n}$$

The transition distribution for the transition interval [0, t) is

$$S(t) \sim N\left(s_0 + \int_0^t \mu B(m(y),q)\,dy,\ \int_0^t \sigma^2 B(m(y),q)\,dy\right) \sim N\left(s_0 + \mu B(M(0,t),q),\ \sigma^2 B(M(0,t),q)\right)$$

and the distribution of a time increment $[t, t+h)$ is

$$S(t+h)-S(t) \sim N\left(\int_t^{t+h} \mu B(m,q)dy,\ \int_t^{t+h} \sigma^2 B(m,q)dy\right) \sim N\left(\mu B(M(t,t+h),q),\ \sigma^2 B(M(t,t+h),q)\right)$$

(3) The Negative Binomial Process

For any differential or non-differential time increment,

$$\int_t^{t+h} NB(m(y),q)dy \sim NB\left(\int_t^{t+h} m(y)dy, q\right) \sim NB(M(t,t+h),q) \text{ where } M(t,t+h)=\int_t^{t+h} m(y)dy$$

is the generator, as an integrated generator basis function, for the time interval $[t, t+h)$. $M(t, t+h)$ is the integrated negative binomial index over that time interval. Like the binomial process, the instantaneous intensity is $q \times m(t)$ and the integrated intensity is $q \times M(t, t+h)$ for the time interval $[t, t+h)$. These are, respectively, the propensity to generate events[73] at time t exact and the propensity to generate events over $[t, t+h)$.

The probability function and likelihood contribution for the increment in the generator is

$$p_n = \frac{(M(t,t+h))_n}{n!}(1-q)^n q^{M(t,t+h)}$$

The transition distribution for the transition interval [0, t) is

$$S(t) \sim N\left(s_0 + \int_0^t \mu\ NB(m(y),q)dy,\ \int_0^t \sigma^2 NB(m(y),q)dy\right) \sim N\left(s_0 + \mu NB(M(0,t),q),\ \sigma^2 NB(M(0,t),q)\right)$$

and the distribution of a time increment $[t, t+h)$ is

$$S(t+h)-S(t) \sim N\left(\int_t^{t+h} \mu\, NB(m,q)dy,\ \int_t^{t+h} \sigma^2 NB(m,q)dy\right) \sim N\left(\mu\, NB(mh,q),\ \sigma^2 NB(mh,q)\right)$$

For only certain time functions of the rate intensity does the compound sum remain a first order diffusion. If for example, the rate intensity is proportional to the square of time, then the integrated intensity is proportional to the cube of time as is then the resulting variance of the normal transition density. This is the variance of integrated Brownian motion, which is a second order diffusion rather than first and is characterized by the second order Markov property rather than the first. Suitable functions of the rate intensity are bounded sinusoids and exponentially damped sinusoids, and gamma functions.

[73] in this case, to convolve an additional diffusion basis component onto the diffusion variable.

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Forward and Backward Equations for a Brownian Motion Based on a Generalized Gaussian Diffusion Basis with Non-Zero Skew and Excess Kurtosis

From the transition distribution with normally distributed stochastic cumulants

$$S(t) \sim N\left(s_0 + \int_0^t \mu N(a,b)dy, \int_0^t \sigma^2 N(a,b)dy \right) \sim N\left(s_0 + \mu N(at,bt),\ \sigma^2 N(at,bt) \right)$$

which is a normal distribution with normal cumulants and consequently a generalized normal which converges to an ordinary normal;

$$\begin{aligned} S(t) &\sim N\left(s_0 + \mu N(at,bt), \sigma^2 N(at,bt)\right) \\ &\sim N\left((s_0 + \mu at), (\sigma^2 a + \mu^2 b)t, 3\mu\sigma^2 bt, 3b\sigma^4 t \right) \\ &\to N\left(s_0 + a\mu t, (\mu^2 b + \sigma^2 a)t, 0, 0 \right) \end{aligned}$$

Prior to convergence, from the relevant statistical functions, (1) the components of diffusion basis $X \sim N(\mu, \sigma^2)$ and (2) the summing generator $N(t) \sim N(at, bt)$ have MGFs

$$M_X(\theta) = \exp\left(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2\right) \qquad \text{and} \qquad M_{N(t)}(\theta) = \exp\left(at\theta + \tfrac{1}{2}bt\theta^2\right)$$

The moment generating function of the transition distribution, conditional on the generator, is

$$M_{S(t)|N(t)}(\theta) = E\left(e^{\theta X_1} e^{\theta X_2} ... e^{\theta X_{N(t)}}\right) = E\left(e^{\theta X_1}\right) E\left(e^{\theta X_2}\right) ... E\left(e^{\theta X_{N(t)}}\right) = \prod_{j=1}^{N(t)} E\left(e^{\theta X_j}\right) = \prod_{j=1}^{N(t)} M_{X_j}(\theta)$$

that is,

$$M_{S(t)|N(t)}(\theta) = M_X(\theta)^{N(t)}$$

Therefore the unconditional transition MGF is

$$M_{S(t)}(\theta) = E\left(M_{S(t)|N(t)}(\theta)\right) = E\left(M_X(\theta)^{N(t)}\right) = \exp\left(N(t)\log M_X(\theta)\right) = \exp\left(N(t)K_X(\theta)\right) = M_{N(t)}\left(K_X(\theta)\right)$$

and since the X are time invariant normal, this is

$$M_{S(t)}(\theta) = M_{N(t)}\left(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2\right)$$

or

$$\begin{aligned} M_{S(t)}(\theta) &= \exp\left(s_0\theta + at(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2) + \tfrac{1}{2}bt(\mu\theta + \tfrac{1}{2}\sigma^2\theta^2)^2\right) \\ &= \exp\left(s_0\theta + a\mu t\theta + \tfrac{1}{2}a\sigma^2 t\theta^2 + \tfrac{1}{2}bt(\mu^2\theta^2 + \mu\sigma^2\theta^3 + \tfrac{1}{4}\sigma^4\theta^4)\right) \\ &= \exp\left(s_0\theta + a\mu t\theta + \tfrac{1}{2}(a\sigma^2 + b\mu^2)t\theta^2 + \tfrac{1}{2}b\mu^2 t\theta^2 + \tfrac{1}{8}b\sigma^4 t\theta^4)\right) \end{aligned}$$

which is the moment generating function of a unit-root linear first order diffusion of a Hilbert space (infinite number of uncorrelated infinitesimal random variables) that have non-zero cumulants to fourth order. In particular, an infinitesimal mean of μat, an infinitesimal variance of $a\sigma^2 + b\mu^2$, an infinitesimal skewness of $3b\mu\sigma^2$, and an infinitesimal excess kurtosis of $3b\sigma^4$. This thesis refers to such a distribution, whereby the number of non-zero cumulants is extended past two, as a generalized Gaussian and extends the random variable notation of an ordinary Gaussian $N(\kappa_1, \kappa_2)$ to that with four non-zero cumulants Gen$N(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. In this notation, an ordinary Gaussian diffusion basis being summed at a rate determined by another Brownian motion is a Brownian motion, or unit-root first order diffusion, based upon a generalized Gaussian diffusion basis with cumulants to order four. The transition distribution of this process is then

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$$S(t) \sim GenN\left(s_0 + \mu at, (\sigma^2 a + \mu^2 b)t, 3\mu\sigma^2 bt, 3b\sigma^4 t\right)$$

The time derivative of the moment generating function of the non-Gaussian Brownian motion is

$$\frac{\partial}{\partial t} M_{S(t)}(\theta) = \left(\mu a\theta + \tfrac{1}{2}(\sigma^2 a + \mu^2 b)\theta^2 + \tfrac{1}{2}\mu\sigma^2 b\theta^3 + \tfrac{1}{8}b\sigma^4\theta^4\right) M_{S(t)}(\theta)$$

By the definition of the diffusion moment generating function in terms of the transition density,

$$M_{S(t)}(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x)dx$$

and from the techniques utilized in FOUNDATIONS I,

$$\theta^n M_{S(t)}(\theta) = (-1)^n \int_{-\infty}^{\infty} e^{\theta x} \frac{\partial^n}{\partial x^n} f_{S(t)}(x)dx$$

the expression for the time derivative of the moment generating function of the diffusion can be stated in terms of the transition density,

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{\theta x} f_{S(t)}(x)dx = -\mu a\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial}{\partial x} f_{S(t)}(x)dx + \tfrac{1}{2}(\sigma^2 a + \mu^2 b)\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial^2}{\partial x^2} f_{S(t)}(x)dx$$
$$-\tfrac{1}{2}\mu\sigma^2 b\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial^3}{\partial x^3} f_{S(t)}(x)dx + \tfrac{1}{8}b\sigma^4\int_{-\infty}^{\infty} e^{\theta x}\frac{\partial^4}{\partial x^4} f_{S(t)}(x)dx$$

Inverting the moment generating function on both sides of the equation results in

$$\frac{\partial}{\partial t} f_{S(t)}(x) = -a\mu\frac{\partial}{\partial x} f_{S(t)}(x) + \tfrac{1}{2}(a\sigma^2 + b\mu^2)\frac{\partial^2}{\partial x^2} f_{S(t)}(x) - \tfrac{1}{2}b\mu\sigma^2\frac{\partial^3}{\partial x^3} f_{S(t)}(x) + \tfrac{1}{8}b\sigma^4\frac{\partial^4}{\partial x^4} f_{S(t)}(x)$$

which is the forward equation of the Brownian motion with infinitesimal basis components that have non-zero skewness and excess kurtosis, as well as the forward equation to a Gaussian diffusion basis that aggregates at a rate determined by a Brownian motion (a first order unit-root aggregation of another Gaussian diffusion basis). This generalizes the heat equation of mathematical physics to include diffusion terms of higher order and is termed by this thesis the generalized heat equation. The adjoint of this equation is the backward equation

$$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = a\mu\frac{\partial}{\partial x_0} f_{S(t)|S(t_0)}(x \mid x_0) + \tfrac{1}{2}(a\sigma^2 + b\mu^2)\frac{\partial^2}{\partial x_0^2} f_{S(t)|S(t_0)}(x \mid x_0)$$
$$+\tfrac{1}{2}b\mu\sigma^2\frac{\partial^3}{\partial x_0^3} f_{S(t)|S(t_0)}(x \mid x_0) + \tfrac{1}{8}b\sigma^4\frac{\partial^4}{\partial x_0^4} f_{S(t)|S(t_0)}(x \mid x_0)$$

and is termed the conjugate generalized heat equation, as the generalization of the basis function of that which pertains to the conjugate of the standard heat equation. Since the diffusion is not Gaussian, the ito-calculus no longer applies and ito-like corrections are obtained by transforming the forward or backward equation. The transformed backward equation to an exponential diffusion (a multiplicative diffusion) has higher order ito-like corrections, which are Stirling numbers. These are derived and discussed in the part of this dissertation regarding replicating portfolios and option markets, BOOK 2.

The forward and backward equations can be solved numerically to obtain the transition density numerically and the equation can be transformed to moments and limited moments and a source (for decay in discounting) added to give the partial differential equation of option premia and related statistics – the generalized partial differential equation for the unbiased premium of an option derivative.

Further Comments Regarding Modeling Random Variance (Stochastic Volatility)

Some key issues to be addressed with regard to stochastic volatility, and stochastic cumulants in general, are twofold;

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1) 'whether it is white or Brown or somewhere in between' i.e., the typical vicinity of the characteristic root. The VIX index, which is a volatility index of the Chicago Board of Options Exchange, is suggestive that it is white on logarithmic differences, i.e., the log-increments of the index is uncorrelated white noise, so that the process (the integration of the differences) on logarithms is Brown, a geometric Brownian motion[74];

2) 'whether it is better to model the volatility or the precision' where precision is the reciprocal of volatility.

If precision, the inverse of variance, is modeled rather than the variance itself, the feasible distributions then have the same range as those for volatility, from zero (inclusive) upwards, and the reciprocal of the range of this variable then has an asymptote. This creates a probability asymptote in the variance at zero and gives rise to unconditional distributions that have infinite (or complex) moments. The family of *t* distributions[75] are the prime, but not exclusive, example. The heavier the probability of a singularity at zero, the more of the moments will be complex or infinite. Consider a chi-square variable with a single degree of freedom, which has a mode very close to zero, and accordingly, most of its density near zero. If the square-root of this variable is divided into a unit normal, the result is a Cauchy variable

$$t_1 = \frac{N(0,1)}{\sqrt{\chi_1^2}}$$

which has no finite moments. If one then considers the same ratio but with a chi-square variable with, say, three degrees of freedom rather than one, the mode is found further out from the origin and has less probability mass near zero. The quotient of a standard normal and the square-root of the chi-square variable is a *t* variable with 3 degrees of freedom

$$t_3 = \frac{N(0,1)}{\sqrt{\chi_3^2}}$$

which has a real mean and variance and no higher real and finite moments. The message here, then, is that the more probability near zero in a variable modeling precision, the less real and finite moments the unconditional distribution will have. This is probably suggestive that modeling the variance is preferable to modeling precision, if one wants to avoid distributions with complex moments and is willing to rule out the set of *t* distributions. The variance gamma process is an example of a stochastic volatility model that models variance rather than precision. It is hypothesized that modeling variance by a variable on $[0,\infty)$ with all real moments will always produce an unconditional distribution that has all real moments. Complex moments occur when there is a non-zero probability of a pole or singularity in the random variable modeling the variance. **Real and finite variance of random variance produces real and finite variance in the random variable and diffusion basis**. This fact also applies to the stochastic modeling of any distributional cumulant – a non zero probability of a singularity in any cumulant that is being modeled stochastically results in an unconditional random variable that has complexity in its set of cumulants and moments.

[74] although this was probably backed out by equating the Black-Scholes formulae to market prices (implied volatility); these formulae are biased for options on market assets so this index may contain considerable statistical error (volatility skew), particularly in times of strong trending markets. It is recommended that implied return-volatility ellipses are trialled by backing out the parameters of the unbiased option formulae equated to market prices.

[75] The distribution developed by William Gosset,

"Student", (1908) The Probable Error of a Mean, *Biometrica*, Vol. 4, pp. 39-48.

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APPENDIX: A COMPENDIUM OF RANDOM WALKS

Motivation

The Compendium of Random Walks consolidates FOUNDATIONS I with FOUNDATIONS II and the topic concerning variables of first order diffusion of BOOK 1.

A flip of a coin forms a component of a walk basis function and the continued flips of a coin form a statistically orthogonal basis function. The aggregation of this basis function then results in a random walk, i.e., a resolved first order diffusion variable. Random outcomes, such as a toss of a coin or a throw of a die, occur everywhere in life; both in the scientific world and the world of civilized society. As soon as variables of such outcomes (simple or otherwise) aggregate, the stochastic calculus is applicable. This is more suggestion of the all pervasiveness of the stochastic calculus as the extension of the calculus of Newton to stochastic variables *S*, and union of statistics and probability with the calculus, and further credence to the assertion of this thesis… *the measure of all things (Kenneth Clarke), all things between Newton and everything.*

Although random walks for a discrete basis (steps) are somewhat simplistic, such variables are excellent for becoming acquainted and accustomed with the technicalities of diffusion variables; particularly, becoming acquainted with a well-defined sample space and accustomed to the relations between Δy and Δt, i.e., the relation between a sultus in a diffusion (time dependent and stochastic) variable and the abscissa of the sultus, and how these aggregate (by means of convolution) onto the diffusion sample space.

A common example of a random walk is that from physics, used frequently by physicist (and others) to demonstrated probability principles related to quantum mechanics, where a ball is let fall through a series of pegs. At each line as it descends, it either moves to the left or to the right of a peg (i.e., a step to the left or the right) and ends in one of a number of possible slots at the end of its descent to the bottom of the pegs. This example shows that, for a large number of these balls, the result (the distribution of balls in the slots) is a normal bell curve. This is a graphical demonstration of the Central Limit Theorem of the random walk (with[76] $u = -d = \Delta y$) as the binomial transition distribution converges to the normal bell, i.e., a normal transition distribution of Brownian motion, where the number of levels of pegs is the transition length. For n pegs, a ball would have travelled (descended by gravity!) a total distance of $T - t_0 = n\Delta t$ by the time it reaches its destination in one of the slots.

This example is an expression of both **(1)** the CLT and **(2)** Brownian motion and the standard heat equation as the domain of attraction of the CLT applied to first order diffusion. The binomial recurrences are the motions of the balls as each descends down through the pegs, limiting with enough lines of pegs, to a dense transition distribution described by the heat equation.

It is hypothesized that with the diffusion calculus of arbitrary order, for increased orders of sophistication in the correlation (convolution) dynamics of balls moving through pegs (correlating the coin tosses), the stochastic calculus will apply to all motions of atomic and subatomic particles; molecular vibrations[77] [forming basis functions] and aggregated molecular vibrations. It will just be a matter of having the expertize[78] to find the right variables to describe the motions, all of which being derivatives of standard Brownian motion and all of which being the calculus of Newton extended from the reals R to stochastic variables S. The increased sophistication of the diffusion calculus to arbitrary order, inclusive of harmonic Brownian motion and cyclic diffusion, is **also that required** for a more technically accurate and flexible toolkit for modeling the dynamics of stock market variables and financial derivatives on such variables. This forms one of the greatest challenges and opportunities for probability theory, feasible only in the context of the integral and differential calculus and the resulting diffusion calculus of arbitrary order of BOOK 3.

Introduction to the Compendium

This compendium derives, states, and tabulates the main formulae describing the mechanics of random walks. A random walk is a diffusion variable composed of a walk variable and a step generator. The walk variable is the diffusion variable and the binomial step generator is, after standardizing to accommodate the step sizes, the diffusion basis. The walk variable is then defined as an aggregation of the steps of the walk. This Compendium describes the additive and multiplicative forms of two types of random walk. These are

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[76] where Δy is half the horizontal distance between each peg and Δt is the vertical distance between each peg.
[77] Edward Teller
[78] for which I am hoping to provide with this thesis.

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(1) ordinary random walk: deterministic cumulative sum of two-point discrete variables
(2) Poisson process random walk[79]: Poisson process cumulative sum of two-point discrete variables

Ordinary random walks, that of a fixed number of additive and multiplicative steps (for a given transition interval), correspond to the variables discussed in FOUNDATIONS I, namely, Brownian motion and geometric Brownian motion as these are first order unit root integrations of a basis function. The Poisson generated walks correspond to the variables discussed in FOUNDATIONS II, namely, additive and multiplicative compound Poisson diffusions which are unit root Poisson summations of a basis function. The defining feature of walk variables, in contrast to the topics of FOUNDATIONS I & II, is that the basis function is an orthogonal vector of (scaled) binomial/Bernoulli variates rather than a continuous Hilbert space of infinitesimal normals like variables of FOUNDATIONS I or an orthogonal vector of non-infinitesimal normals like variables of FOUNDATIONS II. That is, walks are aggregations of basis functions with components of two-valued samples spaces rather than basis functions with components of infinite-valued sample spaces.

If the two valued step (0,1) indicators of an ordinary walk are not scaled to (d,u) before being summed, then an ordinary walk is the Bernoulli process (binomial process with unit index) and is same as the Poisson process except that the waiting time between each unit step is geometric rather than exponential; both the Bernoulli process and the Poisson process are continuous time processes that increment in unit steps (step functions) at random times, one that increments randomly [geometrically] only **at discrete multiplies of Δt** (Bernoulli process) and the other that increments randomly [exponentially] at any point on the time continuum (Poisson process) but **at multiples of Δt on statistical average**. The abscissae are then both on the same scale and order of magnitude. This is the key to familiarizing oneself with the differences and similarities between the two forms of walk and the various generators that can be used to sum the steps of a walk or any other basis function. Note that a walk that has been summed by a Bernoulli process, because $(1 - p)^{th}$ of the total number of sultii are randomly (geometrically) skipped, is an ordinary walk with trinomially distributed steps composed of either an upstep, a downstep, or a null-step. The timing of the steps is then under-dispersed. Multinomial walks, such as the trinomial walk, are considered by one of the Appendices to BOOK 2. Walks with over-dispersion in the timing of the steps, those generated by a negative-binomial process, are considered at the end of this Compendium.

Graphics have been included following the mathematics of **(1)** and **(2)** for visualizing the sample properties and sample spaces of the different forms of random walk. These graphics of simulated walks have been included in order to visualize the differences between

i. the behavior and sample properties of individual paths (the individual trees) and the aggregate behavior of a large number of paths (the trees as the forest) which form the diffusion sample space;
ii. the behavior on different scales of forward transition (the small and large time effects) by comparing the graphics of a small number of steps (50 steps) with the graphics of a large number of steps (1000 steps);
iii. additive Σ and multiplicative Π effects by comparing the graphics of ordinary additive walks (summed Σ-diffusion) with that of multiplicative exponential walks (produced Π-diffusion);
iv. diffusion summed by a generator of fixed length and spacing between abscissae and that summed by a generator of exponential length and spacing between abscissae by comparing the graphics of ordinary random walks with those of compound Poisson random walks.

It is seen clearly with these graphics that the paths of the random walks pertain to variables of first order unit root diffusion and, for a sufficient number of forward steps, are indistinguishable from Brownian motion; ordinary Brownian motion for summed walks and geometric Brownian motion for produced (product) walks. This is because the difference between a binomial basis function and a normal basis function vanishes by the Central Limit Theorem, for sums of homogeneous random variables (with finite variance), as the basis function of the walk aggregates. This feature of convergence is the notion for which the limiting forward and backward equations are obtained, that of the heat and conjugate heat equations, by reference to the recurrence relations of the four main forms of walk discussed in the Compendium, and that observed by letting balls fall through a series of pegs. Simple and concise simulation MATLAB code, which produced the graphics, is included. This should make plotting a random walk a simple a task as plotting, say, a parabola[80]. Note that `cumsum` is the MATLAB code for cumulative sum Σ, and a cumulatively summed vector is converted to a continued product Π by expo-

[79] and may describe, for example, a molecule descending through the atmosphere through a medium of exponentially spaced pegs of lighter molecules or the position and timing of collisions of the particles that make up a gas or liquid; this requires more sophisticated correlation dynamics of BOOK 3.
[80] and help definitively bring forth the marriage (union) of statistics and probability with the calculus (which has been a long time coming).

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nentiation after rescaling the parameters to the logarithmic scale so that the walk variable remains finite and in scale.

The stochastic variables of this topic are X_j, Y_j, Y, S_j

(1) X_j step generator/indicator (walk basis) $X_j \sim B(1, p)$

(2) Y_j scaled and translated indicator (scaled and shifted basis) $Y_j = (u-d)X_j + d$

(3) Y net forward transition $Y = Y_1 + Y_2 + ... + Y_n$

(4) S (conditional) aggregate walk variable $S_t = s_0 + Y = s_0 + \sum_{j=1}^{n} Y$ given $S_0 = s_0$

and, additionally, for Poisson generated random walks X_j, Y_j, Y, S_j, N_t

(5) N_t Poisson process that aggregates the steps of the walk

and, additionally, for Poisson generated random walks with gamma diffusion integrated rate intensity

(6) $\Lambda(t)$ integrated Poisson rate intensity

Each of the remaining variables are those of any diffusion variable; t, n, t_0, T which are the transition length $t = T - t_0$, number of steps in transition interval $n = t/\Delta t$, time at beginning of transition interval t_0, and time at end of transition interval $T = t_0 + n\Delta t$. For continuous diffusion, considered in FOUNDATIONS I, the number of summed differentials for any transition interval of non-infinitesimal length is infinite as the reciprocal calculus limit $n \propto 1/dt \to \infty$. After limits are taken and the time between steps becomes infinitesimal so that the distance between each probability mass becomes negligible, the sample space of the walk variable becomes dense and the pmf of the walk becomes a pdf. Each form of walk is split into the subsections;

The topics of the Compendium are

1. Statistical Construction and Forward Transition Distribution
2. MGF of the Arithmetic Random Walk
3. Transition Mean and Variance
4. Arithmetic Random Walk Simulation Algorithm
5. Limited Moments and Undiscounted Option Premia of the Arithmetic Random Walk
6. Forward and Backward Recurrence Relations of an Additive Random Walk
7. Forward Equation for the Continuous Diffusion Limit
8. Backward Equation for the Continuous Diffusion Limit

Tables of the most relevant formulae

Resolved walk variable
Transition mean and variance
Transition mgf
Forward (convolution) recurrence
Backward (deconvolution) recurrence
Undiscounted call option premia (summation form)
Undiscounted call option premia (expressed in terms of functions of known statistical forms)
Undiscounted option premia (summation form)
Undiscounted option premia (expressed in terms of functions of known statistical forms)

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Table 29: Resolved walk variable

	Resolved walk variable	
additive walk	$S_n = S_0 + uX^{*n} + d(n - X^{*n})$ $= S_0 + nd + (u-d)X^{*n}$	$X^{*n} \sim B(n,p)$
multiplicative walk	$S_n = S_0 u^{X^{*n}} d^{n - X^{*n}}$	$X^{*n} \sim B(n,p)$
Poisson generated additive walk	$S_t = S_0 + uX^{*N_t} + d(N_t - X^{*N_t})$ $= S_0 + d\,N_t + (u-d)X^{*N_t}$	$N_t \sim P(\lambda t) \quad \left(X^{*N_t} \mid N_t = n\right) \sim B(n,p)$
Poisson generated multiplicative walk	$S_t = S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}}$	$N_t \sim P(\lambda t) \quad \left(X^{*N_t} \mid N_t = n\right) \sim B(n,p)$
neg. binomial gen. multiplicative walk	$S_t = S_0 + uX^{*N_t} + d(N_t - X^{*N_t})$ $= S_0 + d\,N_t + (u-d)X^{*N_t}$	$N_t \sim P\left(\Lambda(t)\right) \quad \Lambda(t) \sim \Gamma\left(\alpha t, \beta\right)$ $\left(X^{*N_t} \mid N_t = n, \Lambda(t) = \lambda\right) \sim B(n,p)$
neg. binomial gen. additive walk	$S_t = S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}}$	$N_t \sim P\left(\Lambda(t)\right) \quad \Lambda(t) \sim \Gamma\left(\alpha t, \beta\right)$ $\left(X^{*N_t} \mid N_t = n, \Lambda(t) = \lambda\right) \sim B(n,p)$

Table 30: Transition moment generating function

	transition mgf
additive walk	$M_{S(T)\mid S(t_0)}(\theta) = e^{S_0\theta}\left(pe^{u\theta} + qe^{d\theta}\right)^{(T-t_0)/\Delta t}$
multiplicative walk	$M_{S(T)\mid S(t_0)}(\theta) = \sum_{j=0}^{\infty} \frac{(S_0\theta)^j}{j!}\left(pu^j + qd^j\right)^{(T-t_0)/\Delta t}$
Poisson generated additive walk	$M_{S(T)\mid S(t_0)}(\theta) = e^{S_0\theta + \lambda(T-t_0)(pe^{u\theta} + qe^{d\theta} - 1)}$
Poisson generated multiplicative walk	$M_{S(T)\mid S(t_0)}(\Theta) = \sum_{j=0}^{\infty} \frac{(S_0\theta)^j}{j} e^{\lambda(T-t_0)(pu^j + qd^j - 1)}$
neg. binomial gen. multiplicative walk	$M_{S(T)\mid S(t_0)}(\theta) = e^{S_0\theta}\left(1 + \beta - \beta(pe^{u\theta} + qe^{d\theta})\right)^{-\alpha(T-t_0)}$
neg. binomial gen. additive walk	$M_{S(T)\mid S(t_0)}(\theta) = \sum_{j=0}^{\infty} \frac{(S_0\theta)^j}{j!}\left(1 + \beta - \beta(pu^j + qd^j)\right)^{-\alpha(T-t_0)}$

Table 31: Walk statistics

	Transition mean	Transition variance
additive walk	$E(S_n \mid S_0) = S_0 + n(pu + qd)$	$Var(S_n \mid S_0) = npq(u-d)^2$
multiplicative walk	$E(S_n \mid S_0) = S_0(pu + qd)^n$	$Var(S_n \mid S_0) = S_0^{\,2}\left((pu^2 + qd^2)^n - (pu + qd)^{2n}\right)$
Poisson generated additive walk	$E(S_t \mid S_0) = S_0 + \lambda t(pu + qd)$	$Var(S_t \mid S_0) = \lambda t(pu^2 + qd^2)$
Poisson generated multiplicative walk	$E(S_t \mid S_0) = S_0 \exp\left(\lambda t(pu + qd - 1)\right)$	$Var(S_t \mid S_0) = S_0^{\,2}\left(e^{\lambda t(pu^2 + qd^2 - 1)} - e^{2\lambda t(pu + qd - 1)}\right)$
neg. binomial gen. multiplicative walk	$E(S_t \mid S_0) = S_0 + \alpha\beta t(pu + qd)$	$Var(S_t \mid S_0) = \alpha\beta t\left\{(pu^2 + qd^2) + \beta(pu + qd)^2\right\}$
neg. binomial gen. additive walk	$E(S_t \mid S_0) = S_0\left(1 + \beta - \beta(pu + qd)\right)^{-\alpha t}$	$Var(S_t \mid S_0) = S_0^{\,2}\left(1 + \beta - \beta(pu^2 + qd^2)\right)^{-\alpha t}$ $-S_0^{\,2}\left(1 + \beta - \beta(pu + qd)\right)^{-2\alpha t}$

With $n =$ $n = t/\Delta t$ and $\lambda = 1/\Delta t$ and $\alpha\beta = \lambda$ so that $\alpha\beta = 1/\Delta t$ so $n = \lambda t$, comparing the transition statistics between the different forms of walk shows that while the transition means are level, the transition variances are not and are instead, ascending in size. For the additive walks, the transition means are in equality

$$S_0 + \lambda t(pu + qd) = S_0 + \lambda t(pu + qd) = S_0 + \lambda t(pu + qd)$$

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while the transition means are increasing in ascending order

$$npq(u-d)^2 \le \lambda t(pu^2+qd^2) \le \alpha\beta t\left\{(pu^2+qd^2)+\beta(pu+qd)^2\right\}$$

The increases in variance is a result of each additional source of stochastic/statistical variation.

Table 32: Forward (convolution) recurrence

	Forward (convolution) recurrence	
additive walk	$p_{S(T+\Delta t)\|S(t_0)}(y\,|\,s_0) = p\times p_{S(T)\|S(t_0)}(y-u\,|\,s_0)+q\times p_{S(T)\|S(t_0)}(y-d\,|\,s_0)$	$u,d=\pm\Delta y$
multiplicative walk	$p_{S(T+\Delta t)\|S(t_0)}(y\,|\,s_0) = p\times p_{S(T)\|S(t_0)}(y\div u\,|\,s_0)+q\times p_{S(T)\|S(t_0)}(y\div d\,|\,s_0)$	$u,d=e^{\pm\Delta\log y}$
Poisson generated additive walk	$p_{S(T+\Delta t)\|S(t_0)}(y\,|\,y_0) = \left(p\times p_{S(T)\|S(t_0)}(y-u\,|\,y_0)+q\times p_{S(T)\|S(t_0)}(y-d\,|\,y_0)\right)\lambda\Delta t + p_{S(T)\|S(t_0)}(y\,|\,y_0)(1-\lambda dt)+o(\Delta t)$ $\frac{\partial}{\partial T}p_{S(T)\|S(t_0)}(y\,|\,s_0) = \lambda\left(p\times p_{S(T)\|S(t_0)}(y-u\,|\,s_0)+q\times p_{S(T)\|S(t_0)}(y-d\,|\,s_0)-p_{S(T)\|S(t_0)}(y\,|\,s_0)\right)$	$u,d=\pm\Delta y$
Poisson generated multiplicative walk	$p_{S(T+\Delta t)\|S(t_0)}(y\,|\,y_0) = \left(p\times p_{S(T)\|S(t_0)}(y\div u\,|\,y_0)+q\times p_{S(T)\|S(t_0)}(y\div d\,|\,y_0)\right)\lambda\Delta t + p_{S(T)\|S(t_0)}(y\,|\,y_0)(1-\lambda dt)+o(\Delta t)$ $\frac{\partial}{\partial T}p_{S(T)\|S(t_0)}(y\,|\,y_0) = \lambda\left(p\times p_{S(T)\|S(t_0)}(y\div u\,|\,y_0)+q\times p_{S(T)\|S(t_0)}(y\div d\,|\,y_0)-p_{S(T)\|S(t_0)}(y\,|\,y_0)\right)$	$u,d=e^{\pm\Delta\log y}$

Table 33: Backward (deconvolution) recurrence

	Backward (deconvolution) recurrence	
additive walk	$p_{S(T)\|S(t_0)}(y\,|\,s_0) = p\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,s_0+u)+q\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,s_0+d)$	$u,d=\pm\Delta y$
multiplicative walk	$p_{S(T)\|S(t_0)}(y\,|\,s_0) = p\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,s_0\times u)+q\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,s_0\times d)$	$u,d=e^{\pm\Delta\log y}$
Poisson generated additive walk	$p_{S(T)\|S(t_0)}(y\,|\,y_0) = \left(p\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,y_0+u)+q\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,y_0+d)\right)\lambda\Delta t + p_{S(T)\|S(t_0+\Delta t)}(y\,|\,y_0)(1-\lambda dt)+o(\Delta t)$ $-\frac{\partial}{\partial t_0}p_{S(T)\|S(t_0)}(y\,|\,s_0) = \lambda\left(p\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,s_0+u)+q\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,s_0+d)-p_{S(T)\|S(t_0+\Delta t)}(y\,|\,s_0)\right)$	$u,d=\pm\Delta y$
Poisson generated multiplicative walk	$p_{S(T)\|S(t_0)}(y\,|\,y_0) = \left(p\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,y_0\times u)+q\times p_{S(T)\|S(t_0+\Delta t)}(y\,|\,y_0\times d)\right)\lambda\Delta t + p_{S(T)\|S(t_0+\Delta t)}(y\,|\,y_0)(1-\lambda dt)+o(\Delta t)$ $-\frac{\partial}{\partial t_0}p_{S(T)\|S(t_0)}(y\,|\,y_0) = \lambda\left(p\times p_{S(T)\|S(t_0)}(y\,|\,y_0\times u)+q\times p_{S(T)\|S(t_0)}(y\,|\,y_0\times d)-p_{S(T)\|S(t_0)}(y\,|\,y_0)\right)$	$u,d=e^{\pm\Delta\log y}$

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Table 34: Undiscounted call option premium

additive walk	$\left(S_0 + nd - k\right)\sum_{x=k^*}^{n} {}^{n}C_x p^x q^{n-x} + np(u-d)\sum_{x=k^*-1}^{n-1} {}^{n-1}C_x p^x q^{(n-1)-x}$ where $k^* = (k - S_0 - nd)/(u-d)$
multiplicative walk	$S_0 \sum_{x=k^*}^{n} {}^{n}C_x (pu)^x (qd)^{n-x} - k\sum_{x=k^*}^{n} {}^{n}C_x p^x q^{n-x}$ where $k^* = \log(k/S_0 d^n)/\log(u/d)$
Poisson generated additive walk	$\left(S_0 + qd\lambda t - k\right)\sum_{x=k^*}^{\infty} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} + p(u-d)\lambda t\sum_{x=k^*-1}^{\infty} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$ where $k^* = \left(k - S_0 - qd\lambda t\right)/(u-d)$
Poisson generated multiplicative walk	$S_0 e^{\lambda t((pu+qd)-1)}\sum_{x=k^*}^{\infty} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!} - k\sum_{x=k^*}^{\infty} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$ where $k^* = \left(\log(k/S_0) + \lambda t\, q(1-d)\right)/\log u$
Poisson generated additive walk (gamma diffusion rate intensity)	$(S_0 - k)\sum_{x=k^*}^{\infty} \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t} + \alpha t q d\sum_{x=k^*}^{\infty} \frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$ $+\alpha t\beta p(u-d)\sum_{x=k^*-1}^{\infty} \frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$
Poisson generated multiplicative walk (gamma diffusion rate intensity)	$S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha t}\sum_{x=k^*}^{\infty} \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t}$ $-k\sum_{x=k^*}^{\infty} \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{p\beta}{1+p\beta}\right)^x\left(\frac{1}{1+p\beta}\right)^{\alpha t}$

Table 35: Undiscounted put option premium

additive walk	$\left(k - S_0 - nd\right)\sum_{x=0}^{k^*} {}^{n}C_x p^x q^{n-x} - np(u-d)\sum_{x=0}^{k^*-1} {}^{n-1}C_x p^x q^{(n-1)-x}$ where $k^* = (k - S_0 - nd)/(u-d)$
multiplicative walk	$k\sum_{x=0}^{k^*} {}^{n}C_x p^x q^{n-x} - S_0\sum_{x=0}^{k^*} {}^{n}C_x (pu)^x (qd)^{n-x}$ where $k^* = \log(k/S_0 d^n)/\log(u/d)$
Poisson generated additive walk	$\left(k - S_0 - qd\lambda t\right)\sum_{x=0}^{k^*} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} - p(u-d)\lambda t\sum_{x=0}^{k^*-1} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$ where $k^* = \left(k - S_0 - qd\lambda t\right)/(u-d)$
Poisson generated multiplicative walk	$k\sum_{x=0}^{k^*} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} - S_0 e^{\lambda t((pu+qd)-1)}\sum_{x=0}^{k^*} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!}$ where $k^* = \left(\log(k/S_0) + \lambda t\, q(1-d)\right)/\log u$
Poisson generated additive walk (gamma diffusion rate intensity)	$\left(k - S_0\right)\sum_{x=0}^{k^*} \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t} - \alpha t q d\sum_{x=0}^{k^*} \frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$ $-\alpha t\beta p(u-d)\sum_{x=0}^{k^*-1} \frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$
Poisson generated multiplicative walk (gamma diffusion rate intensity)	$k\sum_{x=0}^{k^*} \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{p\beta}{1+p\beta}\right)^x\left(\frac{1}{1+p\beta}\right)^{\alpha t}$ $-S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha t}\sum_{x=0}^{k^*} \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t}$

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Table 36: Undiscounted call option premium in terms of complementary cdfs

additive walk	$\left(S_0 + nd - k\right)\overline{F}_{B(n,p)}(k^*) + np(u-d)\overline{F}_{B(n-1,p)}(k^*-1)$ where $k^* = (k - S_0 - nd)/(u-d)$
multiplicative walk	$S_0(pu+qd)^n \overline{F}_{B(n,pu/(pu+qd))}(k^*) - k\,\overline{F}_{B(n,p)}(k^*)$ where $k^* = \log(k/S_0 d^n)/\log(u/d)$
Poisson generated additive walk	$\left(S_0 + qd\lambda t - k\right)\overline{F}_{P(p\lambda t)}(k^*) + p(u-d)\lambda t\,\overline{F}_{P(p\lambda t)}(k^*-1)$ where $k^* = \left(k - S_0 - qd\lambda t\right)/(u-d)$
Poisson generated multiplicative walk	$S_0 e^{\lambda t((pu+qd)-1)}\overline{F}_{P(pu\lambda t)}(k^*) - k\,\overline{F}_{P(p\lambda t)}(k^*)$ where $k^* = \left(\log(k/S_0) + \lambda t\, q(1-d)\right)/\log u$
negative binomial generated additive walk	$\left(S_0 - k\right)\overline{F}_{NB(\alpha t, p\beta/(1+p\beta))}\left(k^*\right) + \alpha t q d\,\overline{F}_{NB(\alpha t+1, p\beta/(1+p\beta))}\left(k^*\right) + \alpha t\beta p(u-d)\overline{F}_{NB(\alpha t+1, p\beta/(1+p\beta))}\left(k^*-1\right)$ where $k^* = \dfrac{(k-S_0)(1+\beta p) - \alpha qd}{(u-d)(1+\beta p) + qd}$
negative binomial multiplicative walk	$S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha t}\,\overline{F}_{NB(\alpha t, pu\beta/[1+p(1-qd)])}\left(k^*\right) - k\,\overline{F}_{NB(\alpha t, p\beta/(1+p\beta))}\left(k^*\right)$ where $k^* = \dfrac{\log k - \log S_0 - \alpha\log(1+p\beta)/\left(1+\beta(1-qd)\right)}{\log u + \log(1+p\beta)/\left(1+\beta(1-qd)\right)}$

Table 37: Undiscounted put option premium in terms of cdfs

additive walk	$\left(k - S_0 - nd\right)F_{B(n,p)}(k^*) - np(u-d)F_{B(n-1,p)}(k^*-1)$ where $k^* = (k - S_0 - nd)/(u-d)$
multiplicative walk	$k\,F_{B(n,p)}(k^*) - S_0(pu+qd)^n F_{B(n,pu/(pu+qd))}(k^*)$ where $k^* = \log(k/S_0 d^n)/\log(u/d)$
Poisson generated additive walk	$\left(k - S_0 - qd\lambda t\right)F_{P(p\lambda t)}(k^*) - p(u-d)\lambda t\,F_{P(p\lambda t)}(k^*-1)$ where $k^* = \left(k - S_0 - qd\lambda t\right)/(u-d)$
Poisson generated multiplicative walk	$k\,F_{P(p\lambda t)}(k^*) - S_0 e^{\lambda t((pu+qd)-1)}F_{P(pu\lambda t)}(k^*)$ where $k^* = \left(\log(k/S_0) + \lambda t\, q(1-d)\right)/\log u$
negative binomial generated additive walk	$\left(k - S_0\right)F_{NB(\alpha t, p\beta/(1+p\beta))}\left(k^*\right) - \alpha t q d F_{NB(\alpha t+1, p\beta/(1+p\beta))}\left(k^*\right) - \alpha t\beta p(u-d)F_{NB(\alpha t+1, p\beta/(1+p\beta))}\left(k^*-1\right)$ where $k^* = \dfrac{(k-S_0)(1+\beta p) - \alpha qd}{(u-d)(1+\beta p) + qd}$
negative binomial multiplicative walk	$k\,F_{NB(\alpha t, p\beta/(1+p\beta))}\left(k^*\right) - S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha t}F_{NB(\alpha t, p\beta/[1+p(1-qd)])}\left(k^*\right)$ where $k^* = \dfrac{\log k - \log S_0 - \alpha\log(1+p\beta)/\left(1+\beta(1-qd)\right)}{\log u + \log(1+p\beta)/\left(1+\beta(1-qd)\right)}$

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A. Additive Random Walk

1. Statistical Construction and Forward Transition Distribution

Centering and scaling: an additive random walk moves in additive steps, either upsteps *u* or downsteps *d*. The outcome regarding whether each step *j* is an upstep or a downstep is determined by a Bernoulli zero-one random variable $X_j \sim B(1, p)$, a binomial variable with index 1. The j^{th} indicator is 1 if the step is an upstep *u* and 0 if it is a downstep *d*,

$$X_j = \begin{cases} 0 & \text{with probability } q = 1 - p \\ 1 & \text{with probability } p \end{cases}$$

Scaling and shifting the component Bernoulli variables of the step indicators so that the variables each return a step of size *u* ($u > 0$) if it is an upstep and return *d* ($d < 0$) if it is down step; the rescaled step variables are then discrete two point distributions

$$Y_j = u X_j + d(1 - X_j) = (u - d)X_j + d = \begin{cases} d & \text{with probability } q = 1 - p \\ u & \text{with probability } p \end{cases}$$

which are *u* if the Bernoulli variable indicates an upstep and *d* if the Bernoulli variable indicates a downstep, as required. These are distributed as scaled and shifted binomial variable with index 1,

$$Y_j \sim (u - d)B(1, p) + d$$

The vector of these components forms the walk basis variable. The probability mass function, for any particular single step in time, describes the distribution of a discrete increment of the walk, i.e., one single step. Noting that no Jacobian is required for the transformation since the variable is discrete, the inverse function of the transformed variable is $x_j = (y_j - d)/(u - d)$ and the mass function is that of a Bernoulli variable with translation and scale

$$p_{Y_j}(y_j) = p_{X_j}(x_j) = p_{X_j}\left({(y_j - d)} \Big/ {(u - d)} \right) = p^{(y_j - d)/(u - d)} q^{(u - y_j)/(u - d)} \qquad y_j = d, u$$

Convolving the variables of the individual steps to generate the aggregate steps that form the net transition of the random walk results in a translated and scaled binomial variable, with index equal to the total number of steps *n*

$$\begin{aligned} Y &= Y_1 + Y_2 + \ldots + Y_n \\ &= \left((u - d)X_1 + d\right) + \left((u - d)X_2 + d\right) + \ldots + \left((u - d)X_n + d\right) \\ &\sim (u - d)X + nd \\ &\sim (u - d)B(n, p) + nd \end{aligned}$$

This variable forms the forward transition. The probability mass function is the product of the individual mass functions over all possible combinations (distinct permutations), i.e., over all possible paths

$$p_Y(y) = p_X(x) = p_X\left({(y - nd)} \Big/ {(u - d)} \right) = {}^{n}C_{(y - nd)/(u - d)} p^{(y - nd)/(u - d)} q^{(nu - y)/(u - d)}$$

$$y = nd, (n-1)d + u, (n-2)d + 2u, \ldots, 2d + (n-2)u, d + (n-1)u, nu$$

Adding an initial value s_0 , the point at which the walk starts and the variable which is the backward walk variable, and letting the interval of time between each step be Δt . Therefore, at the n^{th} step, a total of $n\Delta t$ time units will have elapsed since the backward variable at the beginning of the transition interval; so the transition interval is of length *t* and is composed of *n* steps. Then the transition length is $t = n\Delta t$ and the walk variable at time *t*, $S(t)$, given it was $S(0) = S_0$ is the discrete (valued) continuous time variable

$$\begin{aligned} S(t) &= s_0 + Y_1 + Y_2 + \ldots + Y_n \\ &\sim s_0 + nd + (u - d)X \\ &\sim s_0 + nd + (u - d)B(n, p) \end{aligned}$$

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The aggregate walk variable *S* has a distribution that is the scaled and shifted binomial variable further shifted to the starting value s_0 and is defined in continuous time and undergoes discrete steps at every Δt time units. The probability mass function of the variable is the transition probability mass function of the walk,

$$p_{S(t)|S(0)}(y\,|\,s_0) = p_Y(y-s_0) = p_X\left(\left.(y-s_0-nd)\right/(u-d)\right) = {}^nC_{(y-s_0-nd)/(u-d)}\,p^{(y-s_0-nd)/(u-d)}q^{(nu-y_j+s_0)/(u-d)}$$

$$y = s_0+nd, s_0+(n-1)d+u, s_0+(n-2)d+2u, ..., s_0+2d+(n-2)u, s_0+d+(n-1)u, s_0+nu$$

which, like the ordinary binomial probability mass function, satisfies the recurrence relation of Pascal's triangle for both convolutions onto, and deconvolutions from, both the forward and backward walk variables *y* and s_0. The transition length *t* for a given starting time t_0 is then, by definitions in FOUNDATIONS I, the difference between the forward and backward time variables, $t = T - t_0$. Because the walk is time homogenous, $p_{S(t)|S(0)}(y\,|\,s_0)$ is alternatively written as $p_{S(T)|S(t_0)}(y\,|\,s_0)$ where *t* is a length of time and *T* and t_0 are two specific (and variable) points in time.

As a simplification, it is convenient and consistent with ordinary calculus to set the magnitude of upsteps equal to those of downsteps, so that $u = -d$, say $u = \Delta y$ and $d = -\Delta y$. Drift (integrated basis mean) is then built into the probability parameter *p*, rather than the step size (which is the other alternative rather than, say, specifying $u = 1.2\Delta y$ and $d = -\Delta y$ for upward drift). Setting $u = -d$ keeps the sample space of the basis variable symmetric and the walk lattice (the walk sample space) then recombines at the nodes. The step generator becomes

$$Y_j = 2(\Delta y)X_j - \Delta y = \begin{cases} -\Delta y & \text{with probability } q = 1-p \\ +\Delta y & \text{with probability } p \end{cases}$$

with distribution

$$Y_j \sim 2(\Delta y)B(1, p) - \Delta y$$

and probability mass function

$$p_{Y_j}(y_j) = p_{X_j}(x_j) = p_{X_j}\left(\left.(y_j+\Delta y)\right/2\Delta y\right) = p^{(y_j+\Delta y)/2\Delta y}q^{(\Delta y-y_j)/2\Delta y} \qquad y_j = -\Delta y, \Delta y$$

The net transition is the aggregate of the *n* steps of Y_j over the transition interval of length $n\Delta t$, $Y = Y_1 + Y_2 + ... + Y_n$, and the walk variable given the starting value $S(0) = s_0$ at the beginning of the transition interval becomes

$$\begin{aligned} S(t) &= s_0 + Y_1 + Y_2 + ... + Y_n \\ &= s_0 + \left(2(\Delta y)X_1 - \Delta y\right) + \left(2(\Delta y)X_2 - \Delta y\right) + ... + \left(2(\Delta y)X_n - \Delta y\right) \\ &\sim 2(\Delta y)X + s_0 - n\Delta y \\ &\sim 2(\Delta y)B(n, p) + s_0 - n\Delta y \end{aligned}$$

with transition mass function

$$p_{S(T)|S(t_0)}(y\,|\,s_0) = p_X(x) = p_{X_j}\left(\left.(y-s_0+n\Delta y)\right/2\Delta y\right) = {}^nC_{(y-s_0+n\Delta y)/2\Delta y}\,p^{(y-s_0+n\Delta y)/2\Delta y}q^{(n\Delta y-y_j+s_0)/2\Delta y}$$

$$y = s_0 - n\Delta y,\ s_0-(n-2)\Delta y,\ s_0-(n-4)\Delta y,\ ...,\ s_0(n-4)\Delta y,\ s_0+(n-2)\Delta y,\ s_0+n\Delta y$$

2. MGF of the Arithmetic Random Walk

The walk changes by either amount *u* or amount *d* on any step and these two possibilities occur with probabilities *p* and *q* respectively, the moment generating function for which is then $M_{Y_j}(\theta) = pe^{u\theta} + qe^{d\theta}$. For a walk of *n* independent steps, the moment generating function is $M_Y(\theta) = (pe^{u\theta} + qe^{d\theta})^n$. If the walk starts at s_0 then the moment generating function is increased by $e^{s_0\theta}$ resulting in the transition mgf of the aggregate walk *S* at time *T* for transition length *t* from time t_0

$$M_{S(T)|S(t_0)=s_0}(\theta) = e^{s_0\theta}(pe^{u\theta} + qe^{d\theta})^n = \exp\left(s_0\theta + n\log(pe^{u\theta} + qe^{d\theta})\right)$$

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and the cumulant generating function is

$$K_{S(T)|S(t_0)=s_0}(\theta) = s_0\theta + n\log\left(pe^{u\theta} + qe^{d\theta}\right)$$

For $u = -d = \Delta y$, the walk either moves up or down by an amount Δy and the walk either increases by Δy or decreases by Δy, with probabilities p and q respectively and the moment generating function of the step is $pe^{\theta\Delta y} + qe^{-\theta\Delta y}$. For a walk of n steps at time t, the moment generating function of the net transition is $M_Y(\theta) = (pe^{h\theta} + qe^{-h\theta})^n$, and with the backward variable for the aggregate walk variable

$$M_{S(T)|S(t_0)=s_0}(\theta) = e^{s_0\theta}(pe^{\theta\Delta y} + qe^{-\theta\Delta y})^n$$

which is intuitive.

3. Transition Mean and Variance

4. Arithmetic Random Walk Simulation Algorithm

Simulation algorithm: generate n Bernoulli variables, of zeros and ones $X_j \sim B(1, p)$. This can be achieved by generating n unit uniforms and defining a one for any of the uniform variates greater than the fraction p and a zero for any that are less than the fraction p (or binornd in MATLAB). Multiply the vector of indicators by $u - d$ and deduct d from each to create the individual additive steps

$$\begin{aligned} Y_j &= (u-d)X_j + d \\ &\sim (u-d)B(1,p) + d \end{aligned}$$

The components of this vector are the discrete time independent increments with the required up and down magnitudes and form the vector of the walk's steps, i.e.., each element is the successive incremental step $Y_j = (u-d)X_j + d$ and incremental change in the walk $Y_j = S_j - S_{j-1}$. In terms of the indicator step generator, the vector is comprised of the elements

$$Y_1 = (u-d)X_1 + d,\ Y_2 = (u-d)X_2 + d,\ Y_3 = (u-d)X_3 + d \ldots\ Y_n = (u-d)X_n + d$$

which form the statistically orthogonal vector

$$Y_1 = S_1 - S_0,\ Y_2 = S_2 - S_1,\ Y_3 = S_3 - S_2, \ldots,\ Y_n = S_n - S_{n-1}$$

Append an initial value S_0 to the front end of this vector (the infinite history of the steps before time 0), i.e., onto the beginning of the transition interval for the difference components of aggregate walk variable S, composing the vector

$$\left[S_0, Y_1, Y_2, Y_3, ..., Y_n\right] = \left[S_0, S_1 - S_0, S_2 - S_1, S_3 - S_2, ..., S_n - S_{n-1}\right]$$

Cumulatively summing the elements of the vector of increments (steps), which is equivalent to discretely integrating them, gives the ordinates of the cumulative steps of the walk variable

$$\left[S_0, S_1, S_2, ..., S_n\right] = \left[S_0, S_0 + (u-d)X_1 + d, S_0 + (u-d)(X_1 + X_2) + 2d, \ldots, S_0 + (u-d)(X_1 + X_2 + ... + X_n) + nd\right]$$

Plot the resulting vector against a time index, where each random step corresponds to one unit of time. MATLAB code for this simulation algorithm

```
p=p; u=u; d=d; s0=s0; t0=t0; n=n; dt=Δt;      %initialize walk parameters
t=cumsum([t0;ones(n,1)).*dt]);       %step times on transition interval
x=binornd(1,p,n,1); S=cumsum([s0;(u-d).*x+d]);      %simulate steps & walk
plot(t,S);                                                %plot S over t
```

The last line plots S against continuous time t by multiply the step number by the length of time between each step. The path can alternatively be graphed against step number by omitting `t` in the `plot` command. As an example and assuming that the time unit is hours, for a symmetric additive walk of 100 steps starting at value s0=10 at time 2 (o'oclock, say) for a forward transition of length n*dt=100*(5/100)=5 hours to t0+n*dt=2+100*(5/100)=7 (o'clock say), initialize the parameters with the line

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```
p=.5; u=.5; d=-u; s0=10; t0=2; n=100; dt=5/100; %initialize walk parameters
```

where a step occurs once every `dt` = 5/100 hours = 1/20 hours = 3 minutes over the 5 hour simulated transition period.

5. Limited Moments and Undiscounted Option Premia of the Arithmetic Random Walk

The **additive** *n*-step ahead forward walk variable, composed of X^{*n} upsteps and $n - X^{*n}$ downsteps, is

$$\begin{aligned} S_n &= S_0 + uX^{*n} + d(n - X^{*n}) \\ &= S_0 + nd + (u-d)X^{*n} \end{aligned}$$

where $X^{*n} \sim B(n, p)$. The payoff of an option on the walk variable is then

$$\max\left(S_n - k,\ 0\right) = \max\left(S_0 + nd + (u-d)X^{*n} - k,\ 0\right) \qquad \text{if it is an option to call the asset to buy for } k$$

$$\max\left(k - S_n,\ 0\right) = \max\left(k - S_0 - nd - (u-d)X^{*n},\ 0\right) \qquad \text{if it is an option to put the asset to sell for } k$$

For a call option, with $E\left(g(X)\right) = \sum_{x \in X}\ g(x) \times p_X(x)$, the undiscounted premium is

$$\begin{aligned} c(n,k) &= E\left(\max(S_0 - k + nd + (u-d)X^{*n}, 0)\right) \\ &= \sum_{x=0}^{n}\ \max(S_0 - k + nd + (u-d)x, 0)\ {}^nC_x p^x q^{n-x} \end{aligned}$$

The max function is non-zero whenever

$$\begin{aligned} &S_0 - k + nd + (u-d)x \geq 0 \\ &(u-d)x \geq k - S_0 - nd \\ &x \geq (k - S_0 - nd)/(u-d) \end{aligned}$$

so that only terms in the summation for *x* greater than $k^* = (k - S_0 - nd)/(u-d)$ are non-zero,

$$\begin{aligned} c(n,k) &= \sum_{x=k^*}^{n}\ \left(S_0 - k + nd + (u-d)x\right){}^nC_x p^x q^{n-x} \\ &= (u-d)\sum_{x=k^*}^{n}\ x\ {}^nC_x p^x q^{n-x} + (S_0 - k + nd)\sum_{x=k^*}^{n}\ {}^nC_x p^x q^{n-x} \end{aligned}$$

For the weighted sum

$$\begin{aligned} \sum_{x=k^*}^{n}\ x\ {}^nC_x p^x q^{n-x} &= n\sum_{x=k^*}^{n}\ \frac{(n-1)!}{(x-1)![n-1-(x-1)]!}\ p^x q^{n-x} \\ &= np\sum_{y=k^*-1}^{n-1}\ {}^{n-1}C_y\ p^y q^{(n-1)-y} \\ &= np\,\overline{F}_{B(n-1,p)}(k^*-1) \end{aligned} \qquad \text{with} \quad \begin{aligned} x &= y+1 \quad &\Rightarrow y &= x-1 \\ x &= k^* \quad &\Rightarrow y &= k^*-1 \\ x &= n \quad &\Rightarrow y &= n-1 \end{aligned}$$

so

$$\begin{aligned} c(n,k) &= np(u-d)\sum_{y=k^*-1}^{n-1}\ {}^{n-1}C_y p^y q^{(n-1)-y} + \left(S_0 - k + nd\ \right)\sum_{x=k^*}^{n}\ {}^nC_x p^x q^{n-x} \\ &= np(u-d)\overline{F}_{B(n-1,p)}(k^*-1) + (S_0 - k + nd)\overline{F}_{B(n,p)}(k^*) \end{aligned}$$

with k^* being the smallest integer equal to or greater than $(k - S_0 - nd)/(u-d)$ and where $\overline{F}_{B(n,p)}(\cdot)$ and $\overline{F}_{B(n-1,p)}(\cdot)$ are the complementary cumulative distribution functions of binomial variables with parameters *n* and *p* and *n* – 1 and *p* respectively. For a put option

$$\begin{aligned} p(n,k) &= E\left(\max(k - S_0 - nd - (u-d)X^{*n}, 0)\right) \\ &= \sum_{x=0}^{n}\ \max(k - S_0 - nd - (u-d)x, 0)\ {}^nC_x p^x q^{n-x} \end{aligned}$$

and for k^* being the largest integer equal to or smaller than $(k - S_0 - nd)/(u-d)$

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$$p(n,k) = \sum_{x=0}^{k^*} \left(k - S_0 - nd - (u-d)x\right) {}^nC_x p^x q^{n-x}$$
$$= (k - S_0 - nd)\sum_{x=0}^{k^*} {}^nC_x p^x q^{n-x} - (u-d)\sum_{x=0}^{k^*} x\, {}^nC_x p^x q^{n-x}$$

for the weighted sum

$$\sum_{x=0}^{k^*} x\, {}^nC_x p^x q^{n-x} = n\sum_{x=1}^{k^*} \frac{(n-1)!}{(x-1)![n-1-(x-1)]!} p^x q^{n-x} = np\sum_{x=0}^{k^*-1} {}^{n-1}C_y p^y q^{(n-1)-y} = npF_{B(n-1,p)}(k^*-1)$$

Hence

$$p(n,k) = \left(k - S_0 - nd\right)\sum_{x=0}^{k^*} {}^nC_x p^x q^{n-x} - np(u-d)\sum_{y=0}^{k^*-1} {}^{n-1}C_y p^y q^{(n-1)-y}$$
$$= (k - S_0 + nd)F_{B(n,p)}(k^*) - np(u-d)F_{B(n-1,p)}(k^*-1)$$

and where $F_{B(n,p)}(\cdot)$ and $F_{B(n-1,p)}(\cdot)$ are the cumulative distribution functions of binomial variables with parameters *n* and *p* and *n* – 1 and *p* respectively.

6. Forward and Backward Recurrence Relations of an Additive Random Walk

The forward recurrence relation of an arithmetic random walk is an equation in the transition probability mass function of the walk as a function of the time variable and the forward variable of steps *y* for a single forward step into the future. This equation describes the forward convolution of the distribution of the sum of **(1)** the cumulative steps (the sum of Y_1, Y_2, …, Y_n) and **(2)** the forward proceeding two point distribution of the next step (Y_{n+1}). Noting that $S(T) = S(t_0) + Y_1 + Y_2 + ... + Y_n$ and also $S(T + \Delta t) = S(t_0) + Y_1 + Y_2 + ... + Y_n + Y_{n+1}$ the forward convolution recurrence of the walk is

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p_{S(T)+Y_{n+1}|S(t_0)}(y \mid s_0)$$
$$= p \times p_{S(T)+u|S(t_0)}(y \mid s_0) + q \times p_{S(T)+d|S(t_0)}(y \mid s_0)$$
$$= p \times p_{S(T)|S(t_0)}(y-u \mid s_0) + q \times p_{S(T)|S(t_0)}(y-d \mid s_0)$$

which makes clear that this is a convolution formula. The initial condition is the discrete Dirac at the backward variables

$$p_{S(t_0)|S(t_0)}(y \mid s_0) = \begin{cases} 0 & \text{if } y \neq s_0 \\ 1 & \text{if } y = s_0 \end{cases}$$

For a symmetric walk, $u = -d = \Delta y$, it is the statement that over any single forward increment in time, the walk variable either moves up one step from below or moves down one step from above;

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0)}(y - \Delta y \mid s_0) + q \times p_{S(T)|S(t_0)}(y + \Delta y \mid s_0)$$

The probability mass function can be seen to satisfy this relation upon inspection by virtue of the addition formula of Pascal's triangle, like the ordinary binomial probability mass function. The backward recurrence relation is the equivalent relation and conjugate, a backwards deconvolution of the two point steps from the (initial value) the backward walk variable s_0. For a fixed value of the forward time variable *T*, for a forward variation in the backward time variable from t_0 to $t_0 + \Delta t$, noting that $S(T) = S(t_0) + Y_1 + Y_2 + ... + Y_n$ and also $S(T) = S(t_0 + \Delta t) + Y_2 + ... + Y_n$ so that

$$S(T) = S(t_0) + Y_1 + Y_2 + ... + Y_n$$
$$= \left(S(t_0 + \Delta t) - Y_1\right) + Y_1 + Y_2 + ... + Y_n$$

the backward walk recurrence is

$$p_{S(T)|S(t_0)}(y \mid s_0) = p_{S(T)|S(t_0+\Delta t)-Y_1}(y \mid s_0)$$
$$= p \times p_{S(T)|S(t_0+\Delta t)-u}(y \mid s_0) + q \times p_{S(T)|S(t_0+\Delta t)-d}(y \mid s_0)$$
$$= p \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 + u) + q \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 + d)$$

subject to the discrete Dirac at the forward variables

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$$p_{S(T)|S(T)}(y \mid s_0) = \begin{cases} 0 & \text{if } s_0 \neq y \\ 1 & \text{if } s_0 = y \end{cases}$$

For $u = -d = \Delta y$, is a statement that the walk either moves upwards one step or moves downward by one step from the initial value

$$p_{S(T)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 + \Delta y) + q \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 - \Delta y)$$

for which there is one less step (Y_1 is removed from the beginning of the transition interval of S) in the forward transition for a fixed forward time variable. Note that these equations are the convolution and deconvolution equations for forward movements in T and t_0 (increasing and decreasing the length of the transition interval respectively). There are also the deconvolution and convolution equations for backward movements in T and t_0 which result when the sign on the Δt is changed (which contracts the transition interval backward from the upper end of the transition interval T or extends the transition interval backward from the lower end of the transition interval t_0).

7. Forward Equation for the Continuous Diffusion Limit

From the forward convolution recurrence

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0)}(y - \Delta y \mid s_0) + q \times p_{S(T)|S(t_0)}(y + \Delta y \mid s_0)$$

expanding the probability mass functions on the right hand side of the forward recurrence in Taylor series

(1) $$p_{S(T)|S(t_0)}(y - \Delta y \mid s_0) = p_{S(T)|S(t_0)}(y \mid s_0) - (\Delta y)\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) + o(\Delta t)$$

(2) $$p_{S(T)|S(t_0)}(y + \Delta y \mid s_0) = p_{S(T)|S(t_0)}(y \mid s_0) + (\Delta y)\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) + o(\Delta t)$$

where $o\left((\Delta y)^2\right) = o(\Delta t)$. Substituting these into the forward recurrence

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p \times \left(p_{S(T)|S(t_0)}(y \mid s_0) - (\Delta y)\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) \right)$$
$$+ q \times \left(p_{S(T)|S(t_0)}(y \mid s_0) + (\Delta y)\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) \right) + o(\Delta t)$$

that is,

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p_{S(T)|S(t_0)}(y \mid s_0) - (p - q)(\Delta y)\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) + o(\Delta t)$$

Rearranging into an expression for a forward derivative in t on the left hand side and dividing through by Δt

$$\frac{p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) - p_{S(T)|S(t_0)}(y \mid s_0)}{\Delta t} = \tfrac{1}{2}\frac{(\Delta y)^2}{\Delta t}\frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) - (p - q)\frac{(\Delta y)}{\Delta t}\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \frac{o(\Delta t)}{\Delta t}$$

Letting $\Delta t \to dt$ and $o(dt) / dt \to 0$ in the forward recurrence, and setting

$$\lim_{\Delta t \to 0} \frac{(\Delta y)^2}{\Delta t} = \sigma^2 \qquad \text{and} \qquad \lim_{\Delta t \to 0} (p - q)\frac{\Delta y}{\Delta t} = \mu$$

noting that the infinitesimal drift depends on the probability of an upstep and the size of the steps as a proportion of the time interval between each step. The forward diffusion equation of the asymptotic transition probability mass function of the walk is then in the limit

$$\frac{\partial}{\partial T} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) - \mu \frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0)$$

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which is the heat equation and the forward equation of a Brownian motion (and a statement of the central limit theorem).

8. Backward Equation for the Continuous Diffusion Limit

Considering the backward convolution recurrence

$$p_{S(T)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 + \Delta y) + q \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 - \Delta y)$$

and a bivariate Taylor series of the function f for a total increment $f(x+\Delta x, y+\Delta y)$ around the point (x, y),

$$f(x+\Delta x, y+\Delta y) = f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y + \tfrac{1}{2} f_{xx}(x,y)(\Delta x)^2 + \tfrac{1}{2} f_{yy}(x,y)(\Delta y)^2 + f_{xy}(x,y)\Delta x \Delta y$$

Expanding the probability mass functions on the right in bivariate Taylor series around the backward variables t_0 and s_0

(1) $$p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 + \Delta y) = p_{S(T)|S(t_0)}(y \mid s_0) + (\Delta t)\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) + (\Delta y)\frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + o(\Delta t)$$

(2) $$p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 - \Delta y) = p_{S(T)|S(t_0)}(y \mid s_0) + (\Delta t)\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) - (\Delta y)\frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + o(\Delta t)$$

Substituting these into the backward recurrence

$$p_{S(T)|S(t_0)}(y \mid s_0) = p \times \left(1 + (\Delta t)\frac{\partial}{\partial t_0} + (\Delta y)\frac{\partial}{\partial s_0} + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2}\right) p_{S(T)|S(t_0)}(y \mid s_0) + q \times \left(1 + (\Delta t)\frac{\partial}{\partial t_0} - (\Delta y)\frac{\partial}{\partial s_0} + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2}\right) p_{S(T)|S(t_0)}(y \mid s_0) + o(\Delta t)$$

that is

$$p_{S(T)|S(t_0)}(y \mid s_0) = \left(1 + (\Delta t)\frac{\partial}{\partial t_0} + (p-q)(\Delta y)\frac{\partial}{\partial s_0} + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2}\right) p_{S(T)|S(t_0)}(y \mid s_0) + o(\Delta t)$$

Subtracting $p_{S(T)|S(t_0)}(y \mid s_0)$ from both sides and moving the derivative in the backward time variable to the left hand side

$$-(\Delta t)\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = (p-q)(\Delta y)\frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + o(\Delta t)$$

Dividing both sides by Δt

$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = (p-q)\frac{\Delta y}{\Delta t}\frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}\frac{(\Delta y)^2}{\Delta t}\frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + \frac{o(\Delta t)}{\Delta t}$$

and letting the time between each step approach zero, $\Delta t \to dt$, (and the number of steps become infinite $n \to 1/dt$) in the forward recurrence so $o(dt)/dt \to 0$ and, as for the forward equation, setting

$$\lim_{\Delta t \to 0} \frac{(\Delta y)^2}{\Delta t} = \sigma^2 \qquad \text{and} \qquad \lim_{\Delta t \to 0} (p-q)\frac{\Delta y}{\Delta t} = \mu$$

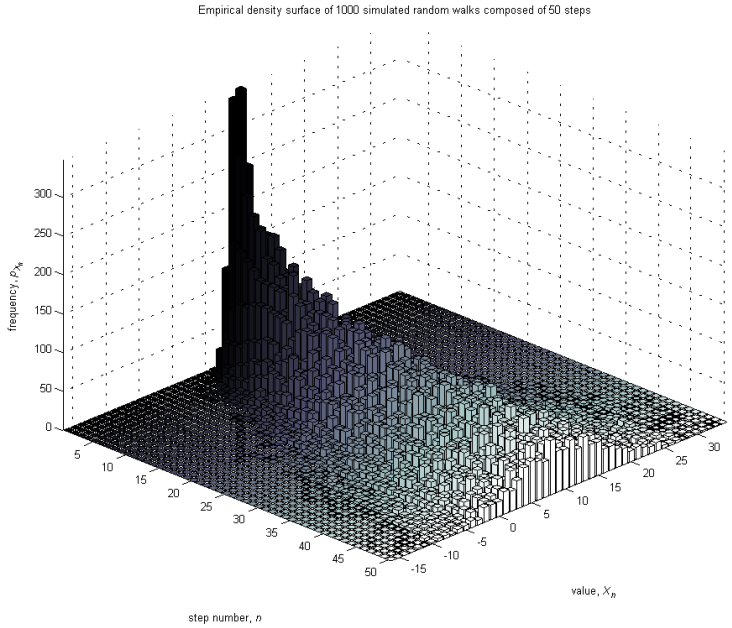
and the forward recurrence becomes

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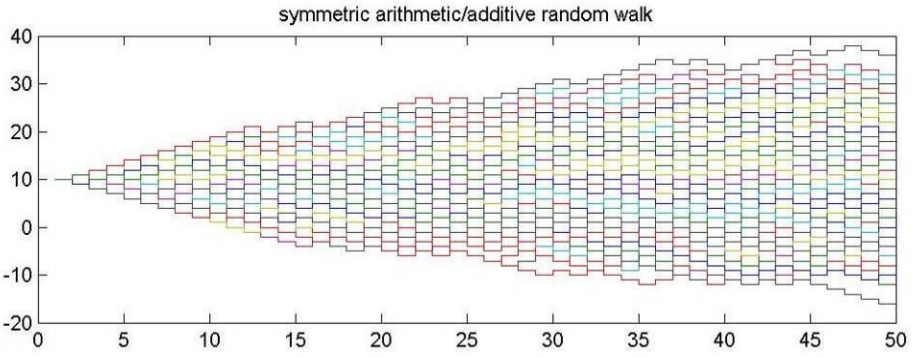
$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + \mu \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$$

which is the conjugate heat equation, the backward equation of Brownian motion. In both the forward case and the backward case, the limiting heat equation is an expression of the Central Limit Theorem and Brownian motion as the domain of attraction for the Central Limit as sums of homogenous random variables aggregate

$$S(T) \mid S(t_0) = s_0$$
$$\sim N\left(s_0 + \mu(T - t_0),\, \sigma^2 (T - t_0)\right)$$



Above is the empirical probability surface of 1000 simulated ordinary additive random walks each composed of 50 steps for which it can be seen resembles closely the normal transition density surface of Brownian motion and the dynamics prototypical to the heat equation, noting the Dirac spike at the backward variable (the starting value $y_0 = 10$ of the walk) common to all diffusion variables. This is 50 histograms side by side in succession, one for each successive step. The simulated variables for which this empirical probability surface was plotted are



which forms (has filled in the majority of) the diffusion sample space over which the transition surface is graphed.

B. Multiplicative Random Walk

1. Statistical Construction and Forward Transition Distribution

A multiplicative random walk is the exponential transformation of the additive arithmetic random walk. This variable corresponds to the log-binomial option valuation model (whether it is demeaned or otherwise). Firstly,

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transforming the individual components of the geometric random walk, the multiplicative secondary distributions, so that each upstep results in a proportional increase of factor u ($u > 1$) and each downstep results in a proportional decrease of factor d ($0 < d < 1$)

$$Y_j = u^{X_j} d^{1-X_j} = d(u/d)^{X_j} = \begin{cases} d & \text{with probability } q = 1-p \\ u & \text{with probability } p \end{cases}$$

where X_j is defined as for the additive walk, the Bernoulli indicator of the j^{th} movement $X_j \sim B(1,p)$. The inverse function of the transformation of the forward walk variable is $x_j = \log(u/d)/\log(u/d) = (\log y_j - \log d)/\log(u/d)$, and noting again that there is no Jacobian since the variable is discrete, the probability mass function is then for each proportional step

$$p_{Y_j}(y_j) = p_{X_j}(x_j) = p_{X_j}\left((\log y_j - \log d) \Big/ \log(u/d) \right) = p^{(\log y_j - \log d)/\log(u/d)} q^{(\log u - \log y_j)/\log(u/d)} \qquad y_j = d, u$$

which is the mass function of a component of a discrete two point multiplicative basis that defines the multiplicative walk. Noting that, on logs, the walk step variables becomes

$$\log Y_j = X_j \log u + (1-X_j)\log d = \begin{cases} \log d & \text{with probability } q = 1-p \\ \log u & \text{with probability } p \end{cases}$$

and the steps of the multiplicative walk revert to those of the additive walk. The product convolved steps of the walk specifies the net forward transition distribution of the walk n steps ahead

$$\begin{aligned} Y &= Y_1 \times Y_2 \times \ldots \times Y_n \\ &= \left(u^{X_1} d^{1-X_1}\right) \times \left(u^{X_2} d^{1-X_2}\right) \times \ldots \times \left(u^{X_n} d^{1-X_n}\right) \\ &\sim u^X d^{n-X} \\ &\sim d^n (u/d)^{B(n,p)} \end{aligned}$$

where X is the convolved sum of the individual X_j, $X = X_1 + X_2 + \ldots + X_n \sim B(n,p)$ and Y is the convolved product of the individual Y_j and is therefore the **continued product** of n scaled log-Bernoulli zero-one variables. The distribution of the transition of the walk for n steps is then an exponentiated binomial, termed by this thesis as a log-binomial (consistent with log-normal), and the probability mass function is

$$p_Y(y) = p_X(x) = p_X\left(\log(y/d^n) \Big/ \log(u/d) \right) = {}^nC_{\log(y/d^n)/\log(u/d)} p^{\log(y/d^n)/\log(u/d)} q^{\log(u^n/y)/\log(u/d)}$$

for $y = d^n, u^1 d^{n-1}, u^2 d^{n-2} u, u^3 d^{n-3} \ldots, u^{n-2} d^2, u^{n-1} d^1, u^n$.

Letting the time interval between each step be Δt so that by the n^{th} step $t = n\Delta t$ time units have elapsed. Defining the variable that measures the walk at time t be S. Introducing the initial value S_0, the backward variable of the walk, for the aggregate walk variable S by convolving this multiplicatively onto the beginning of the aggregate product of the walk's forward steps so that the aggregate walk variable S becomes the log-binomial variable

$$\begin{aligned} S(t) &= S_0 \times Y_1 \times Y_2 \times \ldots \times Y_n \\ &\sim S_0 u^X d^{n-X} \\ &\sim S_0 d^n (u/d)^X \end{aligned}$$

The transition probability mass function of the aggregate walk variable S is then

$$p_{S(t)|S(0)}(y \mid s_0) = p_Y(y/s_0) = p_X\left(\log(y/s_0 d^n) \Big/ \log(u/d) \right) = {}^nC_{\log(y/s_0 d^n)/\log(u/d)} p^{\log(y/s_0 d^n)/\log(u/d)} q^{\log(s_0 u^n/y)/\log(u/d)}$$

for $y = s_0 d^n, s_0 u^1 d^{n-1}, s_0 u^2 d^{n-2}, s_0 u^3 d^{n-3} \ldots, s_0 u^{n-2} d^2, s_0 u^{n-1} d^1, s_0 u^n$

and where, since the walk is time homogenous $S(t) \mid S(0)$ is statistically equivalent to $S(T) \mid S(t_0)$ where t is the transition length and t_0 and T are the backward and forward variables corresponding to the points in time where the transition interval begins and ends. Consequently, $p_{S(t)|S(0)}(y \mid s_0) = p_{S(T)|S(t_0)}(y \mid s_0)$ for $t = T - t_0$.

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Similarly to the additive walk, as a simplification and to keep the walk's sample space symmetric and recombining, the magnitude of the upward and downward steps are set as reciprocals $u = 1 / d = \exp(\Delta\log y)$, i.e., equal in magnitude but opposite in sign on logs $\log u = -\log d = \Delta\log y$. The distribution of the j^{th} step becomes

$$Y_j = e^{(2X_j-1)\Delta\log y} = \begin{cases} e^{-\Delta\log y} & \text{with probability } q = 1-p \\ e^{+\Delta\log y} & \text{with probability } p \end{cases}$$

with log two point distribution

$$Y_j \sim e^{(2B(1,p)-1)\Delta\log y}$$

and probability mass function

$$p_{Y_j}(y_j) = p_{X_j}(x_j) = p_X\left(\log y_j + \Delta\log y \Big/ 2\Delta\log y\right) = p^{(\log y_j + \Delta\log y)/2\Delta\log y} q^{(\Delta\log y - \log y_j)/2\Delta\log y}$$

for $y_j = e^{-\Delta\log y}, e^{\Delta\log y}$. The walk variable is then

$$\begin{aligned} S(t) &= s_0 \times Y_1 \times Y_2 \times \ldots \times Y_n \\ &= s_0 \times \left(e^{(2X_1-1)\Delta\log y}\right) \times \left(e^{(2X_2-1)\Delta\log y}\right) \times \ldots \times e^{(2X_n-1)\Delta\log y} \\ &\sim s_0 \times e^{(2X-n)\Delta\log y} \\ &\sim s_0 \times e^{(2B(n,p)-n)\Delta\log y} \end{aligned}$$

with transition probability mass function

$$p_{S(t)|S(0)}(y \mid s_0) = p_X\left(\log(y e^{n\Delta\log y} / s_0) \Big/ 2\log h\right) = {}^nC_{\log(y e^{n\Delta\log yn} / s_0)/2\log h} p^{\log(y e^{n\Delta\log y}/s_0)/2\log h} q^{\log(s_0 e^{n\Delta\log y}/y)/2\log h}$$

for $y = s_0 e^{-n\Delta\log y}, s_0 e^{-(n-2)\Delta\log y}, s_0 e^{-(n-4)\Delta\log y}, \ldots, s_0 e^{(n-4)\Delta\log y}, s_0 e^{(n-2)\Delta\log y}, s_0 e^{n\Delta\log y}$

Since this is the resulting probability mass function of the exponentiation of the ordinary summed walk, the logarithm of the produced walk results in exactly that of the summed walk and all of its statistical functions including the transition probability mass function.

2. MGF of the Geometric Random Walk

For the multiplicative walk, where the basis is closed as a group only on multiplication and division, the moment generating function is expressible only in terms of power series of moments, as a series expansion in terms of the moment generating function of the additive walk; in other words, the mgf of an exponential diffusion (termed log-diffusion) is only expressible as an exponential series expansion of the underlying additive diffusion. The exponential transformation of the additive random walk to the multiplicative random walk results in a moment generating function conditional on the moment generating function parameter where the parameter is Poisson distributed

$$\begin{aligned} M_{S(T)|S(t_0),\Theta}(\Theta) &= e^{\log S_0 \Theta}(p e^{\Theta\log u} + q e^{\Theta\log d})^n \\ &= S_0{}^{\Theta}(p u^{\Theta} + q d^{\Theta})^n \end{aligned} \qquad \text{where} \qquad \Theta \sim P(\theta)$$

The expression for the unconditional moment generating function of the multiplicative walk, after summing out the random parameter, is then

$$M_{S(T)|S(t_0)}(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j!} M_{\log S(T)|\log S(t_0)}(j)$$

i.e.,

$$M_{S(T)|S(t_0)}(\theta) = \sum_{j=0}^{\infty} \frac{(S_0\theta)^j}{j!}(p u^j + q d^j)^n$$

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The consequence of this is that the cumulants of the additive walk form the raw moments of the exponential walk. If the upstep and downstep ratios are reciprocal, then the mgf of the underlying additive walk is

$$M_{S(T)|S(t_0),\Theta}(\Theta) = S_0^{\Theta}(pe^{\Theta\Delta\log y} + qe^{-\Theta\Delta\log y})^n \qquad \text{where} \qquad \Theta \sim P(\theta)$$

and the unconditional moment generating function of the multiplicative walk is

$$M_{S(T)|S(t_0)}(\theta) = \sum_{j=0}^{\infty} \frac{(S_0\theta)^j}{j!}(pe^{j\Delta\log y} + qe^{-j\Delta\log y})^n$$

3. Transition Mean and Variance

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4. Geometric Random Walk Simulation Algorithm

Simulation algorithm: Generate a vector of n Bernoulli indictor variables, zero and ones as for the arithmetic random walk, $X_j \sim B(1, p)$. Multiply each element of the random vector by the logarithm of the ratio of the up and down factors, $\log(u/d)$, and add $\log d$ to each component to create the individual log-steps,

$$\begin{aligned}\log Y_j &= (\log u - \log d)X_j + \log d \\ &\sim (\log u - \log d)B(1, p)\log d\end{aligned}$$

Each element is the successive log steps $Y_j = (\log u - \log d)X_j + \log d$

$$\left[\log Y_1, \log Y_2, ..., \log Y_n\right] = \left[(\log u - \log d)X_1 + \log d, (\log u - \log d)X_2 + \log d, ..., (\log u - \log d)X_n + \log d\right]$$

and forms the incremental components of the logarithm of the walk $\log Y_j = S_j - S_{j-1}$, i.e., the logarithm of the successive ratios (for example, steps in of asset return). Appending the initial value, $\log S_0$, to the beginning of the vector results in the statistically orthogonal vector of the walk's log increments

$$\left[\log S_0, \log Y_1, \log Y_2, ..., \log Y_n\right] = \left[\log S_0, \log S_1 - \log S_0, \log S_2 - \log S_1, ..., \log S_n - \log S_{n-1}\right]$$

Cumulatively summing results in the vector of the walk for successively increases in transition length

$$\log S = \left[\log S_0, \log S_1, \log S_2, ..., \log S_n\right]$$

Exponentiating each element for the sequence of ordinates of the simulated outcomes for a single path of a geometric random walk consistent with that used in the Binomial model for options,

$$\begin{aligned}S &= \left[S_0, S_1, S_2, ..., S_n\right] \\ &= \left[S_0, S_0 u^{X_1} d^{1-X_1}, S_0 u^{(X_1+X_2)} d^{2-(X_1+X_2)}, ..., S_0 u^{(X_1+X_2+...+X_n)} d^{n-(X_1+X_2+...+X_n)}\right]\end{aligned}$$

The vector can be graphed against the step number, this being the index of time. MATLAB code for this simulation algorithm is

```
p=p; u=u; d=d; s0=s0; t0=t0; n=n; dt=Δt;      %initialize walk parameters
t=cumsum([t0;ones(n,1)).*dt]);        %step times on transition interval
x=binornd(1,p,n,1); S=exp(cumsum([s0; log(u/d).*x+log(d)]));    %simulate
plot(t,S);                                                    %plot S over t
```

where last line includes continuous time t in the graphics, as for the additive walk, by multiplying the step number by the length of time between each step and adding these to the backward time variable t_0 and produces the graph S vs t. The simulated path is graphed against step number with `plot(S) %plot S over n`. As an example, for a log- symmetric multiplicative walk of `n` =100 steps starting at time 2 (o'clock) with value s0=10 for a forward transition of length `n*dt`=100*(5/100)=5 hours to time `t0+n*dt` =7 (o'clock), initialize the parameters with the line

```
p=.5; u=1.05; d=1/u; s0=10; t0=2; n=100; dt=5/100;   %initialize parameters
```

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where a step occurs, like the additive walk, once every `dt`=5/100 hours = 3 minutes over the 5 hour simulated transition period. Observe that the logarithm of u is ten times smaller than the example additive walk in order to scale down the exponential growth in *S* which compounds, in this example, at 5% for every upstep and discounts at 5% for every downstep.

5. Limited Moments and Undiscounted Option Premia of the Geometric Random Walk

The **multiplicative** *n*-period ahead forward walk variable, composed of X^{*n} upsteps and $n - X^{*n}$ downsteps, is

$$S_n = S_0 u^{X^{*n}} d^{n-X^{*n}}$$

where $X^{*n} \sim B(n, p)$. The payoff of an option on the walk variable is then

$\max(S_n - k, 0) = \max(S_0 u^{X^{*n}} d^{n-X^{*n}} - k, 0)$ if it is an option to call the asset to buy for *k*

$\max(k - S_n, 0) = \max(k - S_0 u^{X^{*n}} d^{n-X^{*n}}, 0)$ if it is an option to put the asset to sell for *k*

By statistical formula, $E\left(g(X)\right) = \sum_{x \in X} g(x) \times p_X(x)$, the undiscounted premium for a call is

$$\begin{aligned} c(n,k) &= E\left(\max(S_0 u^{X^{*n}} d^{n-X^{*n}} - k, 0)\right) \\ &= \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} \max(S_0 u^x d^{n-x} - k, 0) \end{aligned}$$

and the max function is non-zero whenever

$$\begin{aligned} &S_0 u^x d^{n-x} - k \geq 0 \\ &(u/d)^x \geq k / S_0 d^n \\ &x \geq \log(k / S_0 d^n) / \log(u/d) \end{aligned}$$

so that only terms in the summation for *x* greater than $k^* = \log(k / S_0 d^n) / \log(u/d)$ are non-zero

$$\begin{aligned} c(n,k) &= \sum_{x=k^*}^{n} \left(S_0 u^x d^{n-x} - k\right) {}^nC_x p^x q^{n-x} \\ &= S_0 \sum_{x=k^*}^{n} {}^nC_x (up)^x (dq)^{n-x} - k \sum_{x=k^*}^{n} {}^nC_x p^x q^{n-x} \\ &= S_0 (up+dq)^n \sum_{x=k^*}^{n} {}^nC_x \left(up/(up+dq)\right)^x \left(dq/(up+dq)\right)^{n-x} - k \sum_{x=k^*}^{n} {}^nC_x p^x q^{n-x} \\ &= S_0 (up+dq)^n \overline{F}_{B(n,up/(up+dq))}(k^*) - k \overline{F}_{B(n,p)}(k^*) \end{aligned}$$

where *k** is the smallest integer greater than $k^* = \log(k / S_0 d^n) / \log(u/d)$ and where $\overline{F}_{B(n,up/(up+dq))}(\cdot)$ and $\overline{F}_{B(n,p)}(\cdot)$ are the complementary cumulative distribution functions of binomial variables with parameters *n* and $up / (up + dq)$ and *n* and *p* respectively. For a put option

$$\begin{aligned} p(n,k) &= E\left(\max(k - S_0 u^{X^{*n}} d^{n-X^{*n}}, 0)\right) \\ &= \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} \max(k - S_0 u^{X^{*n}} d^{n-X^{*n}}, 0) \end{aligned}$$

and for *k** being the largest integer equal to or smaller than $(k - S_0 - nd)/(u-d)$ so that the undiscounted premium is

$$\begin{aligned} p(n,k) &= \sum_{x=0}^{k^*} \left(k - S_0 u^x d^{n-x}\right) {}^nC_x p^x q^{n-x} \\ &= k \sum_{x=0}^{k^*} {}^nC_x p^x q^{n-x} - S_0 \sum_{x=0}^{k^*} {}^nC_x (up)^x (dq)^{n-x} \\ &= k \sum_{x=0}^{k^*} {}^nC_x p^x q^{n-x} - S_0 (up+dq)^n \sum_{x=0}^{k^*} {}^nC_x \left(up/(up+dq)\right)^x \left(dq/(up+dq)\right)^{n-x} \\ &= k F_{B(n,p)}(k^*) - S_0 (up+dq)^n F_{B(n,up/(up+dq))}(k^*) \end{aligned}$$

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where k^* is the greatest integer smaller than $k^* = \log(k / S_0 d^n) / \log(u / d)$ and where $F_{B(n,up/(up+dq))}(\cdot)$ and $F_{B(n,p)}(\cdot)$ are the cumulative distribution functions of binomial variables with parameters n and $up / (up + dq)$ and n and p respectively.

6. Forward and Backward Recurrence Relations of a Geometric Random Walk

The forward recurrence relation of a multiplicative random walk, similarly to an ordinary walk, is an equation in the transition probability mass function of the walk as a function of the forward time variable and the forward variable of proportional steps y for a single forward step into the future. This equation describes the forward product convolution of the distribution of the product of **(1)** the continued product (cumulative ratio) of steps and **(2)** the forward proceeding two point distribution of the next step ratio. Noting that $S(T) = S(t_0) \times Y_1 \times Y_2 \times ... \times Y_n$ and also $S(T + \Delta t) = S(t_0) \times Y_1 \times Y_2 \times ... \times Y_n \times Y_{n+1}$, the forward convolution recurrence of the multiplicative walk is

$$\begin{aligned} p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) &= p_{S(T)\times Y_{n+1}|S(t_0)}(y \mid s_0) \\ &= p \times p_{S(T)\times u|S(t_0)}(y \mid s_0) + q \times p_{S(T)\times d|S(t_0)}(y \mid s_0) \\ &= p \times p_{S(T)|S(t_0)}(y \div u \mid s_0) + q \times p_{S(T)|S(t_0)}(y \div d \mid s_0) \end{aligned}$$

subject to the discrete Dirac at the backward variables

$$p_{S(t_0)|S(t_0)}(y \mid s_0) = \begin{cases} 0 & \text{if } y \neq s_0 \\ 1 & \text{if } y = s_0 \end{cases}$$

consistent with a product convolution formula for a two point distribution. There is, then, one more step ratio Y_{n+1} in the cumulative product of the forward transition for the given starting value s_0 at t_0 and the increase in the forward time variable to $T + \Delta t$. For up and down ratios that are reciprocals $u = -d = \exp(\Delta \log y)$, which are symmetric up and down steps on logs, the forward convolution equation over any single forward increment in time describes that the walk either increases by one step factor from below or decreases by one step factor from above;

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0)}(y \div e^{\Delta \log y} \mid s_0) + q \times p_{S(T)|S(t_0)}(y \times e^{\Delta \log y} \mid s_0)$$

The probability mass function, similarly to the additive walk, obeys the relation of the addition formula of Pascal's triangle which is, in this context, how the binomial indicators convolve and deconvolve for changes in the number of steps in the transition interval. The backward recurrence relation is the equivalent relation and conjugate, a backwards ratio deconvolution of the two point step distribution from the backward walk variable s_0. For a fixed value of the forward time variable T, increasing the backward time variable from t_0 to $t_0 + \Delta t$, noting that $S(T) = S(t_0) \times Y_1 \times Y_2 \times ... \times Y_n$ and also $S(T) = S(t_0 + \Delta t) \times Y_2 \times ... \times Y_n$ so that

$$\begin{aligned} S(T) &= S(t_0) \times Y_1 \times Y_2 \times ... \times Y_n \\ &= \left(S(t_0 + \Delta t) \div Y_1\right) \times Y_1 \times Y_2 \times ... \times Y_n \end{aligned}$$

the backward walk recurrence is

$$\begin{aligned} p_{S(T)|S(t_0)}(y \mid s_0) &= p_{S(T)|S(t_0+\Delta t)\div Y_1}(y \mid s_0) \\ &= p \times p_{S(T)|S(t_0+\Delta t)\div u}(y \mid s_0) + q \times p_{S(T)|S(t_0+\Delta t)\div d}(y \mid s_0) \\ &= p \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 \times u) + q \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 \times d) \end{aligned}$$

subject to the discrete Dirac at the forward variables

$$p_{S(T)|S(T)}(y \mid s_0) = \begin{cases} 0 & \text{if } s_0 \neq y \\ 1 & \text{if } s_0 = y \end{cases}$$

which is the same as for the additive walk. For up and down ratios that are reciprocals, $u = -d = \exp(\Delta \log y)$, the deconvolution recurrence is a statement that the walk either increases or decreases by one step ratio from the initial value as the step ratio Y_1 is removed from the walk variable at the beginning of the transition interval

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$$p_{S(T)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 e^{\Delta \log y}) + q \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 e^{-\Delta \log y})$$

These are the convolution and deconvolution equations corresponding to forward movements in T or t_0, product convolution for increases in T and product deconvolution for increases in t_0 (which result in increases in the transition period for an extension in T and a decrease in the transition period for an increase in t_0). The corresponding product deconvolution and convolution for decreases in T and t_0 are those that correspond to a change in the sign of the Δt (and either removing the time T step or adding a step to precede the step Y_1 at t_0).

7. Forward Equation of the Continuous Diffusion Limit

From the forward product convolution recurrence

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0)}(y \div e^{\Delta \log y} \mid s_0) + q \times p_{S(T)|S(t_0)}(y \times e^{\Delta \log y} \mid s_0)$$

changing the walk variable to its logarithm so that the transition functions can be expanded in power series

$$\begin{aligned} &p_{\log S(T+\Delta t)|\log S(t_0)}(\log y \mid \log s_0) \\ &\quad = p \times p_{\log S(T)|\log S(t_0)}(\log y - \Delta \log y \mid \log s_0) + q \times p_{\log S(T)|\log S(t_0)}(\log y + \Delta \log y \mid \log s_0) \end{aligned}$$

where no Jacobian is required because the variable is discrete. Expanding the transition mass functions on the right-hand-side in a Taylor series around the logarithm of the forward walk variable log y, one expansion for an upstep from below and one for a downstep from above,

(1) $p_{\log S(T)|\log S(t_0)}(\log y - \Delta \log y \mid \log s_0)$

$$\begin{aligned} &= p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) - (\Delta \log y) \frac{\partial}{\partial \log y} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) \\ &\qquad + \tfrac{1}{2} (\Delta \log y)^2 \frac{\partial^2}{(\partial \log y)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + o\left((\Delta \log y)^2\right) \end{aligned}$$

(2) $p_{\log S(T)|\log S(t_0)}(\log y + \Delta \log y \mid \log s_0)$

$$\begin{aligned} &= p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + (\Delta \log y) \frac{\partial}{\partial \log y} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) \\ &\qquad + \tfrac{1}{2} (\Delta \log y)^2 \frac{\partial^2}{(\partial \log y)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + o\left((\Delta \log y)^2\right) \end{aligned}$$

and substituting these into the forward convolution recurrence formula, noting that $o\left((\Delta \log y)^2\right) = o(\Delta t)$

$$\begin{aligned} p_{\log S(T+\Delta t)|\log S(t_0)}(\log y \mid \log s_0) &= p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) \\ &\quad - (p-q)(\Delta \log y) \frac{\partial}{\partial \log y} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) \\ &\quad + \tfrac{1}{2} (\Delta \log y)^2 \frac{\partial^2}{(\partial \log y)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + o(\Delta t) \end{aligned}$$

Subtracting $p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$ from both sides and dividing through by Δt,

$$\begin{aligned} &\frac{p_{\log S(T+\Delta t)|\log S(t_0)}(\log y \mid \log s_0) - p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)}{\Delta t} \\ &= \tfrac{1}{2} \frac{(\Delta \log y)^2}{\Delta t} \frac{\partial^2}{(\partial \log y)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) - (p-q) \frac{\Delta \log y}{\Delta t} \frac{\partial}{\partial \log y} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \frac{o(\Delta t)}{\Delta t} \end{aligned}$$

Letting the time between each step approach zero, $\Delta t \to dt$, so $o(\Delta t) / \Delta t \to 0$ and defining the infinitesimal parameters as the limits

$$\lim_{\Delta t \to 0} \frac{(\Delta \log y)^2}{\Delta t} = \sigma^2 \qquad \text{and} \qquad \lim_{\Delta t \to 0} (p-q) \frac{\Delta \log y}{\Delta t} = \mu$$

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the forward recurrence for the logarithm of the walk becomes

$$\frac{\partial}{\partial T}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$

$$= \tfrac{1}{2}\sigma^2 \frac{\partial^2}{(\partial \log y)^2}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) - \mu \frac{\partial}{\partial \log y}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$

Transforming the transition probability distribution back to that of the original walk variable, $Y_t = \exp(X_t)$, which now requires a Jacobian scale factor since the variable of the walk function is now continuous (a dense sample space), the Jacobian being

$$\frac{dy}{dx} = e^x = y$$

so that the density function becomes $p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\, p_{S(T)|S(t_0)}(y \mid s_0)$. The time derivative of the left-hand-sides becomes

$$\frac{\partial}{\partial T}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y \frac{\partial}{\partial T}\, p_{S(T)|S(t_0)}(y \mid s_0)$$

and using the relations from the product rule, the partial derivatives in the diffusion variable become

$$\frac{\partial}{\partial \log y}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = \frac{\partial y}{\partial \log y}\frac{\partial}{\partial y}\Big(y\; p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

$$= y \frac{\partial}{\partial y}\Big(y\; p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

and[81]

$$\frac{\partial^2}{(\partial \log y)^2}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = \frac{\partial}{\partial \log y}\left(\frac{\partial}{\partial \log y}\, y\, p_{S(T)|S(t_0)}(y \mid s_0)\right)$$

$$= \frac{\partial y}{\partial \log y}\frac{\partial}{\partial y}\left(\frac{\partial y}{\partial \log y}\frac{\partial}{\partial y}\Big(y\, p_{S(T)|S(t_0)}(y \mid s_0)\Big)\right)$$

$$= y \frac{\partial}{\partial y}\left(y \frac{\partial}{\partial y}\Big(y\, p_{S(T)|S(t_0)}(y \mid s_0)\Big)\right)$$

$$= y \frac{\partial}{\partial y}\left(y \frac{\partial}{\partial y}\left(\frac{y^2 p_{S(T)|S(t_0)}(y \mid s_0)}{y}\right)\right)$$

$$= y \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\Big(y^2\, p_{S(T)|S(t_0)}(y \mid s_0)\Big)\right) - y \frac{\partial}{\partial y}\Big(y\, p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

$$= y \frac{\partial^2}{\partial y^2}\Big(y^2\, p_{S(T)|S(t_0)}(y \mid s_0)\Big) - y \frac{\partial}{\partial y}\Big(y\, p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

Substituting these derivatives into the forward equation

$$y \frac{\partial}{\partial T}\, p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 \left\{ y \frac{\partial^2}{\partial y^2}\Big(y^2\, p_{S(T)|S(t_0)}(y \mid s_0)\Big) - y \frac{\partial}{\partial y}\Big(y\, p_{S(T)|S(t_0)}(y \mid s_0)\Big)\right\} - \mu y \frac{\partial}{\partial y}\Big(y\, p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

and canceling the Jacobian y on both sides for

---

[81] the last line of working is alternatively obtained from the mathematical relation

$$\frac{\partial^2}{\partial x^2}\Big(x^2 f(x)\Big) = \frac{\partial^2}{\partial x^2}\, x\,(x\, f(x)) = \frac{\partial}{\partial x}\left(x\, f(x) + x \frac{\partial}{\partial x}\Big(x\, f(x)\Big)\right) = \frac{\partial}{\partial x}\Big(x\, f(x)\Big) + x \frac{\partial^2}{\partial x^2}\Big(x\, f(x)\Big)$$

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$$\frac{\partial}{\partial T} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2}\left(y^2\, p_{S(T)|S(t_0)}(y \mid s_0)\right) - \left(\mu + \tfrac{1}{2}\sigma^2\right)\frac{\partial}{\partial y}\left(y\, p_{S(T)|S(t_0)}(y \mid s_0)\right)$$

which is the forward equation for a proportional Brownian motion.

8. Backward Equation of the Continuous Diffusion Limit

From the backward deconvolution recurrence

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0)}(y \div e^{\Delta \log y} \mid s_0) + q \times p_{S(T)|S(t_0)}(y \times e^{\Delta \log y} \mid s_0)$$

changing the walk variable to its logarithm

$$p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = p \times p_{\log S(T)|\log S(t_0+\Delta t)}(\log y \mid \log s_0 + \Delta \log y) + q \times p_{\log S(T)|\log S(t_0+\Delta t)}(\log y \mid \log s_0 - \Delta \log y)$$

Expanding the transition functions on the right-hand-side in bivariate Taylor series around the backward time variable t_0 and the logarithm of the backward walk variable $\log s_0$

$$\textbf{(1)}\; p_{\log S(T)|\log S(t_0+\Delta t)}(\log y \mid \log s_0 + \Delta \log y) = p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + (\Delta t)\frac{\partial}{\partial t_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$+ (\Delta \log y)\frac{\partial}{\partial \log s_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \tfrac{1}{2}(\Delta \log y)^2 \frac{\partial^2}{(\partial \log s_0)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + o(\Delta t)$$

$$\textbf{(2)}\; p_{\log S(T)|\log S(t_0+\Delta t)}(\log y \mid \log s_0 - \Delta \log y) = p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + (\Delta t)\frac{\partial}{\partial t_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$- (\Delta \log y)\frac{\partial}{\partial \log s_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \tfrac{1}{2}(\Delta \log y)^2 \frac{\partial^2}{(\partial \log s_0)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + o(\Delta t)$$

Substituting these expressions into the backward deconvolution recurrence

$$p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + (p-q)(\Delta \log y)\frac{\partial}{\partial \log s_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$+ \frac{\partial}{\partial t_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \tfrac{1}{2}(\Delta \log y)^2 \frac{\partial^2}{(\partial \log s_0)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + o(\Delta t)$$

Cancelling $p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$ from both sides and dividing through by Δt for

$$-\frac{\partial}{\partial t_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = \tfrac{1}{2}\frac{(\Delta \log y)^2}{\Delta t}\frac{\partial^2}{(\partial \log s_0)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$+ (p-q)\frac{\Delta \log y}{(\Delta t)}\frac{\partial}{\partial \log s_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \frac{o(\Delta t)}{\Delta t}$$

Letting $\Delta t \to dt$ so that $o(dt) / dt \to 0$ and letting the infinitesimal parameters be defined as the limits

$$\lim_{\Delta t \to 0} \frac{(\Delta \log y)^2}{\Delta t} = \sigma^2 \qquad \text{and} \qquad \lim_{\Delta t \to 0} (p-q)\frac{\Delta \log y}{\Delta t} = \mu$$

the backwards recurrence becomes

$$-\frac{\partial}{\partial t_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{(\partial \log s_0)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \mu \frac{\partial}{\partial \log s_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$

Transforming the logarithm of the walk variable to the original variable with exponentiation where the Jacobian is a proportional scale y so that

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$$p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\, p_{S(T)|S(t_0)}(y \mid s_0)$$

The partial derivatives in the independent variable from $\log y_0$ to y_0 transform to

$$\frac{\partial}{\partial t_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y \frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0)$$

$$\frac{\partial}{\partial \log s_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = \frac{\partial s_0}{\partial \log s_0} \frac{\partial}{\partial s_0} y\, p_{S(T)|S(t_0)}(y \mid s_0) = y\, s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$$

$$\begin{aligned}\frac{\partial^2}{(\partial \log s_0)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) &= \frac{\partial s_0}{\partial \log s_0} \frac{\partial}{\partial s_0}\left(\frac{\partial s_0}{\partial \log s_0} \frac{\partial}{\partial s_0} y\, p_{S(T)|S(t_0)}(y \mid s_0) \right) \\ &= y s_0 \frac{\partial}{\partial s_0}\left(s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) \right) \\ &= y\, {s_0}^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + y s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)\end{aligned}$$

which, upon substitution

$$-y \frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 \left(y\, {s_0}^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + y s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) \right) + \mu y s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$$

and cancelling the Jacobian

$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 {s_0}^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + \left(\mu + \tfrac{1}{2}\sigma^2 \right) s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$$

which is the backward equation of a proportional Brownian motion.

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C. Simulated Paths of Random Walks

SIMULATED ORDINARY ARITHMETIC RANDOM WALKS

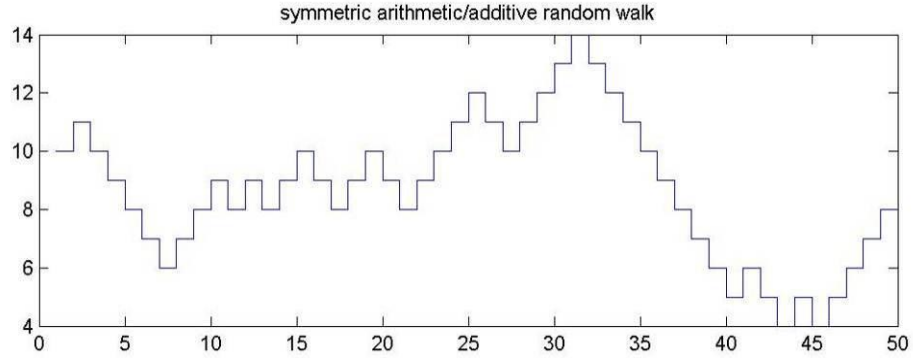


Figure 36: Arithmetic random walk - single simulated paths (50 steps)

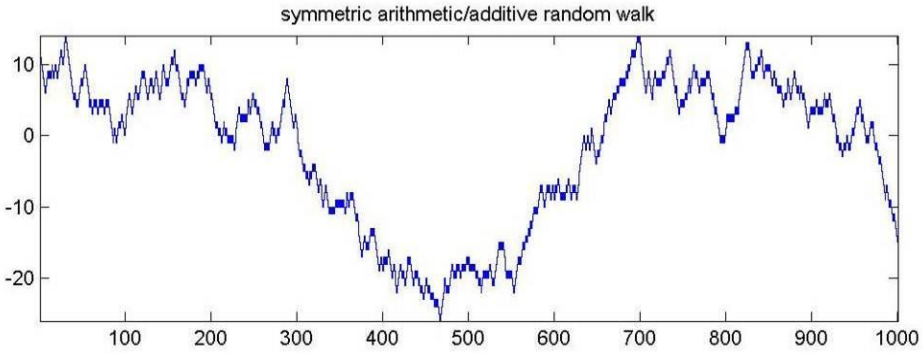


Figure 37: Arithmetic random walk - single simulated path (1,000 steps)

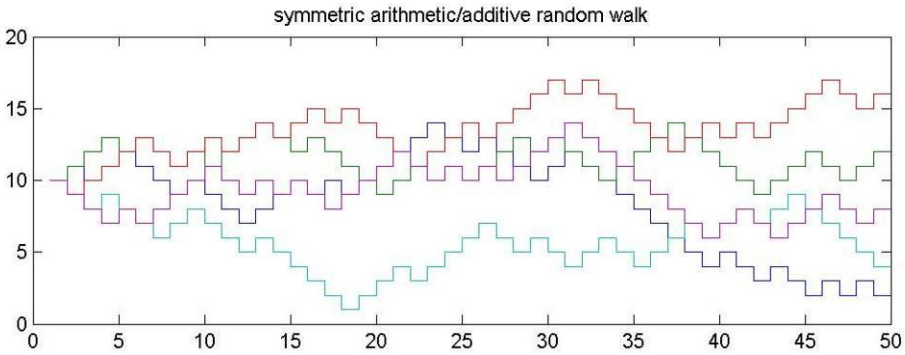


Figure 38: Arithmetic random walk - 5 simulated paths (50 steps)

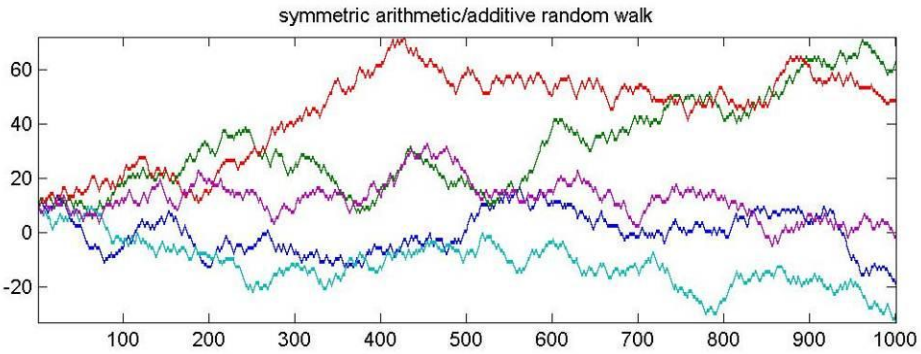


Figure 39: Arithmetic random walk - 5 simulated paths (1,000 steps)

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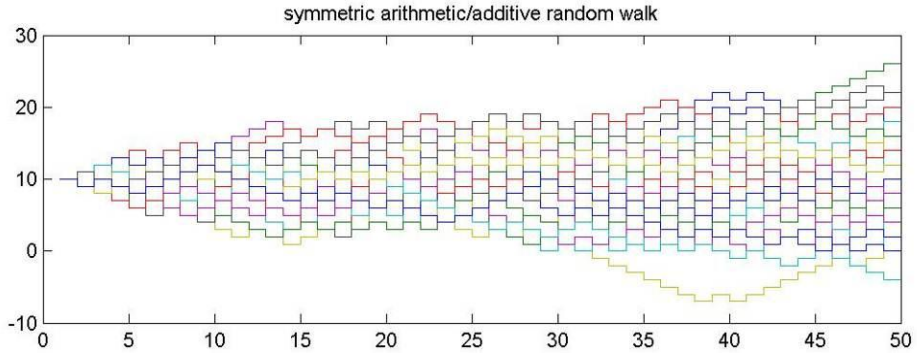


Figure 40: Arithmetic random walk - 50 simulated paths (50 steps)

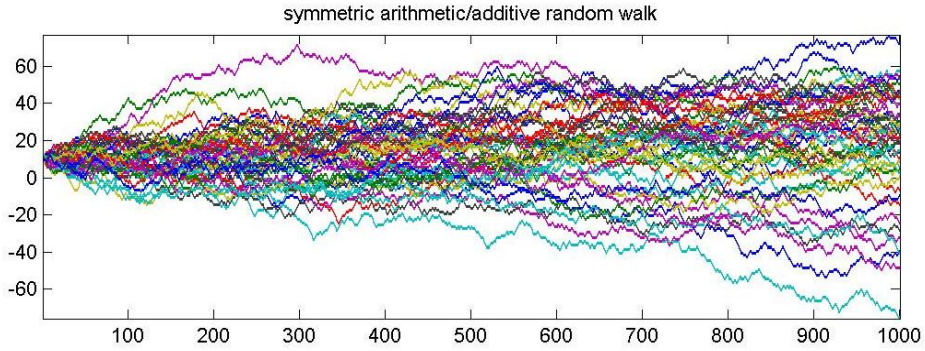


Figure 41: Arithmetic random walk - 50 simulated paths (1,000 steps)

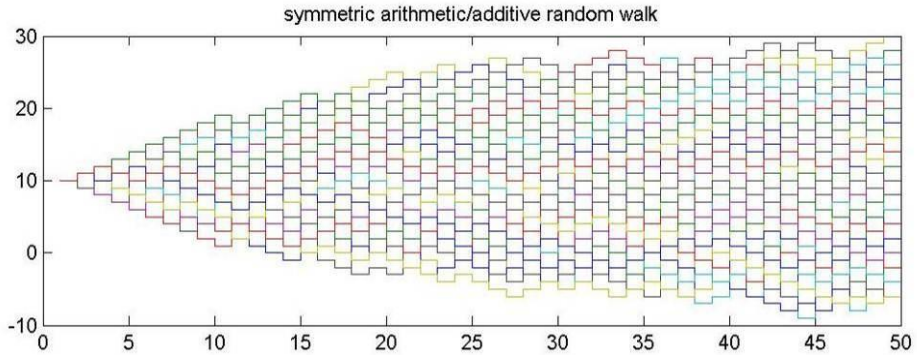


Figure 42: Arithmetic random walk - 500 simulated paths (50 steps)

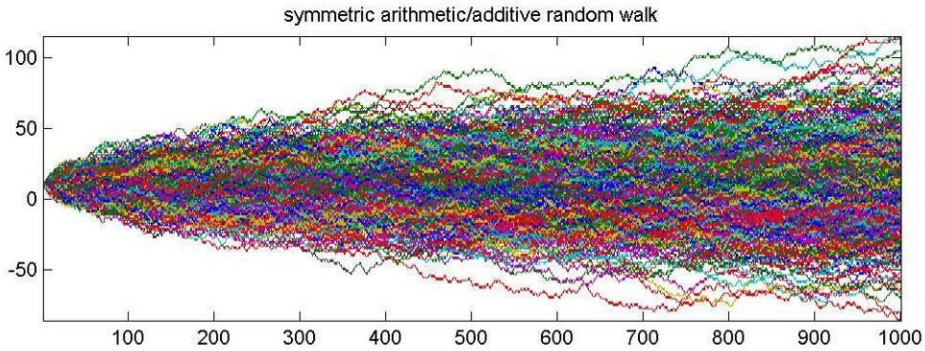


Figure 43: Arithmetic random walk - 500 simulated paths (1,000 steps)

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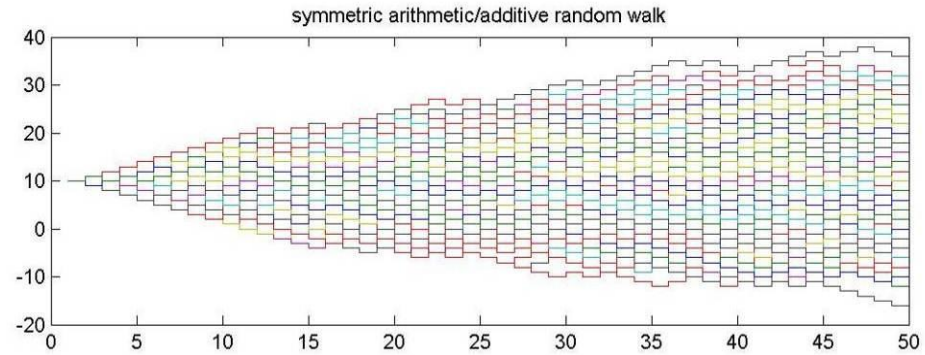


Figure 44: Arithmetic random walk - 5,000 simulated paths (50 steps)

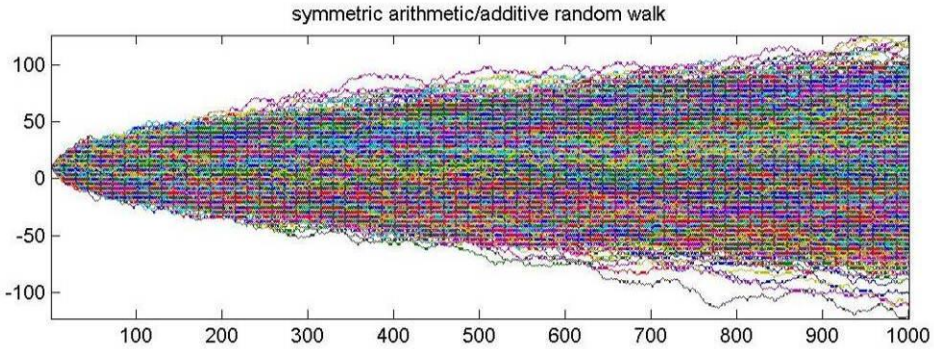


Figure 45: Arithmetic random walk - 5,000 simulated paths (1,000 steps)

SIMULATED ORDINARY GEOMETRIC RANDOM WALKS

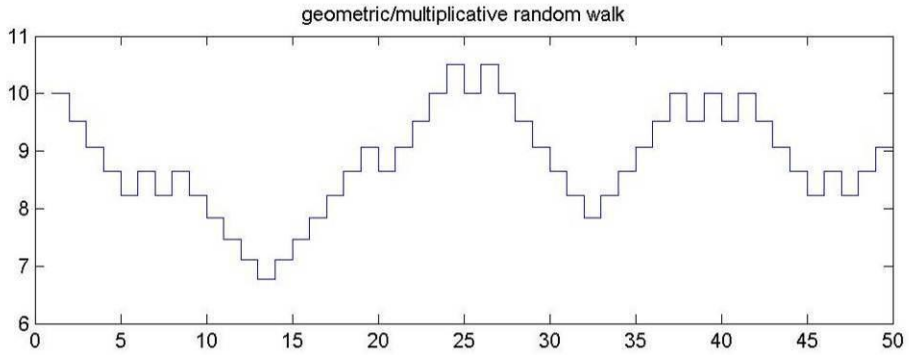


Figure 46: Geometric random walk - single simulated path (50 steps)

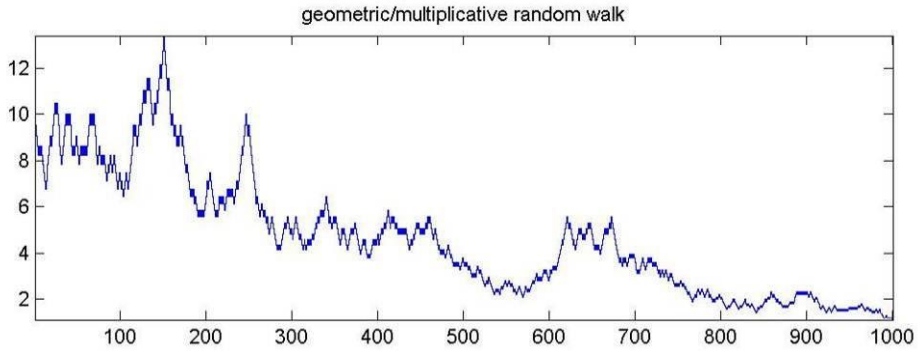


Figure 47: Geometric random walk - single simulated path (1,000 steps)

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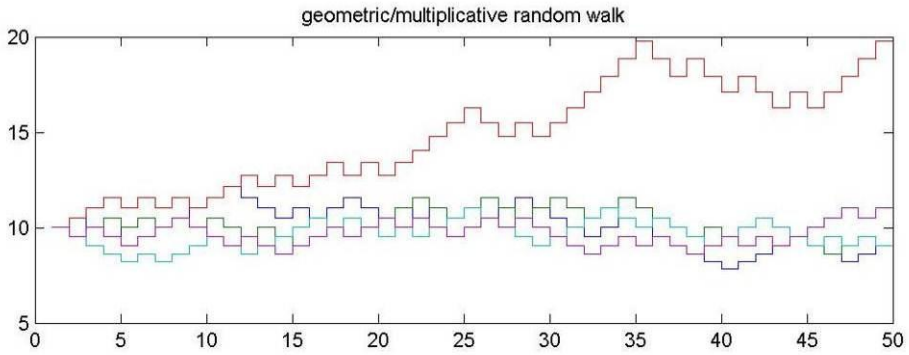


Figure 48: Geometric random walk - 5 simulated paths (50 steps)

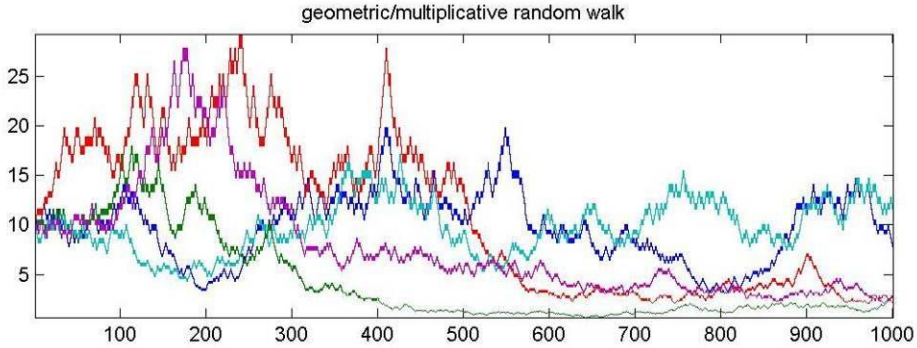


Figure 49: Geometric random walk - 5 simulated paths (1,000 steps)

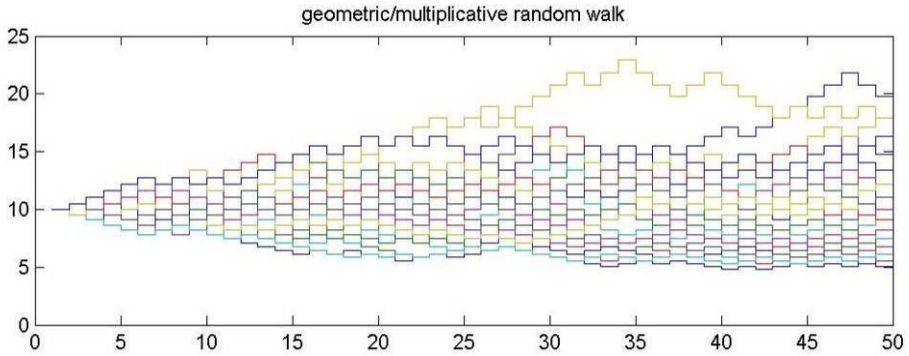


Figure 50: Geometric random walk - 50 simulated paths (50 steps)

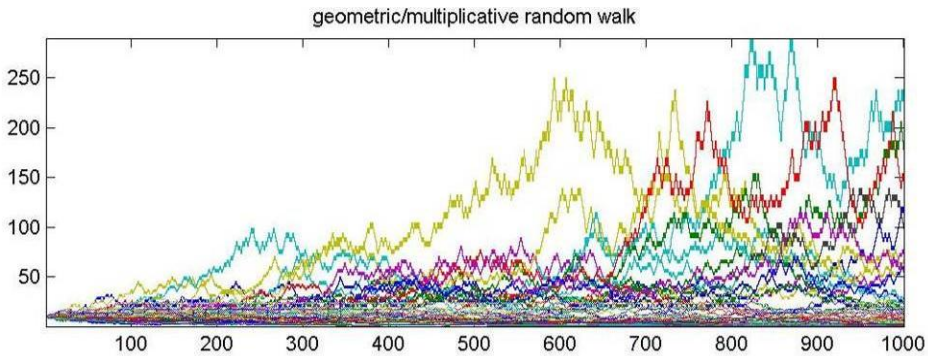


Figure 51: Geometric random walk - 50 simulated paths (1,000 steps)

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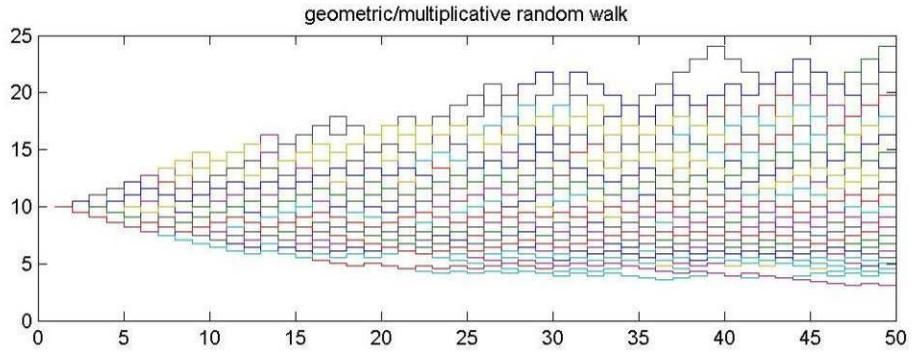


Figure 52: Geometric random walk - 500 simulated paths (50 steps)

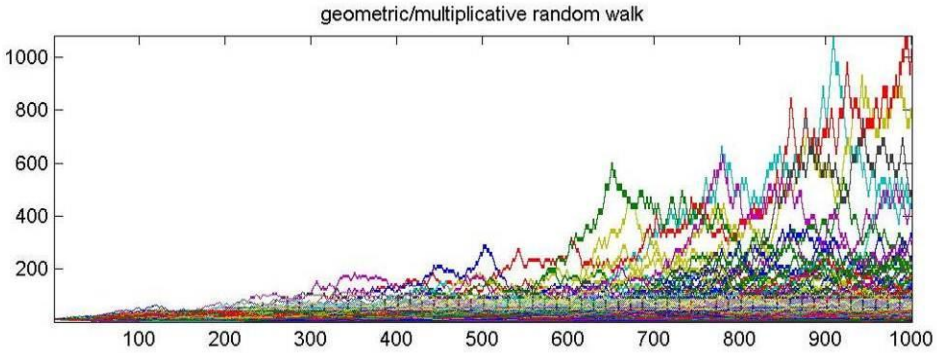


Figure 53: Geometric random walk - 500 simulated paths (1,000 steps)

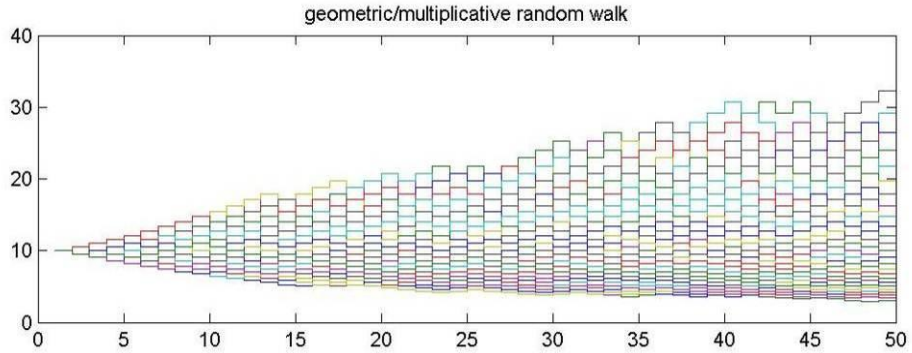


Figure 54: Geometric random walk - 5,000 simulated paths (50 steps)

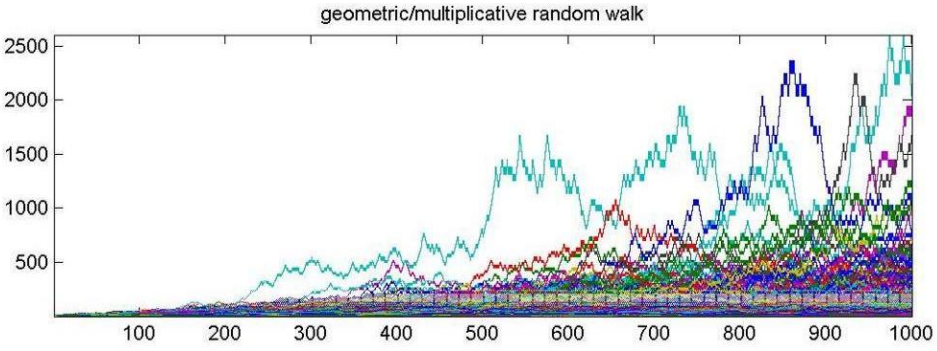


Figure 55: Geometric random walk - 5,000 simulated paths (1,000 steps)

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A. Additive Compound Poisson Random Walk

1. Statistical Construction and Forward Transition Distribution

As for the ordinary walks, the random variable determining what the outcome is at each step, whether it is a step of size *u* or a step of size *d*, is Bernoulli distributed $X_j \sim B(1,p)$,

$$X_j = \begin{cases} 0 & \text{with probability } q = 1-p \\ 1 & \text{with probability } p \end{cases}$$

which is now, in contrast to the ordinary walks, conditional on (given) a step occurring at a particular time which is determined by a Poisson process, a first order integration of a Poisson basis function, rather than a predetermined occurrence once every δt splitting the transition interval into a contiguous length of equal parts. Instead, the transition interval is now split into a contiguous length of exponentially distributed parts, each with mean proportional to δt. In the limit, the ratio of the sultus to the abscissa remains in scale and well defined as a proper infinitesimal derivative inclusive of statistical variation that is also in scale and finite.

Proceeding as per the ordinary walk by centering and scaling the zero-one variables so that they are transformed to two point distributions that return a step size of *u* ($u > 0$) with probability *p* and a step size of *d* ($d < 0$) with probability $q = 1 - q$

$$Y_j = u X_j + d(1-X_j) = (u-d)X_j + d = \begin{cases} d & \text{with probability } q = 1-p \\ u & \text{with probability } p \end{cases}$$

These variables form the components of the basis variable of the walk, i.e., the walk's steps. The density of each step of the walk is the same as that for an ordinary walk of fixed time interval between each step

$$p_{Y_j}(y_j) = p_{X_j}(x_j) = p_{X_j}\left({(y_j - d)} \big/ {(u-d)} \right) = p^{(y_j-d)/(u-d)} q^{(u-y_j)/(u-d)} \qquad y_j = d, u$$

The Poisson sum of these steps is then a Poisson indexed Binomial variable, scaled by the difference between the magnitude of the two alternative steps, that is centered by the Poisson count of the number downsteps,

$$\begin{aligned} Y &= Y_1 + Y_2 + \ldots + Y_{N(t)} \\ &= \left((u-d)X_1 + d\right) + \left((u-d)X_2 + d\right) + \ldots + \left((u-d)X_{N(t)} + d\right) \\ &= u\,X + d\left(N(t) - X\right) \\ &= (u-d)X + d \times N(t) \\ &\sim (u-d)\,B\left(N(t), p\right) + d \times N(t) \\ &\sim (u-d)\,B\left(P(\lambda t), p\right) + d \times P(\lambda t) \end{aligned}$$

This is the variable of the net walk forward transition. The Poisson indexed binomial $X \mid N_t \sim B(N_t, p)$ with $N_t \sim P(\lambda t)$ is equivalently a Bernoulli compound Poisson sum (as per the walk) and a binomial with a Poisson mixed index (as per the convolved Bernoullis). The resulting unconditional distribution is a Poisson distribution, here, another Poisson process with rate $p\lambda t$ $X \sim P(p\lambda t)$, which is shown below by either the probability mass function or the moment generating function

$$p_{X|N_t=n}(x) = \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} p^x q^{n-x} \qquad M_{X|N_t}(\theta) = E\left(e^X \mid N_t\right) = (q + pe^\theta)^{N_t}$$

$$p_{N_t}(n) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \qquad M_X(\theta) = E\left(e^X\right) = E\left(E\left(e^X \mid N_t\right)\right) = E\left((q + pe^\theta)^{N_t}\right)$$

$$= M_{N_t}\left((q + pe^\theta)^{N_t}\right)$$

hence

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$$p_X(x) = \sum_{n=x}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\times\frac{n!}{x!(n-x)!}p^x q^{n-x}$$
$$= \frac{e^{-\lambda t}p^x}{x!}\sum_{n=x}^{\infty}\frac{(\lambda t)^{n-x+x}}{(n-x)!}q^{n-x}$$
$$= \frac{e^{-\lambda t}(p\lambda t)^x}{x!}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}q^n$$
$$= \frac{e^{-\lambda t}(p\lambda t)^x}{x!}e^{(1-p)\lambda t}$$
$$= \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

$$M_X(\theta) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}(q+pe^{\theta})^n$$
$$= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{\left(\lambda t(q+pe^{\theta})\right)^n}{n!}$$
$$= e^{-\lambda t}e^{\lambda t(q+pe^{\theta})}\sum_{n=0}^{\infty} e^{-\lambda t(q+pe^{\theta})}\frac{\left(\lambda t(q+pe^{\theta})\right)^n}{n!}$$
$$= e^{-\lambda t}e^{\lambda t(1-p+pe^{\theta})}$$
$$= e^{-\lambda t+\lambda t-\lambda pt+\lambda pte^{\theta}}$$
$$= e^{p\lambda t(e^{\theta}-1)}$$

for $x = 0,1,2,\ldots$

which is the mass function and moment generating function, respectively, of the distribution $X \sim P(p\lambda t)$ of the number of upsteps (steps of size *u*) of the Poisson generated Binomial sum. By similar arguments, it is similarly shown that the distribution of the number of downsteps (steps of size *d*) of the sum is distributed as a Poisson process with rate $q\lambda t$,

$$N(t) - X \sim P(q\lambda t)$$

which is in both cases intuitive; the proportion of the number of the Poisson steps *N* that are upsteps is *p* and (hence average to $p\lambda t$) and the proportion to the number of the Poisson steps *N* that are downsteps is *q* (hence average to $q\lambda t$). Hence the distribution of the net transition is the linear combination of two Poisson processes

$$Y \sim (u-d)P(p\lambda t) + d\,P(\lambda t)$$
$$\sim u\,P(p\lambda t) + d\,P(q\lambda t)$$

The resulting distribution of the aggregate of the Poisson steps of the walk is, therefore, the sum of two scaled Poisson variables. Although Poisson variables are closed to convolution, with heterogeneous rate parameters, Poisson variables do not sum and remain Poisson if they are on different scales (i.e., heterogeneously scaled before being convolved). Although the fact that the transition distribution does not remain Poisson (although has an mgf that is only slightly different) and the resulting distribution is not of a common statistical form is inconvenient, it does not pose any difficulties for the purposes of this thesis and Compendium; since **(1)** the undiscounted premiums need only be manipulated and derived in terms of the component Poisson process and the step distribution, **(2)** the simulation algorithm is built in terms of the components of the Poisson generator and the binomial steps, and once simulated, are manipulated on *R* rather than *S* (and do not require a distribution function), **(3)** does not affect any of the recurrences, only complicates their solution which is already known anyway, and **(4)** the useful case where the walk steps with up and down steps of equal magnitude, $u = -d = \Delta y$, the mass function can be expressed in terms of a Bessel function of the first kind *I* for which there are tables (Abramowitz and Stegun for example) and function calls in the mathematical software[82]. The case where $u = -d = \Delta t$ is considered shortly, as for the ordinary walks.

The density function given the value of the Poisson process $Y \mid N(t) = n$ is the *n*-fold convolution of the component density functions of the individual component steps of the walk

$$p_{Y|N(t)}(y \mid n) = p_{X|N(t)}(x \mid n) = p_X\left(\left.(y-nd)\right/(u-d)\right) = {}^nC_{(y-nd)/(u-d)}\,p^{(y-nd)/(u-d)}q^{(nu-y)/(u-d)}$$

$$y = nd,\ (n-1)d+u,\ (n-2)d+2u,\ \ldots,\ 2d+(n-2)u,\ d+(n-1)u,\ nu$$

where $x = x_1 + x_2 + \ldots + x_n$ and $y = y_1 + y_2 + \ldots + y_n$. Summing out the Poisson process from the density function results in the unconditional transition density function of the Poisson-Binomial random walk

[82] MATLAB reproduces the tables from Abramowitz and Stegun with a single function call; type `help besseli` in MATLAB base.

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$$
\begin{aligned}
p_Y(y) &= \sum_{n=0}^{\infty} p_{N(t)}(n)\, p_{Y|N(t)}(y \mid n) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,{}^{n}C_{(y-nd)/(u-d)}\, p^{(y-nd)/(u-d)} q^{(nu-y)/(u-d)} \\
&= \sum_{n=y/u}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{n!}{\big((y-nd)/(u-d)\big)!\big((nu-y)/(u-d)\big)}\, p^{(y-nd)/(u-d)} q^{(nu-y)/(u-d)} \\
&= e^{-\lambda t}\sum_{n=y/u}^{\infty} \frac{(p\lambda t)^{(y-nd)/(u-d)}(q\lambda t)^{(nu-y)/(u-d)}}{\big((y-nd)/(u-d)\big)!\big((nu-y)/(u-d)\big)}
\end{aligned}
$$

$$
\begin{aligned}
&{}^{n}C_{x} = 0 \text{ for } x > n \\
&n \geq (y-nd)/(u-d) \\
&n(u-d) \geq (y-nd) \\
&n(u-d) + nd \geq y \\
&nu \geq y \\
&n \geq y/u
\end{aligned}
$$

The variable of summation *n* occurs in the summand so the summation in *n* is circular and so does not sum into a closed form, although it can be seen to be a convolution sum for two scaled and translated Poisson processes, as discussed above. When the walk is symmetric and $u = -d = \Delta y$, the circularity in the sum vanishes and the unconditional transition can be expressed in terms of a Bessel function after a change of variable

$$v = (nu - y)/(u - d) = (n\Delta y - y)/2\Delta y$$

Adding the variable of the net walk transition, composed of $N(t)$ Poisson steps, to the initial value, s_0 , of the walk results in the aggregate walk variable $S(t)$ given its value at the beginning of the walk transition interval, i.e., given $S(0)$. Given $S(t) = S_0$, the aggregate (and resolved form) walk variable is

$$
\begin{aligned}
S(t) &= S_0 + Y_1 + Y_2 + \ldots + Y_{N(t)} \\
&= S_0 + \big((u-d)X_1 + d\big) + \big((u-d)X_2 + d\big) + \ldots + \big((u-d)X_{N(t)} + d\big) \\
&\sim S_0 + (u-d)X - d \times N(t) \\
&\sim S_0 + (u-d)B\big(N(t), p\big) + d \times N(t)
\end{aligned}
$$

and is also the difference between two scaled Poisson processes shifted upwards by the walk's starting value

$$S(t) \sim S_0 + (u-d)\,P(p\lambda t) + d \times P(\lambda t)$$

one with rate λt and the other with rate $p\lambda t$, the former corresponding to the generating Poisson process the other to the Binomial-Poisson compound from the Poisson summed Bernoulli indicators; alternatively,

$$
\begin{aligned}
S(t) &\sim S_0 + u \times P(p\lambda t) + d \times \big(P(\lambda t) - P(p\lambda t)\big) \\
&\sim S_0 + u \times P(p\lambda t) + d \times P(q\lambda t)
\end{aligned}
$$

which is the starting value of the walk at the beginning of the transition interval plus $N_1(t) \sim P(p\lambda t)$ Poisson steps of size *u* and $N_2(t) \sim P(q\lambda t)$ Poisson steps of size *d*. The probability mass function of the unconditional walk transition variable *S* is

$$p_{S(t)|S(0)}(y \mid s_0) = p_Y(y - s_0)$$

Since the unconditional net transition variable *Y* is not expressible as a closed form function, neither is the unconditional walk transition variable *S*. Although inconvenient and somewhat unsatisfying, the fact that the probability mass function is not closed form (only expressible as an infinite sum) does not matter because the moment generating function is. The mgf can be transformed to the probability generating function with the relation

$$P_S(z) = E\big(z^S\big) = E\big(e^{S \log z}\big) = M_S(\log z)$$

which can return any required probability with a Laurent series expansion and picking out coefficients.

To this thesis's initial dismay, probabilities cannot be obtained by differentiation and setting *z*=0 because there is probability mass on the negative integers and the pgf returns a singularity at *z* =0 for any derivative of any order. The singularity that results is an indicator that the range of the support is unbounded for both sides of the natural numbers. (This helps answer a 'research problem' poised in Graham, Knuth, and Patashnik [page 424] and may prove to be helpful to applied probablists that work with generating functions, such as anyone studying this thesis: HYPOTHESIS; a differentiated pgf evaluated at z=0 is singular unless the variable is defined on $\mathbb{N}$ [or can

be transformed to $\mathbb{N}$, i.e., singular (non-removable pole) if defined on $\mathbb{Z}$ or is dense on $\mathbb{R}$ and cannot be transformed to $\mathbb{N}$ [at lease one essential pole]).

Because a Poisson generated walk is time homogenous, like an ordinary walk (and any diffusion variable considered by this thesis), the variable $S(t)$ given $S(0)$ for a transition length t is identical in distribution to the variable at time T, $S(T)$, given $S(t_0)$ for which the transition interval begins at time t_0 and where $t = T - t_0$. The probability function of $p_{S(t)|S(0)}(y \mid S_0)$ is then the same as $p_{S(T)|S(t_0)}(y \mid S_0)$ with $T - t_0$ taking the place of t (since they are equivalent by definition). The applicable generating Poisson process is Poisson distributed with rate $\lambda t = \lambda(T - t_0)$

$$\begin{aligned} N(t) &= N(t) \mid N(0) \\ &\sim P(\lambda t) \\ &\overset{d}{=} N(T) \mid N(t_0) \\ &\sim P(\lambda(T - t_0)) \end{aligned}$$

For the special and most relevant case for application, where the up and downsteps are of the same size (of equivalent magnitude and opposite sign) $u = -d = \Delta y$, the distribution of the walk's steps is the same as that for the ordinary walk with fixed times between each step

$$Y_j = 2(\Delta y)X_j - \Delta y = \begin{cases} -\Delta y & \text{with probability } q = 1 - p \\ +\Delta y & \text{with probability } p \end{cases}$$

with distribution

$$Y_j \sim 2(\Delta y)B(1, p) - \Delta y$$

and mass function

$$p_{Y_j}(y_j) = p_{X_j}(x_j) = p_{X_j}\left({(y_j + \Delta y)} \big/ {2\Delta y} \right) = p^{(y_j + \Delta y)/2\Delta y} q^{(\Delta y - y_j)/2\Delta y} \qquad y_j = \pm \Delta y$$

The net transition $Y = Y_1 + Y_2 + \ldots + Y_{N(t)}$ for n steps for a given value of the Poisson generator $N(t) = n$, $Y = Y_1 + Y_2 + \ldots + Y_n$ is

$$p_{Y|N(t)}(y \mid n) = p_X\left({(y + n\Delta y)} \big/ {2\Delta y} \right) = {}^{n}C_{(y+n\Delta y)/2\Delta y} p^{(y+n\Delta y)/2\Delta y} q^{(n\Delta y - y)/2\Delta y}$$

for $y = -n\Delta y, -(n-2)\Delta y, -(n-4)\Delta y, \ldots, +(n-4)\Delta y, +(n-2)\Delta y, +n\Delta y$. The distribution of the net transition Y, for the Poisson process $N(t) \sim P(\lambda t)$, is

$$\begin{aligned} Y &= Y_1 + Y_2 + \ldots + Y_{N(t)} \\ &= (2X_1 - 1)\Delta y + (2X_2 - 1)\Delta y + \ldots + (2X_{N(t)} - 1)\Delta y \\ &\sim (2X - N(t))\Delta y \\ &\sim (2B(N(t), p) + N(t))\Delta y \end{aligned}$$

which is the scaled difference between two independent Poisson processes with infinitesimal rate parameters $p\lambda dt$ and $q\lambda dt$

$$Y \sim (P(p\lambda t) - P(q\lambda t))\Delta y$$

This appears to be just a simple deconvolution of one Poisson variable from another. While independent Poissons convolve, they do not deconvolve unless it is the deconvolution of an independent variable which has already been added (and hence are not independent), i.e., **a deconvolution that unravels a previous convolution [and hence must not be independent!]**. Since the two Poisson processes are independent, the difference between the two does not equate to a deconvolution into another Poisson process. Most notably, the domain of the mass distribution is over the entire real line of integers -∞, ..., -2, -1, 0, 1, 2, …, ∞, not just positive integers 0, 1,

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2, …, ∞ as for a Poisson variable. This feature makes this distribution, as an extension of a Poisson variable, potentially a very useful addition to the inventory of discrete distributions which mostly pertain to counting on $\mathbb{N}$. As an addition to this inventory, this distribution could even still be called a Poisson variable variant for $\mathbb{Z}$ (where the standard Poisson is the variant for $\mathbb{N}$) because of the form of mgf … where the conventional Poisson, the variant for $\mathbb{N}$, has mgf $M_Y(\theta)=\exp\left(\lambda(e^{\theta}-1)\right)$ and the variant for $\mathbb{Z}$ has mgf[83] $M_Y(\theta)=\exp\left(\lambda(pe^{\theta}+qe^{-\theta}-1)\right)$ … and with symmetry over the two halves of $\mathbb{Z}$, $p = q = 0.5$, $M_Y(\theta)=\exp\left(0.5\lambda(e^{\theta}+e^{-\theta}-1)\right)$ … standardizing discrete variables with *u* and *d*.

The resulting mass function is, instead of an exponential series, expressible as a modified Bessel series of the first kind[84] *I*

$$I_v(x)=\sum_{n=0}^{\infty}\frac{(x/2)^n}{n!}\frac{(x/2)^{n+v}}{(n+v)!} \qquad I_{-v}(x)=I_v(x)$$

the Bessel function is very similar to an exponential series (two nested exponential series) and, when thought of as an extension of an ordinary exponential series, gives credence to the mgf $M_Y(\theta)=\exp\left(\lambda(pe^{\theta}+qe^{-\theta}-1)\right)$ being the mgf of a form of Poisson variable, such as 'Poisson variable variant for $\mathbb{Z}$; the only reason it is not more common is that it does not sum into an analytical function; has this stopped anyone with normal cdf's?

From the unconditional mass function of the net transition of the walk, summing out the Poisson process $N(t)=n$ from the conditional mass function

$$\begin{aligned}
p_Y(y)&=\sum_{n=0}^{\infty}p_{N(t)}(n)\,p_{Y|N(t)}(y\,|\,n)\\
&=\sum_{n=0}^{\infty}p_{N(t)}(n)\,{}^{n}C_{(y+n\Delta y)/2\Delta y}\,p^{(y+n\Delta y)/2\Delta y}q^{(n\Delta y-y)/2\Delta y}\\
&=\sum_{n=y/\Delta y}^{\infty}p_{N(t)}(n)\,{}^{n}C_{(y+n\Delta y)/2\Delta y}\,p^{(y+n\Delta y)/2\Delta y}q^{(n\Delta y-y)/2\Delta y}\\
&=\sum_{n=y/\Delta y}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{n!}{\left((y+n\Delta y)/2\Delta y\right)!\left((n\Delta y-y)/2\Delta y\right)!}\,p^{(y+n\Delta y)/2\Delta y}q^{(n\Delta y-y)/2\Delta y}\\
&=e^{-\lambda t}\sum_{n=y/\Delta y}^{\infty}\frac{(p\lambda t)^{(y+n\Delta y)/2\Delta y}(q\lambda t)^{(n\Delta y-y)/2\Delta y}}{\left((y+n\Delta y)/2\Delta y\right)!\left((n\Delta y-y)/2\Delta y\right)!}
\end{aligned}$$

and using the change of variable of summation to $v=(n\Delta y-y)/2\Delta y$ so that $(y+n\Delta y)/2\Delta y=v+y/\Delta y$ and the circularity in *n* in the variable of summation and the summand vanishes, the summation in the new variable starts from zero and results in

$$\begin{aligned}
p_Y(y)&=e^{-\lambda t}\sum_{v=0}^{\infty}\frac{(p\lambda t)^{v+y/\Delta y}(q\lambda t)^{v}}{\left(v+y/\Delta y\right)!v!}\\
&=e^{-\lambda t}\sum_{v=0}^{\infty}\frac{(pq\lambda^2t^2)^{v}(p\lambda t)^{y/\Delta y}}{\left(v+y/\Delta y\right)!v!}
\end{aligned}$$

---

[83] which is the generating function for the modified Bessel function of the first kind, which is formula 9.6.33 on page 376 of
Abramowitz, M. and Stegun, I. A., (1968) *Handbook of Mathematical Functions*. Dover Publications, Inc.

[84] **i.** Feller, W., (1966) An Introduction to Probability Theory and Its Applications, Volume 2. John Wiley & Sons; (page)

ii. Feller, W., (1966) Infinitely Divisible Distributions and Bessel Functions Associated with Random Walks. *SIAM Journal on Applied Mathematics*, Vol. 14, No. 4 (Jul., 1966), pp. 864-875.

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$$= e^{-\lambda t}(p\lambda t)^{y/\Delta y}(\sqrt{pq}\lambda t)^{-y/\Delta y}\sum_{v=0}^{\infty}\frac{(\sqrt{pq}\lambda t)^{v}}{v!}\frac{(\sqrt{pq}\lambda t)^{v}(\sqrt{pq}\lambda t)^{y/\Delta y}}{\left(v+y/\Delta y\right)!}$$

$$= e^{-\lambda t}(\sqrt{p/q})^{y/\Delta y}\sum_{v=0}^{\infty}\frac{(\sqrt{pq}\lambda t)^{v}}{v!}\frac{(\sqrt{pq}\lambda t)^{v+y/\Delta y}}{\left(v+y/\Delta y\right)!v!}$$

Hence, the difference between two scaled Poisson distributions is expressible as a Bessel function so the unconditional mass function can be written as

$$p_Y(y) = e^{-\lambda t}(\sqrt{p/q})^{y/\Delta y} I_{y/\Delta y}\left(2\sqrt{pq}\,\lambda t\right) \qquad \text{for } y = -\infty, ..., -2\Delta y, -\Delta y, 0, +\Delta y, +2\Delta y, ..., +\infty.$$

The walk variable, given $S(0) = s_0$, and for the Poisson process $N(t) \sim P(\lambda t)$ is then the starting value of the walk at $S(0)$ plus the net transition $S(t) = S_0 + Y = S_0 + Y_1 + Y_2 + ... + Y_{N(t)}$ which is a shifted version of Y

$$\begin{aligned} S(t) &= S_0 + Y_1 + Y_2 + ... + Y_{N(t)} \\ &= S_0 + \left(2(\Delta y)X_1 - \Delta y\right) + \left(2(\Delta y)X_2 - \Delta y\right) + ... + \left(2(\Delta y)X_{N(t)} - \Delta y\right) \\ &\sim S_0 + 2(\Delta y)X - N(t)\times\Delta y \\ &\sim S_0 + 2(\Delta y)B\left(N(t), p\right) + N(t)\times\Delta y \end{aligned}$$

and also, for the statistical part, the scaled difference between two independent Poisson processes with infinitesimal rate parameters $p\lambda dt$ and $q\lambda dt$

$$S(t) \sim S_0 + \left(P(p\lambda t) - P(q\lambda t)\right)\Delta y$$

The probability mass function of the walk is then the shifted mass function of the net transition

$$p_{S(t)|S(0)}(y\,|\,s_0) = p_Y(y - s_0) = e^{-\lambda t}(\sqrt{p/q})^{(y-s_0)/\Delta y}\sum_{v=0}^{\infty}\frac{(\sqrt{pq}\lambda t)^{v}}{v!}\frac{(\sqrt{pq}\lambda t)^{v+(y-s_0)/\Delta y}}{\left(v+(y-s_0)/\Delta y\right)!v!}$$

i.e.,

$$p_{S(t)|S(0)}(y\,|\,s_0) = e^{-\lambda t}(\sqrt{p/q})^{(y-s_0)/\Delta y} I_{(y-s_0)/\Delta y}\left(2\sqrt{pq}\,\lambda t\right)$$

defined for $y = -\infty, ..., s_0 - 2\Delta y, s_0 - \Delta y, s_0, s_0 + \Delta y, s_0 + 2\Delta y, ..., +\infty$. For example, the probability that the walk is net one step (net Δy, for example, 10 steps up and one step down) above its starting value s_0 at time t is

$$\begin{aligned} p_{S(t)|S(0)}(s_0 + \Delta y\,|\,s_0) &= \Pr\left(S(t) = s_0 + \Delta y\,|\,S(0) = s_0\right) \\ &= e^{-\lambda t}(\sqrt{p/q})I_1\left(2\sqrt{pq}\,\lambda t\right) = e^{-\lambda t}(\sqrt{p/q})\sum_{v=0}^{\infty}\frac{(\sqrt{pq}\lambda t)^{v}}{v!}\frac{(\sqrt{pq}\lambda t)^{v+1}}{(v+1)!} \end{aligned}$$

and the probability that the walk is net one step (net Δy) below its starting value s_0 at time t is

$$\begin{aligned} p_{S(t)|S(0)}(s_0 - \Delta y\,|\,s_0) &= \Pr\left(S(t) = s_0 - \Delta y\,|\,S(0) = s_0\right) \\ &= e^{-\lambda t}(\sqrt{p/q})^{-1} I_{-1}\left(2\sqrt{pq}\,\lambda t\right) = e^{-\lambda t}(\sqrt{p/q})^{-1}\sum_{v=0}^{\infty}\frac{(\sqrt{pq}\lambda t)^{v}}{v!}\frac{(\sqrt{pq}\lambda t)^{v-1}}{(v-1)!} \end{aligned}$$

where the Bessel function $I_{+1} = I_{-1}$ can be evaluated with a MATLAB function call to `besseli(1, `$2\sqrt{pq}\,\lambda t$`)`. Code that plots an example mass function (to keep things simple, it is not normalized) is

```
t=[cumsum([0;ones(20,1)])-10]; stem(t,(.65).^(-t).*besseli(t,10));
```

The Bessel function can be evaluated from tables or a function call, unfortunately, the probability generating function cannot be differentiated to the required order for the required probability (with software if needbe) since the pgf can't be differentiated and then evaluated for z = 0 because there are then singularities because of probabilities on an infinite number of negative integers.

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2. MGF of the Arithmetic Compound Poisson Random Walk

The moment generating function of the scaled and centered discrete binomial step j is

$$M_{Y_j}(\theta) = M_{(u-d)X_j+d}(\theta) = E\left(e^{\left((u-d)X_j+d\right)\theta}\right) = e^{\theta d}E\left(e^{(u-d)X_j\theta}\right) = e^{\theta d}M_{X_j}\left((u-d)\theta\right) = \left(qe^{d\theta} + pe^{u\theta}\right)$$

and the moment generating function of the Poisson process compound sum, conditional on the Poisson process, is

$$M_{\sum_{j=1}^{N(t)} Y_j|N(t)}(\theta) = M_{Y_j}(\theta)^{N(t)} = \left(qe^{d\theta} + pe^{u\theta}\right)^{N(t)}$$

This gives, upon summing out the Poisson distribution or by nested expectation, the unconditional moment generating function of the Poisson sum of the discrete two point binomial steps

$$M_Y(\theta) = E\left((qe^{d\theta} + pe^{u\theta})^{N(t)}\right) = M_{N(t)}\left(\log(qe^{d\theta} + pe^{u\theta})\right) = e^{\lambda t\left(qe^{d\theta}+pe^{u\theta}-1\right)}$$

which is the product of the mgfs of two scaled Poisson processes, i.e., the sum of two Poisson processes, one with rate $q\lambda t$ that increments (decrements) by d (since d is ordinarily negative) at each point increase (decrease) and another with rate $p\lambda t$ that increments by u at each point increase. This is demonstrative of the fact that $Y \sim d\,P\left(q\lambda t\right) + u\,P\left(p\lambda t\right)$. Inclusive of the initial value, the backward diffusion variable, the transition mgf is

$$M_{S(T)|S(t_0)}(\theta) = e^{S_0\theta+\lambda t(pe^{u\theta}+qe^{d\theta}-1)}$$

The cumulant generating function is then the logarithm

$$K_{S(T)|S(t_0)}(\theta) = S_0\theta + \lambda t\left(pe^{u\theta} + qe^{d\theta} - 1\right)$$

The probability generating function is

$$P_{S(T)|S(t_0)}(\theta) = M_{S(T)|S(t_0)}(\log z) = e^{S_0\log z+\lambda t(pz^u+qz^d-1)}$$

For a walk that has upsteps and downsteps of equal magnitude, $u = -d = \Delta y$, for any step as for the ordinary summed walk $M_{Y_j}(\theta) = \left(qe^{-\theta\Delta y} + pe^{\theta\Delta y}\right)$ for which the mgf for total number of Poisson steps is $M_{\sum_{j=1}^{N(t)} Y_j|N(t)}(\theta) = \left(qe^{-\theta\Delta y} + pe^{\theta\Delta y}\right)^{N(t)}$. Independently of the Poisson process, the transition mgf is $M_Y(\theta) = e^{\lambda t\left(qe^{-\theta\Delta y}+pe^{\theta\Delta y}-1\right)}$ which is the mgf of the deconvolution (convolution) of a Poisson process that increments (decrements) in steps of Δy with rate $q\lambda t$ from a Poisson process that increments in steps of Δy at rate $p\lambda t$, i.e.,

$$Y \sim \left(P\left(p\lambda t\right) - P\left(q\lambda t\right)\right)\Delta y$$

Why is this interesting? because it is defined all integers (scaled to size Δy) on $(-\infty,\infty)$ rather than $[0,\infty)$. For $p = q$, this is a symmetric distribution and is comparable with, say a binomial distribution centered at the origin with the difference of having probabilities over an infinite set of points (scaled integers) rather than a set restricted to n points. For a walk, the transition distribution is not restricted to an under-dispersed transition distribution as it is when it is binomial for an ordinary walk.

For the aggregate walk variable S conditioned on and starting at the backward variable S_0

$$M_{S(T)|S(t_0)}(\theta) = e^{S_0\theta+\lambda t(pe^{\theta\Delta y}+qe^{-\theta\Delta y}-1)}$$

the cgf for which is $K_{S(T)|S(t_0)}(\theta) = S_0\theta + \lambda t\left(pe^{\theta\Delta y} + qe^{-\theta\Delta y} - 1\right)$ and pgf $P_{S(T)|S(t_0)}(z) = e^{S_0\log z+\lambda t(pz^{\Delta y}+qz^{-\Delta y}-1)}$.

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Differentiating the pgf and setting $z = 0$ results in a singularity (composed of an infinite number of poles) which is an indicator of probability mass on negative integers. The poles are essential in this case since they could only be removed if the support was finite in at least one of either tails, which case the poles could be removed with a change of location in the pgf parameter.

3. Transition Mean and Variance

INSERT HERE FROM TABLES

4. Compound Poisson Process Arithmetic Random Walk Simulation Algorithm

Simulate the exponential diffusion for the abscissae: simulate the individual renewal times by generating n exponential variables with mean $\beta = 1 / \lambda$

$$\begin{aligned} t_{(j)} - t_{(j-1)} &\sim Exp\left(1/\lambda\right) \\ &\sim Exp\left(\beta\right) \end{aligned}$$

where $\lambda \propto 1/\Delta t$ adjusts for time scale so that the exponential mean β is directly proportional to the discrete time differential $\beta \propto \Delta t$, and where order statistics notation is utilized to signify the ordering of the step times. The parameter of the Poisson process $\lambda \propto 1/\Delta t$ is the average number of Poisson events per Δt of unit time and $\beta \propto \Delta t$ is the average time between each Poisson event. Discretely integrate the individual renewal times by cumulatively summing the simulated variables

$$\begin{aligned} \sum_{j=1}^{n} \left(t_{(j)} - t_{(j-1)}\right) &= t_{(n)} - t_0 \\ &\sim \Gamma\left(n, 1/\lambda\right) \\ &\sim \Gamma\left(n, \beta\right) \end{aligned}$$

where the n^{th} renewal corresponds to the n^{th} Poisson generated event, which is gamma with parameters n and β. Adding the backward time variable for a random transition interval for a given number of Poisson steps.

$$\begin{aligned} t_{(j)} &= t_0 + \sum_{j=1}^{n} \left(t_{(j)} - t_{(j-1)}\right) \\ &\sim t_0 + \Gamma\left(n, 1/\lambda\right) \\ &\sim t_0 + \Gamma\left(n, \beta\right) \end{aligned}$$

which makes it clear that the Poisson process has a first passage time process which is a first order unit root exponential diffusion, as discussed in FOUNDATIONS II. The components, $t_{(J)}$, are the values of a Poisson process with mean rate of occurrence λ and are the abscissa of the simulated path. Generate the vector of n Bernoulli step indicators $X_j \sim B(1, p)$ and scale for up and down steps $(u-d)X_j + d$ and append the backward walk variable to the beginning of the vector. Cumulatively sum this vector. This vector forms the ordinates of the walk, as per the ordinary additive walk. Plot the components of the vector of the additive random walk (vector of ordinates) against the components of the vector of the Poisson process (vector of abscissae). The MATLAB code for this simulation algorithm is

```
p=p; u=u; d=d; s0=s0; t0=t0; n=n; b=β;          %initialize walk parameters
t=cumsum([t0; exprnd(b,n,1)]);                    %simulate step times
x=binornd(1,p,n,1); S=cumsum([s0; (u-d).*x+d]);  %simulate steps & walk
plot(t,S);                                          %plot S over t
```

where, here, the parameter initialization is exactly that of the ordinary walk with the addition of the mean renewal time `b=`β (inverse of the rate intensity) in place of the discrete time differential `dt=`Δt . Both `b=`β and `dt=`Δt are in the same units and both are on the same scale, one stochastic the other not.

For a compound Poisson symmetric additive walk of 100 steps starting at value 10 at time 2, initialize the parameters with the line

```
p=.5; u=.5; d=-u; s0=10; t0=2; n=100; b=5/100;  %initialize walk parameters
```

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Choosing a time unit of one hour. The average transition length of these 100 steps is `n*b`=100*(5/100) = 5 hours and, for the transition interval that begins at `t0`=2 (say 2 o'clock), the average timing of the 100th step (transaction) is at time `t0+nb`=7 (say 7 o'clock) which is where the forward time variable T in the example for the next algorithm is located, for an average of 100 steps (transactions) for a fixed transition interval of `T-t0`=5 hours where steps (transactions) occur, on average, every 5/100=1/20 hours, i.e., every 3 minutes.

The last algorithm simulates a path for a given number of Poisson steps (and random transition length). The alternatively algorithm simulates a path for a given forward time variable and transition length (for a random number of Poisson steps). This algorithm (1) simulates the total number of Poisson steps for a given transition length and (2) then distributes the step times over the transition interval. Given the fixed forward time variable, the Poisson process determines the number of steps *n* within the transition interval, and then the *n* Poisson steps are uniformly distributed over the transition interval

$$t_j \sim U(t_0, T)$$

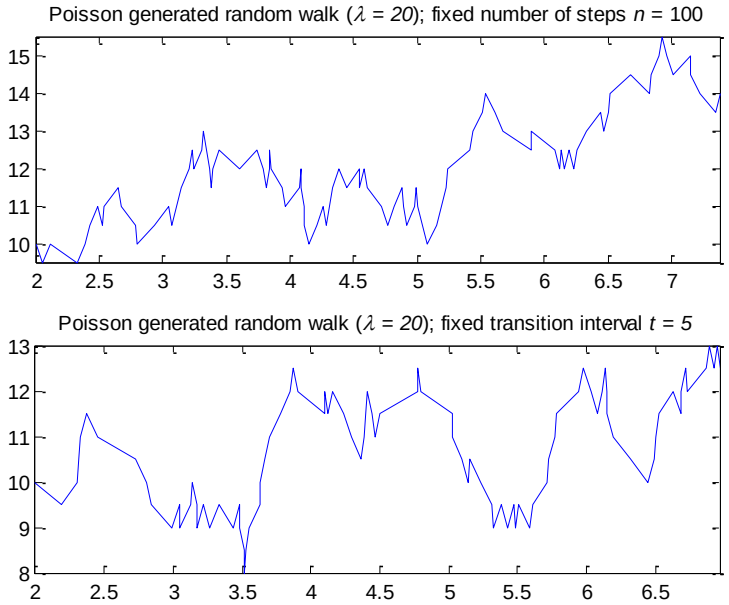
which are then sorted into a vector of ascending order with a backward variable of the exponential diffusion appended to the beginning. The MATLAB code is

```
p=p; u=u; d=d; s0=s0; t0=t0; T=T; b=β;          %initialize walk parameters
n=poissrnd((1/b).*(T-t0),1,1); t=[t0; t0+sort(unifrnd(0,T-t0,n,1))];
x=binornd(1,p,n,1); S=cumsum([s0; (u-d).*x+d]);       %simulate steps & walk
plot(t,S);                                                %plot S over t
```

An example for a random number of steps, consistent with the example last for a random transition length, is initialized with the line

```
p=.5; u=.5; d=-u; s0=10; t0=2; T=7; b=5/100;     %initialize walk parameters
```

For a time unit of one hour, the transition interval begins at time `t0`=2 (say 2 o'clock) and ends at time `T` =7 (say 7 o'clock) exact. The transition length is `T-t0`=5 hours. A Poisson step (transaction) occurs once every `b` =5/100=1/20 hours =3 minutes on average which amounts to `lambda`=1/(5/100)=20 steps per hour on average and an average of $\lambda(T-t_0)$ =20*5=100 steps (transactions) in total throughout the duration of the transition interval $T-t_0=5$ of the simulation.



Both algorithms are instructive, particularly for visualizing the excess dispersion arising from the random number of steps for any transition interval of fixed length and are easily modifiable for any part of FOUNDATIONS II. The former algorithm can be converted, somewhat inefficiently, to the latter with **accept-reject**, i.e., simulate a paths until one exceeds the required transition length and throw away the others and that part of the path that exceeds the required forward time variable.

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5. Limited Moments and Undiscounted Option Premia

The **additive** *t*-period ahead forward walk variable, composed of $X *^{N_t}$ upsteps and $N_t - X *^{N_t}$ downsteps, is

$$\begin{aligned} S_n &= S_0 + uX *^{N_t} + d(N_t - X *^{N_t}) \\ &= S_0 + d\,N_t + (u-d)X *^{N_t} \end{aligned}$$

where $N_t \sim P(\lambda t)$ and, given $N_t = n$, $X *^n \sim B(n, p)$. The payoff of an option on the walk variable is then

$\max\left(S_t - k,\ 0\right) = \max\left(S_0 + d\,N_t + (u-d)X *^{N_t} - k,\ 0\right)$ if it is an option to call the asset to buy for *k*

$\max\left(k - S_t,\ 0\right) = \max\left(k - S_0 - d\,N_t - (u-d)X *^{N_t},\ 0\right)$ if it is an option to put the asset to sell for *k*

For a call option, with $E\left(g(X)\right) = \sum_{x \in X}\ g(x) \times p_X(x)$, the undiscounted premium is

$$\begin{aligned} c(t,k) &= E\left(\max(S_0 + (u-d)X *^{N_t} + d\,N_t - k), 0)\right) \\ &= \sum_{n=0}^{\infty}\ e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{x=0}^{n}\ {}^nC_x p^x q^{n-x} \times \max\left(S_0 - k + nd + (u-d)x, 0\right) \\ &= \sum_{x=0}^{\infty}\ \sum_{n=x}^{\infty}\ e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times {}^nC_x p^x q^{n-x} \times \max\left(S_0 - k + nd + (u-d)x, 0\right) \\ &= \sum_{x=0}^{\infty}\ \max\left(\left(S_0 - k + (u-d)x\right)\sum_{n=x}^{\infty}\ e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,{}^nC_x p^x q^{n-x} + d\sum_{n=x}^{\infty}\ n e^{-\lambda t}\frac{(\lambda t)^n}{n!}\,{}^nC_x p^x q^{n-x},\ 0\right) \end{aligned}$$

noting that

(1) $\displaystyle\sum_{n=x}^{\infty}\ e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times {}^nC_x p^x q^{n-x} = \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$

(2) $\displaystyle\sum_{n=x}^{\infty}\ n e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times {}^nC_x p^x q^{n-x} = q\lambda t\frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$

$$\begin{aligned} &\sum_{n=x}^{\infty}\ e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times \frac{n!}{x!(n-x)!}p^x q^{n-x} \\ &= \frac{e^{-\lambda t}p^x}{x!}\sum_{n=x}^{\infty}\ \frac{(\lambda t)^{n-x+x}}{(n-x)!}q^{n-x} \\ &= \frac{e^{-\lambda t}(p\lambda t)^x}{x!}\sum_{n=0}^{\infty}\ \frac{(\lambda t)^n}{n!}q^n \\ &= \frac{e^{-\lambda t}(p\lambda t)^x}{x!}e^{q\lambda t} \\ &= \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} \end{aligned}$$

$$\begin{aligned} &\sum_{n=x}^{\infty}\ n \times e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times \frac{n!}{x!(n-x)!}p^x q^{n-x} \\ &= \frac{e^{-\lambda t}p^x}{x!}\sum_{n=x}^{\infty}\ n\frac{(\lambda t)^{n-x+x}}{(n-x)!}q^{n-x} \\ &= \frac{e^{-\lambda t}(p\lambda t)^x}{x!}\sum_{n=0}^{\infty}\ n\frac{(\lambda t)^n}{n!}q^n \\ &= \frac{e^{-\lambda t}(p\lambda t)^x}{x!}\sum_{n=1}^{\infty}\ n\frac{(q\lambda t)^n}{n!} \\ &= \frac{e^{-\lambda t}(p\lambda t)^x}{x!}(q\lambda t)\sum_{n=1}^{\infty}\ \frac{(q\lambda t)^{n-1}}{(n-1)!} \\ &= \frac{e^{-\lambda t}(p\lambda t)^x}{x!}(q\lambda t)\sum_{n=0}^{\infty}\ \frac{(q\lambda t)^n}{n!} \\ &= q\lambda t\frac{e^{-\lambda t}(p\lambda t)^x}{x!}e^{q\lambda t} \\ &= q\lambda t\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} \end{aligned}$$

the call premium then becomes

$$c(t,k) = \sum_{x=0}^{\infty}\ \max\left(\left(S_0 - k + qd\lambda t\right)\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} + (u-d)x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!},\ 0\right)$$

The max function is non-zero whenever

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$$\left(S_0 + qd\lambda t - k\right)\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} + (u-d)x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} \geq 0$$
$$\left(S_0 + qd\lambda t - k\right) + (u-d)x \geq 0$$
$$(u-d)x \geq \left(k - S_0 - qd\lambda t\right)$$
$$x \geq \left(k - S_0 - qd\lambda t\right)/(u-d)$$

so that only terms in the summation for *x* greater than $\left(k - S_0 - qd\lambda t\right)/(u-d)$ are non-zero,

$$c(t,k) = \sum_{x=k^*}^{\infty}\left(\left(S_0 + qd\lambda t - k\right)\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} + (u-d)x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!}\right)$$
$$= \left(S_0 + qd\lambda t - k\right)\sum_{x=k^*}^{\infty}\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} + (u-d)\sum_{x=k^*}^{\infty}x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

noting that the weighted sum in *x* is

$$\sum_{x=k^*}^{\infty}x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} = \sum_{x=k^*}^{\infty}\frac{e^{-p\lambda t}(p\lambda t)^x}{(x-1)!} = p\lambda t\sum_{x=k^*}^{\infty}\frac{e^{-p\lambda t}(p\lambda t)^{x-1}}{(x-1)!} = p\lambda t\sum_{x=k^*-1}^{\infty}\frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

so that

$$c(t,k) = \left(S_0 + qd\lambda t - k\right)\sum_{x=k^*}^{\infty}\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} + (u-d)p\lambda t\sum_{x=k^*-1}^{\infty}\frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$
$$= \left(S_0 + qd\lambda t - k\right)\overline{F}_{P(p\lambda t)}(k^*) + p(u-d)\lambda t\,\overline{F}_{P(p\lambda t)}(k^*-1)$$

with k^* being the smallest integer equal to or greater than $\left(k - S_0 - qd\lambda t\right)/(u-d)$ and where $\overline{F}_{P(p\lambda t)}(\cdot)$ is the complementary cumulative distribution functions of a Poisson variable with parameter $p\lambda t$. This expression can also be written in terms of two incomplete gamma functions (rather than Poisson ccdfs). For a put option

$$p(t,k) = E\left(\max(k - S_0 - d\,N_t - (u-d)X^{*N_t}, 0)\right)$$
$$= \sum_{n=0}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{x=0}^{n}{}^nC_x p^x q^{n-x} \times \max\left(k - S_0 - nd - (u-d)x, 0\right)$$
$$= \sum_{x=0}^{\infty}\sum_{n=x}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times {}^nC_x p^x q^{n-x} \times \max\left(k - S_0 - nd - (u-d)x, 0\right)$$
$$= \sum_{x=0}^{\infty}\max\left(\left(k - S_0 - (u-d)x\right)\sum_{n=x}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!}{}^nC_x p^x q^{n-x} - d\sum_{n=x}^{\infty}ne^{-\lambda t}\frac{(\lambda t)^n}{n!}{}^nC_x p^x q^{n-x},\ 0\right)$$

with the summations for the binomial density functions with a Poisson process mixture derived for a call

(1) $$\sum_{n=x}^{\infty}e^{-\lambda t}\frac{(\lambda t)^n}{n!} \times {}^nC_x p^x q^{n-x} = \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

(2) $$\sum_{n=x}^{\infty}ne^{-\lambda t}\frac{(\lambda t)^n}{n!} \times {}^nC_x p^x q^{n-x} = q\lambda t\frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

applicable similarly here

$$p(t,k) = \sum_{x=0}^{\infty}\max\left(\left(k - S_0 - qd\lambda t\right)\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} - (u-d)x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!},\ 0\right)$$

and for k^* being the largest integer equal to or smaller than $\left(k - S_0 - qd\lambda t\right)/(u-d)$

$$p(t,k) = \sum_{x=0}^{k^*}\left(\left(k - S_0 - qd\lambda t\right)\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} - (u-d)x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!}\right)$$
$$= \left(k - S_0 - qd\lambda t\right)\sum_{x=k^*}^{\infty}\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} - (u-d)\sum_{x=k^*}^{\infty}x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

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and with

$$\sum_{x=0}^{k^*} x\frac{e^{-p\lambda t}(p\lambda t)^x}{x!} = \sum_{x=1}^{k^*} \frac{e^{-p\lambda t}(p\lambda t)^x}{(x-1)!} = p\lambda t\sum_{x=1}^{k^*} \frac{e^{-p\lambda t}(p\lambda t)^{x-1}}{(x-1)!} = p\lambda t\sum_{x=0}^{k^*-1} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

the undiscounted put option premium becomes

$$\begin{aligned} p(t,k) &= \left(k - S_0 - qd\lambda t\right)\sum_{x=0}^{k^*} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} - p(u-d)\lambda t\sum_{x=0}^{k^*-1} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} \\ &= \left(k - S_0 - qd\lambda t\right)F_{P(p\lambda t)}(k^*) - p(u-d)\lambda t\, F_{P(p\lambda t)}(k^*-1) \end{aligned}$$

where $F_{P(p\lambda t)}(\cdot)$ is the cumulative distribution functions of a Poisson variable with parameter $p\lambda t$.

6. Forward and Backward Recurrence Relations of a Compound Poisson Process Additive Random Walk

Since the generated is now continuous time, the recurrence relation unlike the ordinary random walk with fixed step length involves a time derivative. It is therefore a differential equation in time and a difference equation in the walk variable and requires the forward and backward recurrence relations of the ordinary walk and the forward and backward equations of the Poisson process generating the steps. The forward and backward recurrence relations of the ordinary walk are, from $S(T) = S(t_0) + Y_1 + Y_2 + ... + Y_{N(T-t_0)}$ given $N(T-t_0) = n$,

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0)}(y-u \mid s_0) + q \times p_{S(T)|S(t_0)}(y-d \mid s_0)$$

and

$$p_{S(T)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0+u) + q \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0+d)$$

and the forward and backward recurrence equations of the Poisson process $N(T) \mid N(t_0) \sim P\left(\lambda(T-t_0)\right)$ are

$$p_{N(T+\Delta t)|N(t_0)}(n \mid n_0) = p_{N(T)|N(t_0)}(n+1 \mid n_0)\,\lambda\Delta t + p_{N(T)|N(t_0)}(n \mid n_0)\left(1-\lambda\Delta t\right) + o(\Delta t)$$

and

$$p_{N(T)|N(t_0)}(n \mid n_0) = p_{N(T)|N(t_0+\Delta t)}(n \mid n_0 - 1)\,\lambda\Delta t + p_{N(T)|N(t_0+\Delta t)}(n \mid n_0)\left(1-\lambda\Delta t\right) + o(\Delta t)$$

With these two sets of backward and forward recurrence equations in mind, the equivalent set of forward and backward equations for the Poisson walk are developed and written down by combining the two sets of underlying equations by logical reasoning[85].

The Poisson random walk, over any transition differential (differential increment in time), a step either occurs or it does not occur, with probabilities $1 - e^{-\lambda\Delta t} = \lambda\Delta t + o(\Delta t)$ $1 - e^{-\lambda dt} = \lambda dt$ and $e^{-\lambda\Delta t} = 1 - \lambda\Delta t + o(\Delta t)$ $e^{-\lambda dt} = 1 - \lambda dt$, and the probability of more than one step is $o(\Delta t)$ and is negligible as $\Delta t \to dt \to 0$. If a step does occur, it is either a step of size u (upwards) or it is a step of size d (downwards), with the binomial probabilities, as before, p and $q = 1 - p$ respectively. Stating this mathematically in terms of the transition function as the dependent variable of the expression, results in the forward (convolution) equation for the random walk summed by the Poisson process[86],

[85] From the author.

[86] It may be of value to review the Poisson process forward equation,

$$p_n(t+dt) = p_{n-1}(t)\,p_1(dt) + p_n(t)\,p_0(dt) + o(dt) = p_{n-1}(t)\lambda dt + p_n(t)(1-\lambda dt) + o(dt)$$

$$\frac{\partial p_n(t)}{\partial t} = \lambda[\,p_{n-1}(t) - p_n(t)]$$

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Forward transition function

= *Pr*(no Poisson event occurs)*zero-fold Σ-convolution onto transition function ← add no steps

+*Pr*(one Poisson event occurs)*one-fold Σ-convolution onto transition function ← add one step

+*Pr*(two Poisson events occur)*two-fold Σ-convolution onto transition function ← add two steps

+…

…etc.

or, in mathematical symbols, assuming (correctly) that two or more Poisson events are $o(\Delta t)$

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = \left(p \times p_{S(T)|S(t_0)}(y-u \mid s_0) + q \times p_{S(T)|S(t_0)}(y-d \mid s_0) \right) \lambda \Delta t + p_{S(T)|S(t_0)}(y \mid s_0)\left(1-\lambda \Delta t\right) + o(\Delta t)$$

Subtracting $p_{S(T)|S(t_0)}(y \mid s_0)$ from both sides and dividing both sides by the transition differential Δt

$$\frac{p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) - p_{S(T)|S(t_0)}(y \mid s_0)}{\Delta t} = \lambda \left(p \times p_{S(T)|S(t_0)}(y-u \mid s_0) + q \times p_{S(T)|S(t_0)}(y-d \mid s_0) - p_{S(T)|S(t_0)}(y \mid s_0) \right) + \frac{o(\Delta t)}{\Delta t}$$

and taking the limit for the transition differential so that $o(\Delta t) / \Delta t \to 0$, the forward recurrence of the random walk summed by the Poisson process becomes

$$\frac{\partial}{\partial T} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda \left(p \times p_{S(T)|S(t_0)}(y-u \mid s_0) + q \times p_{S(T)|S(t_0)}(y-d \mid s_0) - p_{S(T)|S(t_0)}(y \mid s_0) \right)$$

noting that the magnitude of the steps, *u* and *d*, need not scale down with the transition differential since the walk still steps at discretely points (and *u* and *d* need only be comparative to $1 / \lambda$) and it is only necessary to scale down the magnitude of the steps to keep each of the differential components in scale as the Poisson rate intensity becomes infinite and the steps of the discrete diffusion occur infinitely often and merge into continuous diffusion. For a walk with steps that are symmetric in magnitude, $u = -d = \Delta y$, the forward recurrence is

$$\frac{\partial}{\partial T} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda \left(p \times p_{S(T)|S(t_0)}(y-\Delta y \mid s_0) + q \times p_{S(T)|S(t_0)}(y+\Delta y \mid s_0) - p_{S(T)|S(t_0)}(y \mid s_0) \right)$$

where Δy is in proportion with the $1 / \lambda$ and scale down to infinitesimals with $1 / \lambda$ rather than directly through Δt, since $\lambda \propto 1/\Delta t$ is the measure of average elapsed time between each discrete step (**the average abscissa of each sultus**) . It is interesting to contrast this equation with the Poisson process forward equation[87] as well as that for an arbitrary compound Poisson sum[88], of those considered in the earlier parts of FOUNDATIONS II;

$p \times p_{S(T)|S(t_0)}(y-u \mid s_0) + q \times p_{S(T)|S(t_0)}(y-d \mid s_0)$ is the equivalent two point convolution formula corresponding to

$$\int_{-\infty}^{\infty} f_X(y)\, f_{S(T)|S(t_0)}(x-y \mid s_0) dy$$

Considering a backward step at the beginning of the walk which, if a Poisson event (a step in the walk) has occurred, is either a backwards step down from an upstep (which occurred with probability *p*) or a backwards step up from a downstep (which occurred with probability *q*). If no step had occurred, then the transition function remains unchanged. The Poisson event occurred with probability λdt . The backward (deconvolution) equation for the walk[89] is then,

---

[87] $\frac{\partial p_n(t)}{\partial t} = \lambda [\, p_{n-1}(t) - p_n(t)]$

[88] which is $\frac{\partial}{\partial t} f_{S(t)}(x) = \lambda \int_{-\infty}^{\infty} f_X(y)\, f_{S(t)}(x-y) dy - \lambda\ f_{S(t)}(x)$

[89] Likewise, it may be of value to review the Poisson process backward equation,
$p_n(t-dt \mid n_0) = p_n(t \mid n_0 + 1)\lambda dt + p_n(t \mid n_0)(1-\lambda dt) + o(dt)$

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Backward transition function

= *Pr*(no Poisson event occurred)*zero-fold Σ-deconvolution from transition function ← remove no steps

+*Pr*(one Poisson event occurred)*one-fold Σ-deconvolution from transition function ← remove one step

+*Pr*(two Poisson events occurred)*two-fold Σ-deconvolution from transition function ← remove two steps

+… etc.

or, in mathematical symbols, assuming that two or more Poisson events were $o(\Delta t)$

$$p_{S(T)|S(t_0)}(y \mid s_0) = \left(p \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 + u) + q \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 + d)\right)\lambda\Delta t + p_{S(T)|S(t_0+\Delta t)}(y \mid s_0)\left(1 - \lambda\Delta t\right) + o(\Delta t)$$

and noting that, by time homogeneity over any transition interval, this expression is equivalent to

$$p_{S(T)|S(t_0-\Delta t)}(y \mid s_0) = \left(p \times p_{S(T)|S(t_0)}(y \mid s_0 + u) + q \times p_{S(T)|S(t_0)}(y \mid s_0 + d)\right)\lambda\Delta t + p_{S(T)|S(t_0)}(y \mid s_0)\left(1 - \lambda\Delta t\right) + o(\Delta t)$$

Subtracting $p_{S(T)|S(t_0)}(y \mid s_0)$ from both sides and dividing both sides by the transition differential

$$\frac{p_{S(T)|S(t_0-\Delta t)}(y \mid s_0) - p_{S(T)|S(t_0)}(y \mid s_0)}{\Delta t} = \lambda\left(p \times p_{S(T)|S(t_0)}(y \mid s_0 + u) + q \times p_{S(T)|S(t_0)}(y \mid s_0 + d) - p_{S(T)|S(t_0)}(y \mid s_0)\right) + \frac{o(\Delta t)}{\Delta t}$$

and taking the limit for the transition differential $\Delta t \to 0$ so that $o(\Delta t) / \Delta t \to 0$, the backward recurrence of the random walk summed by the Poisson process becomes

$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda\left(p \times p_{S(T)|S(t_0)}(y \mid s_0 + u) + q \times p_{S(T)|S(t_0)}(y \mid s_0 + d) - p_{S(T)|S(t_0)}(y \mid s_0)\right)$$

For a walk that has steps that are symmetric in magnitude, $u = -d = \Delta y$, the backward recurrence is

$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda\left(p \times p_{S(T)|S(t_0)}(y \mid s_0 + \Delta y) + q \times p_{S(T)|S(t_0)}(y \mid s_0 - \Delta y) - p_{S(T)|S(t_0)}(y \mid s_0)\right)$$

where $\Delta y \propto 1/\lambda$. It is interesting to contrast this with the Poisson process backward equation[90] and the corresponding equation for an arbitrary increment (secondary) variable[91] discussed in the earlier parts of FOUNDATIONS II for which these equations are the equivalents for alternative basis functions, the walk discrete binomial and that of FOUNDATIONS II continuous; the sole difference being that expressions in **Σ-convolutions** rather than **Π-convolutions**.

7. Forward Equation of the Continuous Diffusion Limit

From the forward convolution relation

$$\frac{\partial}{\partial T} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda\left(p \times p_{S(T)|S(t_0)}(y - \Delta y \mid s_0) + q \times p_{S(T)|S(t_0)}(y + \Delta y \mid s_0) - p_{S(T)|S(t_0)}(y \mid s_0)\right)$$

expanding the first two terms in the brackets in univariate Taylor series about the forward variable

(1) $$p_{S(T)|S(t_0)}(y - \Delta y \mid s_0) = p_{S(T)|S(t_0)}(y \mid s_0) - (\Delta y)\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) + o\left((\Delta y)^2\right)$$

(2) $$p_{S(T)|S(t_0)}(y + \Delta y \mid s_0) = p_{S(T)|S(t_0)}(y \mid s_0) + (\Delta y)\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) + o\left((\Delta y)^2\right)$$

---

[90] $$-\frac{\partial p_n(t \mid n_0)}{\partial t} = \lambda[p_n(t \mid n_0 + 1) - p_n(t \mid n_0)]$$

[91] which is $$-\frac{\partial}{\partial t_0} f_{S(t)|S(t_0)}(x \mid x_0) = \lambda\int_{-\infty}^{\infty} f_X(y) f_{S(t)|S(t_0)}(x \mid x_0 + y)dy - \lambda\, f_{S(t)|S(t_0)}(x \mid x_0)$$

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These expansions do not consider time as infinitesimal changes in the transition length are already incorporated into the time side of the recurrence relation for the Poisson process; limiting arguments are for the rate intensity of the steps which is a limit in λ rather than contiguous increments of the transition interval. Substituting these expansions into the forward convolution formula and dividing by the time differential and subsequently simplifying results in

$$\frac{\partial}{\partial T} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\lambda(\Delta y)^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) - \lambda(p-q)(\Delta y)\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0) + \lambda o\left((\Delta y)^2\right)$$

where $(\Delta y)^2$ is $o(1/\lambda)$. Letting the rate intensity become infinite $\lambda \to 1/dt \to \infty$ so that steps occur infinitely often and spaced infinitesimally next to each other $1/\lambda \to dt \to 0$ so that and $\lambda o\left((\Delta y)^2\right) \to 0$ and defining the infinitesimal parameters of the continuous diffusion limit as

$$\lim_{1/\lambda \to 0} \frac{(\Delta y)^2}{1/\lambda} = \sigma^2 \qquad \text{and} \qquad \lim_{1/\lambda \to 0} (p-q)\frac{\Delta y}{1/\lambda} = \mu$$

which have the same definition as those for the ordinary walk with $\Delta t = 1/\lambda$ as the variables and abscissa of the infinitesimal limit. The forward recurrence becomes

$$\frac{\partial}{\partial T} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial y^2} p_{S(T)|S(t_0)}(y \mid s_0) - \mu\frac{\partial}{\partial y} p_{S(T)|S(t_0)}(y \mid s_0)$$

which is the heat equation and the forward equation for a Brownian motion and a statement of the central limit theorem.

8. Backward Equation of the Continuous Diffusion Limit

From the backward deconvolution recurrence,

$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda\left(p \times p_{S(T)|S(t_0)}(y \mid s_0 + \Delta y) + q \times p_{S(T)|S(t_0)}(y \mid s_0 - \Delta y) - p_{S(T)|S(t_0)}(y \mid s_0)\right)$$

expanding the first two terms in the brackets in univariate Taylor series about the backward variable s_0

(1) $$p_{S(T)|S(t_0)}(y \mid s_0 + \Delta y) = p_{S(T)|S(t_0)}(y \mid s_0) + (\Delta y)\frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + o\left((\Delta y)^2\right)$$

(2) $$p_{S(T)|S(t_0)}(y \mid s_0 - \Delta y) = p_{S(T)|S(t_0)}(y \mid s_0) - (\Delta y)\frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + o\left((\Delta y)^2\right)$$

Substituting these into the backward deconvolution formula and subsequently simplifying

$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda(p-q)(\Delta y)\frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) + \tfrac{1}{2}\lambda(\Delta y)^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + \lambda o\left((\Delta y)^2\right)$$

Letting the Poisson rate intensity become infinite $\lambda \to 1/dt \to \infty$ so that $1/\lambda \to dt \to 0$ and $\lambda o\left((\Delta y)^2\right) \to 0$ and defining the infinitesimal parameters as the limits

$$\lim_{1/\lambda \to 0} \frac{(\Delta y)^2}{1/\lambda} = \sigma^2 \qquad \text{and} \qquad \lim_{1/\lambda \to 0} (p-q)\frac{\Delta y}{1/\lambda} = \mu$$

results in the backward recurrence of the continuous diffusion limit

$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + \mu\frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$$

which is the conjugate heat equation and the backward equation of a Brownian motion with the infinitesimal parameters consistent with those of the forward equation.

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B. Multiplicative Compound Poisson Process Random Walk

1. Statistical Construction and Forward Transition Distribution

Transforming from the additive process to the geometric equivalent for the Poisson generated walk follows the same principles as the transformation for the conventional random walk; the step indictor and walk basis variable component X_j is a Bernoulli zero-one indicator

$$X_j = \begin{cases} 0 & \text{with probability } q = 1-p \\ 1 & \text{with probability } p \end{cases}$$

and the scaled and shifted basis function Y_j is a two point Bernoulli with probability at u ($u > 1$) and d ($0 < d < 1$), converted to that which is closed to (forms a group for) multiplication (rather than addition)

$$Y_j = u^{X_j} d^{1-X_j} = (u/d)^{X_j} d = \begin{cases} d & \text{with probability } q = 1-p \\ u & \text{with probability } p \end{cases}$$

the mass function for which is the same as that for the geometric walk

$$p_{Y_j}(y_j) = p_{X_j}(x_j) = p_{X_j}\left(\log(y_j/d) \Big/ \log(u/d) \right) = p^{\log(y_j/d)/\log(u/d)} q^{\log(u/y_j)/\log(u/d)} \qquad y_j = d, u$$

The net transition variable is the product of each proportional step up to the forward time variable, which is here determined by the Poisson process that sums the walk

$$\begin{aligned} Y &= Y_1 \times Y_2 \times \ldots \times Y_{N(t)} \\ &= \left(u^{X_1} d^{1-X_1}\right) \times \left(u^{X_2} d^{1-X_2}\right) \times \ldots \times \left(u^{X_{N(t)}} d^{1-X_{N(t)}}\right) \\ &\sim u^X d^{N(t)-X} \\ &\sim d^n (u/d)^{B(N(t),p)} \end{aligned}$$

which is, by the theory of the additive walk, the product (ratio) of two log-Poisson variables, with rate intensities $p\lambda t$ and $q\lambda t$, respectively

$$Y \sim u^{P(p\lambda t)} d^{P(q\lambda t)}$$

one counting the number of proportional steps of size u and the other counting the number of size d. The conditional mass function, given a fixed value of the Poisson process, is that of the ordinary multiplicative walk which is log binomial

$$p_{Y|N(t)}(y \mid n) = p_X\left(\log(y/d^n) \Big/ \log(u/d) \right) = {}^nC_{\log(y/d^n)/\log(u/d)} p^{\log(y/d^n)/\log(u/d)} q^{\log(u^n/y)/\log(u/d)}$$

for $y = d^n, u^1 d^{n-1}, u^2 d^{n-2} u, u^3 d^{n-3} \ldots, u^{n-2} d^2, u^{n-1} d^1, u^n$. The transition mass function is

$$\begin{aligned} p_Y(y) &= \sum_{n=0}^{\infty} p_{N(t)}(n)\, p_{Y|N(t)}(y \mid n) \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} {}^nC_{\log(y/d^n)/\log(u/d)} p^{\log(y/d^n)/\log(u/d)} q^{\log(u^n/y)/\log(u/d)} \\ &= \sum_{n=\log y/\log u}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} {}^nC_{\log(y/d^n)/\log(u/d)} p^{\log(y/d^n)/\log(u/d)} q^{\log(u^n/y)/\log(u/d)} \end{aligned}$$

$${}^nC_x = 0 \text{ for } x > n$$

$$n \geq (\log y - n \log d)/\log(u/d)$$

$$n \log(u/d) \geq (\log y - n \log d)$$

$$n \log u \geq \log y$$

$$n \geq \log y / \log u$$

and is not of a closed form unless the step ratios are reciprocals, $u = 1/d$ which is considered shortly. The aggregate walk variable S is then the net transition product convolved onto the backward variable $S(t) = S_0 \times Y = S_0 \times Y_1 \times Y_2 \times \ldots \times Y_n$, i.e.,

$$S_0 u^X d^{N(t)-X} = S_0 \exp\left(X \log u + [N(t) - X] \log d \right)$$

which is log-binomial given the Poisson process and (Poisson variant for Z) with mass function

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$$p_{S(t)|S(0),N(t)}(y\,|\,s_0,n)=p_{Y|N(t)}(y/s_0\,|\,n)={}^{n}C_{\log(y/s_0d^n)/\log(u/d)}p^{\log(y/s_0d^n)/\log(u/d)}q^{\log(s_0u^n/y)/\log(u/d)}$$

and log-Poisson unconditionally

$$\begin{aligned}S_0u^Xd^{N(t)-X}&=S_0\exp\big(X\log u+[N(t)-X]\log d\big)\\&\sim S_0\exp\Big(\log(u/d)B\big(N(t),p\big)+(\log d)P(\lambda t)\Big)\\&\sim S_0\exp\big((\log u)P(p\lambda t)+(\log d)P(q\lambda t)\big)\end{aligned}$$

Taking the binomial walk ratios as reciprocals, $u=1/d=e^{\Delta\log y}$ where $\Delta\log y$ is a constant until limits are taken and then for which it scales down as a Newtonian differential with dt. With $u=1/d=e^{\Delta\log y}$, the multiplicative basis function of the walk becomes

$$Y_j=e^{(2X_j-1)\Delta y}=\begin{cases}e^{-\Delta\log y} & \text{with probability } q=1-p\\ e^{+\Delta\log y} & \text{with probability } p\end{cases}$$

with log-Bernoulli distribution

$$Y_j\sim e^{(2B(1,p)-1)\Delta y}$$

and probability function

$$p_{Y_j}(y_j)=p_{X_j}(x_j)=p_{X_j}\left(\log(y_je^{\Delta\log y})\Big/2\Delta\log y\right)=p^{\log(y_je^{\Delta\log y})/2\Delta\log y}q^{\log(e^{\Delta\log y}/y_j)/2\Delta\log y}\qquad y_j=e^{\pm\Delta\log y}$$

The net forward transition distribution $Y=Y_1\times Y_2\times\ldots\times Y_{N(t)}$ conditional on the Poisson generator $N(t)=n$ is

$$Y=Y_1\times Y_2\times\ldots\times Y_{N(t)}$$

with log-binomial probability function

$$p_{Y|N(t)}(y\,|\,n)=p_X\left(\log(ye^{n\Delta\log y})\Big/2\Delta\log y\right)={}^{n}C_{\log(ye^{n\Delta\log y})/2\Delta\log y}p^{\log(ye^{n\Delta\log y})/2\Delta\log y}q^{\log(e^{n\Delta\log y}/y)/2\Delta\log y}$$

for $y=e^{-n\Delta\log y},e^{-(n-2)\Delta\log y},e^{-(n-4)\Delta\log y},\ldots,e^{+(n-4)\Delta\log y},e^{+(n-2)\Delta\log y},e^{+n\Delta\log y}$. The distribution of the net transition for randomly variable $N(t)$ is

$$\begin{aligned}Y&=Y_1\times Y_2\times\ldots\times Y_{N(t)}\\&=e^{(2X_1-1)\Delta\log y}\times e^{(2X_2-1)\Delta\log y}\times\ldots\times e^{(2X_{N(t)}-1)\Delta\log y}\\&\sim\exp\big([2X-N(t)]\Delta\log y\big)\\&\sim\exp\Big([2B\big(N(t),p\big)+N(t)]\Delta\log y\Big)\end{aligned}$$

which is also

$$Y\sim\exp\Big(\big(P(p\lambda t)-P(q\lambda t)\big)\Delta\log y\Big)$$

the mass function for which expressible in terms of a modified Bessel function of the first kind, differing from the additive equivalent by a log argument for the forward variable walk *y*. Summing out the Poisson variable from the conditional walk transition function

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$$
\begin{aligned}
p_Y(y) &= \sum_{n=0}^{\infty} p_{N(t)}(n)\, p_{Y|N(t)}(y \mid n) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\, {}^{n}C_{\log(ye^{n\Delta\log y})}\, p^{\log(ye^{n\Delta\log y})/2\Delta\log y} q^{\log(e^{n\Delta\log y}/y)/2\Delta\log y} \\
&= \sum_{n=\log y/\Delta\log y}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\, {}^{n}C_{\log(ye^{n\Delta\log y})}\, p^{\log(ye^{n\Delta\log y})/2\Delta\log y} q^{\log(e^{n\Delta\log y}/y)/2\Delta\log y} \\
&= e^{-\lambda t}\sum_{n=\log y/\Delta\log y}^{\infty} \frac{(p\lambda t)^{\log(ye^{n\Delta\log y})/2\Delta\log y}(q\lambda t)^{\log(e^{n\Delta\log y}/y)/2\Delta\log y}}{\left(\log(ye^{n\Delta\log y})/2\Delta\log y\right)!\left(\log(e^{n\Delta\log y}/y)/2\Delta\log y\right)!}
\end{aligned}
$$

changing the variable of summation to $v = \log(e^{n\Delta\log y}/y)/2\Delta\log y$

$$
\begin{aligned}
p_Y(y) &= \sum_{n=0}^{\infty} p_{N(t)}(n)\, p_{Y|N(t)}(y \mid n) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\, {}^{n}C_{\log(ye^{n\Delta\log y})}\, p^{\log(ye^{n\Delta\log y})/2\Delta\log y} q^{\log(e^{n\Delta\log y}/y)/2\Delta\log y} \\
&= \sum_{n=\log y/\Delta\log y}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\, {}^{n}C_{\log(ye^{n\Delta\log y})}\, p^{\log(ye^{n\Delta\log y})/2\Delta\log y} q^{\log(e^{n\Delta\log y}/y)/2\Delta\log y} \\
&= e^{-\lambda t}\sum_{v=0}^{\infty} \frac{(p\lambda t)^{v+\log(y)/\Delta\log y}(q\lambda t)^{v}}{\left(v+\log y/\Delta\log y\right)!v!}
\end{aligned}
$$

and rearranging (see the additive equivalent for the additional lines of working)

$$
\begin{aligned}
p_Y(y) &= e^{-\lambda t}(\sqrt{p/q})^{\log(y)/\Delta\log y}\sum_{v=0}^{\infty} \frac{(\sqrt{pq}\lambda t)^{2v+\log(y)/\Delta\log y}}{\left(v+\log y/\Delta\log y\right)!v!} \\
&= e^{-\lambda t}(\sqrt{p/q})^{\log(y)/\Delta\log y} I_{\log(y)/\Delta\log y}\left(2\sqrt{pq}\lambda t\right)
\end{aligned}
$$

for $y = -\infty, ..., e^{-2\Delta t}, e^{-\Delta t}, 1, e^{+\Delta t}, e^{+2\Delta t}, ..., +\infty$. Product convolving the net forward transition Y onto the starting value of the walk $S(0) = s_0$ for the aggregate walk variable $S(t) = S_0 \times Y = S_0 \times Y_1 \times Y_2 \times ... \times Y_{N(t)}$ results in the distribution of the Poisson generated produced walk which has transition mass function

$$
p_{S(t)|S(0)}(y \mid s_0) = p_Y(y/s_0) = e^{-\lambda t}(\sqrt{p/q})^{\log(y/s_0)/\Delta\log y} I_{\log(y/s_0)/\Delta\log y}\left(2\sqrt{pq}\lambda t\right)
$$

for the walk sample space at the forward time variable of

$y = -\infty, ..., s_0 e^{-2\Delta\log y}, s_0 e^{-\Delta\log y}, s_0, s_0 e^{+\Delta\log y}, s_0 e^{+2\Delta\log y}, ..., +\infty$.

For example, the probability that the walk is one step ratio greater than its starting value at the start of the walk

$$
\begin{aligned}
p_{S(t)|S(0)}(s_0 e^{\Delta\log y} \mid s_0) &= \Pr\left(S(t) = s_0 e^{\Delta\log y} \mid S(0) = s_0\right) \\
&= e^{-\lambda t}(\sqrt{p/q})\, I_1\left(2\sqrt{pq}\,\lambda t\right) = e^{-\lambda t}(\sqrt{p/q})\sum_{v=0}^{\infty} \frac{(\sqrt{pq}\lambda t)^{v}}{v!}\frac{(\sqrt{pq}\lambda t)^{v+1}}{\left(v+1\right)!}
\end{aligned}
$$

and the probability that it is one step ratio less than its starting value is

$$
\begin{aligned}
p_{S(t)|S(0)}(s_0 e^{-\Delta\log y} \mid s_0) &= \Pr\left(S(t) = s_0 e^{-\Delta\log y} \mid S(0) = s_0\right) \\
&= e^{-\lambda t}(\sqrt{p/q})^{-1} I_{-1}\left(2\sqrt{pq}\,\lambda t\right) = e^{-\lambda t}(\sqrt{p/q})\sum_{v=0}^{\infty} \frac{(\sqrt{pq}\lambda t)^{v}}{v!}\frac{(\sqrt{pq}\lambda t)^{v-1}}{\left(v-1\right)!}
\end{aligned}
$$

where $I_{+1} = I_{-1}$ and can be evaluated with a table lookup or function call to MATLAB

`besseli(1,` $2\sqrt{pq}\,\lambda t$ `)`

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There are other more obtuse mathematical functions, hypergeometric functions[92], for other ratios such as an up-tick being 2/3 and a downtick being 1/3 of the total possible range between the possible up and down movements – such functions are not considered further here.

2. MGF of the Multiplicative Compound Poisson Random Walk

The moment generating function of the corresponding additive walk on logarithms is

$$M_{\log S(T)|\log S(t_0)}(\Theta) = e^{\Theta \log S_0 + \lambda t\left(pe^{\Theta \log u} + qe^{\Theta \log d} - 1\right)} = S_0{}^{\Theta} e^{\lambda t\left(pu^{\Theta} + qd^{\Theta} - 1\right)} \qquad \text{where} \qquad \Theta \sim P(\theta)$$

and the moment generating function for the multiplicative walk as that of the additive walk weighted by Poisson probabilities in the moment generating function parameter

$$M_{S(T)|S(t_0)}(\theta) = \sum_{j=0}^{\infty} \frac{\theta^j}{j!} M_{\log S(T)|\log S(t_0)}(j) \qquad \text{where} \qquad \Theta \sim P(\theta)$$

so that

$$M_{S(T)|S(t_0)}(\Theta) = \sum_{j=0}^{\infty} \frac{(S_0\theta)^j}{j} e^{\lambda t\left(pu^j + qd^j - 1\right)}$$

When the up ratios are reciprocal with the down ratios $u = 1/d = e^{\Delta \log y}$, the mgf for the logarithm of the walk is

$$M_{\log S(T)|\log S(t_0)}(\Theta) = S_0{}^{\Theta} e^{\lambda t\left(pe^{\Theta \Delta \log y} + qe^{-\Theta \Delta \log y} - 1\right)} \qquad \text{where} \qquad \Theta \sim P(\theta)$$

and that of the original walk is

$$M_{S(T)|S(t_0)}(\Theta) = \sum_{j=0}^{\infty} \frac{(S_0\theta)^j}{j} e^{\lambda t\left(pe^{j\Delta \log y} + qe^{-j\Delta \log y} - 1\right)}$$

3. Transition Mean and Variance

INSERT HERE FROM TABLES

4. Compound Poisson Process: Multiplicative Random Walk Simulation Algorithm

The simulation algorithms of the multiplicative walks parallel those of the additive walk exactly apart from the fact that the simulated basis components product (Π) convolve, with a continued product of the basis vector of steps, rather than sum (Σ) convolve with a cumulatively sum of the basis vector.

Simulate the exponential diffusion for the abscissae: generate the individual renewals with n exponential variables with mean $\beta = 1 / \lambda$

$$\begin{aligned} t_{(j)} - t_{(j-1)} &\sim Exp\left(1/\lambda\right) \\ &\sim Exp\left(\beta\right) \end{aligned}$$

where $\lambda \propto 1/\Delta t$ is the mean number of Poisson steps on Δt and $\beta \propto \Delta t$ is the average time between each Poisson step. Discretely integrated and appending the backward time variable to the beginning of the vector

$$\begin{aligned} t_{(j)} &= t_0 + \sum_{j=1}^{n} \left(t_{(j)} - t_{(j-1)}\right) \\ &\sim t_0 + \Gamma\left(n, 1/\lambda\right) \\ &\sim t_0 + \Gamma\left(n, \beta\right) \end{aligned}$$

[92] Abramowitz, M. and Stegun, I. A., (1968) *Handbook of Mathematical Functions*. Dover Publications, Inc.

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which, as per the additive Poisson generated walk, are the values of a Poisson process which form the abscissa of the simulated path. Generate the vector of a multiplicative random walk with *n* steps. These form the ordinates of the process. Plot the components of the vector of the multiplicative random walk (vector of ordinates) against the components of the vector of the Poisson process (vector of abscissae). MATLAB code for this simulation algorithm is

```
p=p; u=u; d=d; s0=s0; t0=t0; n=n; b=β;          %initialize walk parameters
t=cumsum([t0; exprnd(b,n,1)]);                    %simulate step times
x=binornd(1,p,n,1); S=s0.*exp(cumsum(log(u/d).*x+log(d));    %simulate walk
plot(t,S);                                                    %plot S over t
```

For a compound Poisson symmetric additive walk of n =100 steps starting at s_0 =10 at time t_0 =2, initialize the parameters with the line

```
p=.5; u=1.05; d=1/u; s0=10; t0=2; n=100; b=5/100;     %initialize parameters
```

For a time unit of one hour, that statistical functions regarding the timing of the sulti of the simulated path are exactly those of the additive walk; a proportional sultus occurs once every 5/100 hours = 3 minutes on average, and for a fixed number 100 steps exact, the average transition length is 100*3 minutes = 5 hours. If the transition interval begins at 2 (o'clock), then the 100th sultus is gamma distributed and occurs at 7 (o'clock) on average. At this time, there will be a Poisson number of compound factors of 5% with a total of p*(n/b)=0.5*100=50 on average and a Poisson number of compound factors of -5% with a total of (1-p)*(n/p)=50 on average. Because the walk is symmetric in both probability and step factors, there will be no change in the backward walk variable s0=10 on average (but distributed as a Bessel function).

The simulation algorithm can alternatively simulate for a fixed transition length and a random number of steps by either accept-reject or by simulating the number of Poisson steps for the fixed transition length and then distributing the $N(T-t_0)=n$ abscissae (identically distributed steps) over the transition interval.

```
p=p; u=u; d=d; s0=s0; t0=t0; T=T; b=b;          %initialize walk parameters
n=poissrnd((1/b).*(T-t0),1,1); t=[t0; t0+sort(unifrnd(0,T-t0,n,1))];
x=binornd(1,p,n,1); S=s0.*exp(cumsum(log(u/d).*x+log(d));     %simulate walk
plot(t,S);                                                    %plot S over t
```

As an example, for a simulated path of a Poisson generated path for a fixed transition length of $T-t_0$ =5 starting at time t_0 =2 at value s_0 =10, initialize the parameters with the line

```
p=.5; u=1.05; d=1/u; s0=10; t0=2; T=7; b=5/100; %initialize walk parameters
```

If the time unit is hours then, say, the simulated path begins at 2 (o'clock) and ends at 7 (o'clock) with Poisson steps occurring once every `b=5/100=1/20` hours =3 minutes on average which amounts to `lambda=1/(5/100)=20` steps per hour on average and a total average of `(T-t0)/b=5/(5/100)=100` steps of proportional increases or decreases of 5% over the 5 hour transition period.

5. Limited Moments and Undiscounted Option Premia

The **multiplicative** *t* period ahead forward walk variable, composed of X^{*N_t} upsteps and $N_t - X^{*N_t}$ downsteps, is

$$S_t = S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}}$$

where $N_t \sim P(\lambda t)$ and, given $N_t = n$, $X^{*n} \sim B(n,p)$. The payoff of an option on the walk variable is then

$\max(S_t - k, 0) = \max(S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}} - k, 0)$ if it is an option to call the asset to buy for *k*

$\max(k - S_t, 0) = \max(k - S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}}, 0)$ if it is an option to put the asset to sell for *k*

By statistical formula, $E\big(g(X)\big) = \sum_{x \in X} g(x) \times p_X(x)$, the undiscounted premium for a call is

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$$
\begin{aligned}
c(t,k) &= E\left(\max(S_0 u^{X^*N_t} d^{N_t - X^*N_t} - k), 0)\right) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} \times \max\left(S_0 u^x d^{n-x} - k, 0\right) \\
&= \sum_{x=0}^{\infty} \sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^nC_x p^x q^{n-x} \times \max\left(S_0 u^x d^{n-x} - k, 0\right) \\
&= \sum_{x=0}^{\infty} \max\left(S_0 \sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^n C_x (pu)^x (qd)^{n-x} - k \sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^n C_x p^x q^{n-x},\ 0 \right)
\end{aligned}
$$

noting that

(1) $$\sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^n C_x p^x q^{n-x} = \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

(2) $$\sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^nC_x (pu)^x (qd)^{n-x} = e^{\lambda t((pu+qd)-1)} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!}$$

$$
\begin{aligned}
&\sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \frac{n!}{x!(n-x)!} p^x q^{n-x} \\
&= \frac{e^{-\lambda t} p^x}{x!} \sum_{n=x}^{\infty} \frac{(\lambda t)^{n-x+x}}{(n-x)!} q^{n-x} \\
&= \frac{e^{-\lambda t}(p\lambda t)^x}{x!} \sum_{n=0}^{\infty} \frac{(q\lambda t)^n}{n!} \\
&= \frac{e^{-\lambda t}(p\lambda t)^x}{x!} e^{q\lambda t} \\
&= \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times \frac{n!}{x!(n-x)!} (pu)^x (qd)^{n-x} \\
&= \frac{e^{-\lambda t}(pu)^x}{x!} \sum_{n=x}^{\infty} \frac{(\lambda t)^{n-x+x}}{(n-x)!} (qd)^{n-x} \\
&= \frac{e^{-\lambda t}(pu\lambda t)^x}{x!} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} (qd)^n \\
&= \frac{e^{-\lambda t}(pu\lambda t)^x}{x!} e^{qd\lambda t} \\
&= e^{\lambda t((pu+qd)-1)} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!}
\end{aligned}
$$

the call premium then becomes

$$c(t,k) = \sum_{x=0}^{\infty} \max\left(S_0 e^{\lambda t((pu+qd)-1)} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!} - k \frac{e^{-p\lambda t}(p\lambda t)^x}{x!},\ 0 \right)$$

The max function is non-zero whenever

$$
\begin{aligned}
&S_0 e^{\lambda t[(pu+qd)-1]} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!} - k \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} \geq 0 \\
&S_0 e^{\lambda t[qd+p-1]} u^x \geq k \\
&u^x \geq e^{\lambda t q(1-d)} k / S_0 \\
&x \geq \left(\lambda t\, q(1-d) + \log k - \log S_0\right) / \log u
\end{aligned}
$$

so that only terms in the summation for x greater than $\left(\lambda t\, q(1-d) - \log(S_0/k)\right)/\log u$ are non-zero. The undiscounted call option premium is

$$c(t,k) = S_0 e^{\lambda t((pu+qd)-1)} \sum_{x=k^*}^{\infty} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!} - k \sum_{x=k^*}^{\infty} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

where k^* is the smallest integer equal to or larger than $\left(\lambda t\, q(1-d) - \log(S_0/k)\right)/\log u$. Written in terms of Poisson complementary cdfs, the premium is

$$c(t,k) = S_0 e^{\lambda t((pu+qd)-1)} \overline{F}_{P(pu\lambda t)}(k^*) - k\, \overline{F}_{P(p\lambda t)}(k^*)$$

where $\overline{F}_{P(p\lambda t)}(\cdot)$ and $\overline{F}_{P(p\lambda t)}(\cdot)$ are the complementary cdfs of Poisson distributions with means $p\lambda t$ and $pu\lambda t$ respectively. Similarly to the option premium for the Poisson generated additive walk, the Poisson cdfs and complementary cdfs can be written in terms of incomplete gamma functions. These expressions are omitted. For a put option

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$$
\begin{aligned}
p(t,k) &= E\left(\max(k - S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}}), 0)\right) \\
&= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} \times \max\left(k - S_0 u^x d^{n-x}, 0\right) \\
&= \sum_{x=0}^{\infty} \sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^nC_x p^x q^{n-x} \times \max\left(k - S_0 u^x d^{n-x}, 0\right) \\
&= \sum_{x=0}^{\infty} \max\left(k \sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^n C_x p^x q^{n-x} - S_0 \sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^n C_x (pu)^x (qd)^{n-x},\ 0 \right)
\end{aligned}
$$

and with the summations

(1) $$\sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^n C_x p^x q^{n-x} = \frac{e^{-p\lambda t}(p\lambda t)^x}{x!}$$

(2) $$\sum_{n=x}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \times {}^n C_x (pu)^x (qd)^{n-x} = e^{\lambda t((pu+qd)-1)} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!}$$

the formulae simplifies to

$$p(t,k) = \sum_{x=0}^{\infty} \max\left(k \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} - S_0 e^{\lambda t((pu+qd)-1)} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!},\ 0 \right)$$

For k^* being the largest integer equal to or smaller than $\left(\lambda t\, q(1-d) - \log(S_0/k)\right)/\log u$, the undiscounted put option premium becomes

$$p(t,k) = k \sum_{x=0}^{k^*} \frac{e^{-p\lambda t}(p\lambda t)^x}{x!} - S_0 e^{\lambda t((pu+qd)-1)} \sum_{x=0}^{k^*} \frac{e^{-pu\lambda t}(pu\lambda t)^x}{x!}$$

Written in terms of Poisson cdfs, the leading term with rate $p\lambda t$ and the trailing term with rate $pu\lambda t$, the undiscounted premium is

$$p(t,k) = k\ F_{P(p\lambda t)}(k^*) - S_0 e^{\lambda t((pu+qd)-1)} F_{P(pu\lambda t)}(k^*)$$

where $F_{P(p\lambda t)}(\cdot)$ and $F_{P(p\lambda t)}(\cdot)$ are the cdfs of Poisson distributions with means $p\lambda t$ and $pu\lambda t$ respectively.

6. Forward and Backward Recurrence Relations of a Compound Poisson Process Multiplicative Random Walk

The principle of aggregation of the basis function is the continued product of the steps (which are ratios) rather than the continued sum. The recurrence relation for the multiplicative walk, therefore, differs from that of the additive walk in that the convolutions (deconvolutions) of the density functions are multiplicative; the convolutions (deconvolutions) of the transition function are with regard to the multiplication (division), rather than addition (subtraction), of two random variables. In terms of the two point distribution of the steps, which form the increments of the random walk, if a step in S occurs then the cumulative product of the steps is multiplied by either u with probability p or by d with probability $q = 1 - p$ and similarly at the beginning of the transition interval at the backward variables. From $S(T) = S(t_0) + Y_1 + Y_2 + ... + Y_{N(T-t_0)}$ given $N(T - t_0) = n$, the forward and backward recurrences for the steps, from the ordinary multiplicative walk, are

$$p_{S(T+\Delta t)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0)}(y \div u \mid s_0) + q \times p_{S(T)|S(t_0)}(y \div d \mid s_0)$$

and

$$p_{S(T)|S(t_0)}(y \mid s_0) = p \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 \times u) + q \times p_{S(T)|S(t_0+\Delta t)}(y \mid s_0 \times d)$$

and those of the Poisson generator $N(T) \mid N(t_0) \sim P\left(\lambda(T - t_0)\right)$ are

$$p_{N(T+\Delta t)|N(t_0)}(n \mid n_0) = p_{N(T)|N(t_0)}(n+1 \mid n_0)\,\lambda\Delta t + p_{N(T)|N(t_0)}(n \mid n_0)\left(1 - \lambda\Delta t\right) + o(\Delta t)$$

and

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$$p_{N(T)|N(t_0)}(n\,|\,n_0) = p_{N(T)|N(t_0+\Delta t)}(n\,|\,n_0-1)\,\lambda\Delta t + p_{N(T)|N(t_0+\Delta t)}(n\,|\,n_0)\big(1-\lambda\Delta t\big) + o(\Delta t)$$

where a unit increment in n then corresponds to a step onto $S(T)$ and a unit decrement in n_0 corresponds to a step from $S(t_0)$; multiple steps in S, either $S(T)$ or $S(t_0)$, over length Δt are $o(\Delta t)$. Combining these notions for the forward recurrence

Forward transition function
= *Pr*(no Poisson event occurs)*zero-fold Π-convolution onto transition function ← walk unchanged
+*Pr*(one Poisson event occurs)*one-fold Π-convolution onto transition function ← multiply one step factor
+*Pr*(two Poisson events occur)*two-fold Π-convolution onto transition function ← multiply two step factors
+… etc.

Therefore, from the fact that $S(T) = S(t_0) + Y_1 + Y_2 + ... + Y_{N(T-t_0)}$ given $N(T-t_0) = n$,

$$p_{S(T+\Delta t)|S(t_0)}(y\,|\,s_0) = \Big(p\times p_{S(T)|S(t_0)}(y\div u\,|\,s_0) + q\times p_{S(T)|S(t_0)}(y\div d\,|\,s_0)\Big)\lambda\Delta t + p_{S(T)|S(t_0)}(y\,|\,s_0)\big(1-\lambda\Delta t\big) + o(\Delta t)$$

Subtracting $p_{S(T)|S(t_0)}(y\,|\,s_0)$ from both sides and dividing through by Δt for

$$\frac{p_{S(T+\Delta t)|S(t_0)}(y\,|\,s_0) - p_{S(T)|S(t_0)}(y\,|\,s_0)}{\Delta t} = \lambda\Big(p\times p_{S(T)|S(t_0)}(y\div u\,|\,s_0) + q\times p_{S(T)|S(t_0)}(y\div d\,|\,s_0) - p_{S(T)|S(t_0)}(y\,|\,s_0)\Big) + \frac{o(\Delta t)}{\Delta t}$$

and letting $\Delta t\to 0$ so that $o(\Delta t)/\Delta t\to 0$ results in the forward convolution recurrence

$$\frac{\partial}{\partial T}p_{S(T)|S(t_0)}(y\,|\,s_0) = \lambda\Big(p\times p_{S(T)|S(t_0)}(y\div u\,|\,s_0) + q\times p_{S(T)|S(t_0)}(y\div d\,|\,s_0) - p_{S(T)|S(t_0)}(y\,|\,s_0)\Big)$$

If the binomial step ratios are reciprocals, $u = 1/d = e^{\Delta\log y}$, then for a Poisson generated step the walk either increases from below (decreased from above) proportionately by $e^{\Delta\log y}$ or decreases from above (increased from below) proportionately by $e^{-\Delta\log y}$

$$\frac{\partial}{\partial T}p_{S(T)|S(t_0)}(y\,|\,s_0) = \lambda\Big(p\times p_{S(T)|S(t_0)}(y\div e^{\Delta\log y}\,|\,s_0) + q\times p_{S(T)|S(t_0)}(y\times e^{\Delta\log y}\,|\,s_0) - p_{S(T)|S(t_0)}(y\,|\,s_0)\Big)$$

For the backward recurrence

Backward transition function
= *Pr*(no Poisson event occurred)*zero-fold Π-deconvolution from transition function ← walk unchanged
+*Pr*(one Poisson event occurred)*one-fold Π-deconvolution from transition function ← reduce one step factor
+*Pr*(two Poisson events occurred)*two-fold Π-deconvolution from transition function ← reduce by two step factors
+… etc.

so that

$$p_{S(T)|S(t_0)}(y\,|\,s_0) = \Big(p\times p_{S(T)|S(t_0+\Delta t)}(y\,|\,s_0\times u) + q\times p_{S(T)|S(t_0+\Delta t)}(y\,|\,s_0\times d)\Big)\lambda\Delta t + p_{S(T)|S(t_0+\Delta t)}(y\,|\,s_0)\big(1-\lambda\Delta t\big) + o(\Delta t)$$

noting that by time homogeneity this can be written as

$$p_{S(T)|S(t_0-\Delta t)}(y\,|\,s_0) = \Big(p\times p_{S(T)|S(t_0)}(y\,|\,s_0\times u) + q\times p_{S(T)|S(t_0)}(y\,|\,s_0\times d)\Big)\lambda\Delta t + p_{S(T)|S(t_0)}(y\,|\,s_0)\big(1-\lambda\Delta t\big) + o(\Delta t)$$ Su

btracting $p_{S(T)|S(t_0)}(y\,|\,s_0)$ from both sides and dividing through by Δt for

$$-\frac{p_{S(T)|S(t_0)}(y\,|\,s_0) - p_{S(T)|S(t_0-\Delta t)}(y\,|\,s_0)}{\Delta t} = \lambda\Big(p\times p_{S(T)|S(t_0)}(y\,|\,s_0\times u) + q\times p_{S(T)|S(t_0)}(y\,|\,s_0\times d) - p_{S(T)|S(t_0)}(y\,|\,s_0)\Big) + \frac{o(\Delta t)}{\Delta t}$$

and letting $\Delta t\to 0$ so that $o(\Delta t)/\Delta t\to 0$, the backward recurrence becomes

$$-\frac{\partial}{\partial t_0}p_{S(T)|S(t_0)}(y\,|\,s_0) = \lambda\Big(p\times p_{S(T)|S(t_0)}(y\,|\,s_0\times u) + q\times p_{S(T)|S(t_0)}(y\,|\,s_0\times d) - p_{S(T)|S(t_0)}(y\,|\,s_0)\Big)$$

which is the backward deconvolution recurrence. When the binomial step ratios are proportional $u = 1/d = e^{\Delta\log y}$ the recurrence becomes

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$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda\Big(p \times p_{S(T)|S(t_0)}(y \mid s_0 \times e^{\Delta \log y}) + q \times p_{S(T)|S(t_0)}(y \mid s_0 \div e^{\Delta \log y}) - p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

7. Forward Equation of the Continuous Diffusion Limit

From the forward convolution recurrence of the Poisson generated random walk

$$\frac{\partial}{\partial T} p_{S(T)|S(t_0)}(y \mid s_0) = \lambda\Big(p \times p_{S(T)|S(t_0)}(y \div e^{\Delta \log y} \mid s_0) + q \times p_{S(T)|S(t_0)}(y \times e^{\Delta \log y} \mid s_0) - p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

changing the walk variable to its logarithm so the transition functions can be expanding in power series

$$\frac{\partial}{\partial T} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$= \lambda\Big(p \times p_{\log S(T)|\log S(t_0)}(\log y - \Delta \log y \mid \log s_0) + q \times p_{\log S(T)|\log S(t_0)}(\log y + \Delta \log y \mid \log s_0) - p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)\Big)$$

Expanding the first two bracketed transition functions in univariate power series around the logarithm of the forward diffusion variable, $\log y$

(1) $p_{\log S(T)|\log S(t_0)}(\log y - \Delta \log y \mid \log s_0)$

$$= p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) - (\Delta \log y) \frac{\partial}{\partial \log y} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$+ \tfrac{1}{2}(\Delta \log y)^2 \frac{\partial^2}{(\partial \log y)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \lambda o\Big((\Delta \log y)^2\Big)$$

(2) $p_{\log S(T)|\log S(t_0)}(\log y + \Delta \log y \mid \log s_0)$

$$= p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + (\Delta \log y) \frac{\partial}{\partial \log y} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$+ \tfrac{1}{2}(\Delta \log y)^2 \frac{\partial^2}{(\partial \log y)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \lambda o\Big((\Delta \log y)^2\Big)$$

noting that $o\Big((\Delta \log y)^2\Big) = o(\Delta t)$ and noting that $\lambda \propto 1/\Delta t$. Substituting the expansions into the forward recurrence and simplifying for

$$\frac{\partial}{\partial T} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$= \tfrac{1}{2}\lambda(\Delta \log y)^2 \frac{\partial^2}{(\partial \log y)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) - (p-q)\lambda(\Delta \log y) \frac{\partial}{\partial \log y} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \lambda o(\Delta t)$$

and letting $\lambda \to 1/\Delta t$ and $\Delta t \to 0$ so that $\lambda o(\Delta t) \to 0$ and defining the infinitesimal parameters as the limits

$$\lim_{\substack{\lambda \to 1/\Delta t \\ \Delta t \to 0}} \frac{(\Delta \log y)^2}{1/\lambda} = \sigma^2 \qquad \text{and} \qquad \lim_{\substack{\lambda \to 1/\Delta t \\ \Delta t \to 0}} (p-q)\frac{(\Delta \log y)}{1/\lambda} = \mu$$

for

$$\frac{\partial}{\partial T} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = \tfrac{1}{2}\sigma^2 \frac{\partial^2}{(\partial \log y)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) - \mu \frac{\partial}{\partial \log y} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$

which is the forward equation of a Brownian motion, the heat equation, in the log-variables. Transforming back to the original walk variable by exponentiating, the relations between the transition density functions being $p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\, p_{S(T)|S(t_0)}(y \mid s_0)$, where the right-hand-side contains the Jacobian scale factor since the walk variable is now on a dense sample space as that of a continuous diffusion variable. With the transformed time and diffusion derivatives, derived in the section of the forward equation for the ordinary geometric random walk

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(1) $\dfrac{\partial}{\partial T}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\dfrac{\partial}{\partial T}\, p_{S(T)|S(t_0)}(y \mid s_0)$

(2) $\dfrac{\partial}{\partial \log y}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\dfrac{\partial}{\partial y}\Big(y\; p_{S(T)|S(t_0)}(y \mid s_0)\Big)$

(3) $\dfrac{\partial^2}{(\partial \log y)^2}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\dfrac{\partial^2}{\partial y^2}\Big(y^2\; p_{S(T)|S(t_0)}(y \mid s_0)\Big) - y\dfrac{\partial}{\partial y}\Big(y\; p_{S(T)|S(t_0)}(y \mid s_0)\Big)$

the forward equation becomes, following substitution and a division through by y ,

$$y\frac{\partial}{\partial T}\, p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2\left\{y\frac{\partial^2}{\partial y^2}\Big(y^2\; p_{S(T)|S(t_0)}(y \mid s_0)\Big) - y\frac{\partial}{\partial y}\Big(y\; p_{S(T)|S(t_0)}(y \mid s_0)\Big)\right\} - \mu y\frac{\partial}{\partial y}\Big(y\; p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

Canceling the Jacobian

$$\frac{\partial}{\partial T}\, p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}\Big(y^2\; p_{S(T)|S(t_0)}(y \mid s_0)\Big) - \Big(\mu + \tfrac{1}{2}\sigma^2\Big)\frac{\partial}{\partial y}\Big(y\; p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

which is the forward equation of a proportional Brownian motion.

8. Backward Equation of the Continuous Diffusion Limit Motion

From the backward convolution recurrence of the Poisson generated random walk

$$-\frac{\partial}{\partial t_0}\, p_{S(T)|S(t_0)}(y \mid s_0) = \lambda\Big(p \times p_{S(T)|S(t_0)}(y \mid s_0 \times e^{\Delta \log y}) + q \times p_{S(T)|S(t_0)}(y \mid s_0 \div e^{\Delta \log y}) - p_{S(T)|S(t_0)}(y \mid s_0)\Big)$$

changing the walk variable to its logarithm

$$-\frac{\partial}{\partial t_0}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$= \lambda\Big(p \times p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0 + \Delta \log y) + q \times p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0 - \Delta \log y) - p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)\Big)$$

and expanding the transition probability functions around the logarithm of the backward diffusion variable in univariate power series,

(1) $p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0 + \Delta \log y) = p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + (\Delta \log y)\dfrac{\partial}{\partial \log s_0}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$
$$+ \tfrac{1}{2}(\Delta \log y)^2\frac{\partial^2}{(\partial \log s_0)^2}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \lambda\, o\Big((\Delta \log y)^2\Big)$$

(2) $p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0 - \Delta \log y) = p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) - (\Delta \log y)\dfrac{\partial}{\partial \log s_0}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$
$$+ \tfrac{1}{2}(\Delta \log y)^2\frac{\partial^2}{(\partial \log s_0)^2}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \lambda\, o\Big((\Delta \log y)^2\Big)$$

and substituting these into the backwards recurrence,

$$-\frac{\partial}{\partial t_0}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = \tfrac{1}{2}\lambda(\Delta \log y)^2\frac{\partial^2}{(\partial \log s_0)^2}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$
$$+ \lambda(p - q)(\Delta \log y)\frac{\partial}{\partial \log s_0}\, p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \lambda\, o\Big((\Delta \log y)^2\Big)$$

Letting $\lambda \to 1/\Delta t$ and $\Delta t \to 0$ so that $\lambda\, o\big(\Delta t\big) \to 0$ and again defining the infinitesimal parameters as the limits

$$\lim_{\substack{\lambda \to 1/\Delta t \\ \Delta t \to 0}} \frac{(\Delta \log y)^2}{1/\lambda} = \sigma^2 \qquad \text{and} \qquad \lim_{\substack{\lambda \to 1/\Delta t \\ \Delta t \to 0}} (p - q)\frac{(\Delta \log y)}{1/\lambda} = \mu$$

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The conjugate heat equation in the log walk variable results

$$-\frac{\partial}{\partial t_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$

$$= \tfrac{1}{2}\sigma^2 \frac{\partial^2}{(\partial \log s_0)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) + \mu \frac{\partial}{\partial \log s_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0)$$

Transforming the logarithmic variable back by exponentiation, inclusive of the Jacobian scale factor y $p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\, p_{S(T)|S(t_0)}(y \mid s_0)$ where, from the ordinary multiplicative walk,

(1) $\dfrac{\partial}{\partial t_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y \dfrac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0)$

(2) $\dfrac{\partial}{\partial \log s_0} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\, s_0 \dfrac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$

(3) $\dfrac{\partial^2}{(\partial \log s_0)^2} p_{\log S(T)|\log S(t_0)}(\log y \mid \log s_0) = y\, {s_0}^2 \dfrac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + y\, s_0 \dfrac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$

the equation becomes

$$-y \frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 \left\{ y\, {s_0}^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + y\, s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0) \right\} + \mu y\, s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$$

Canceling the Jacobian and rearranging

$$-\frac{\partial}{\partial t_0} p_{S(T)|S(t_0)}(y \mid s_0) = \tfrac{1}{2}\sigma^2 {s_0}^2 \frac{\partial^2}{\partial {s_0}^2} p_{S(T)|S(t_0)}(y \mid s_0) + \left(\mu + \tfrac{1}{2}\sigma^2\right) s_0 \frac{\partial}{\partial s_0} p_{S(T)|S(t_0)}(y \mid s_0)$$

which is the backward equation of a proportional Brownian motion.

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C. Simulated Paths of Compound Poisson Random walks

SIMULATED ARITHMETIC COMPOUND POISSON RANDOM WALKS

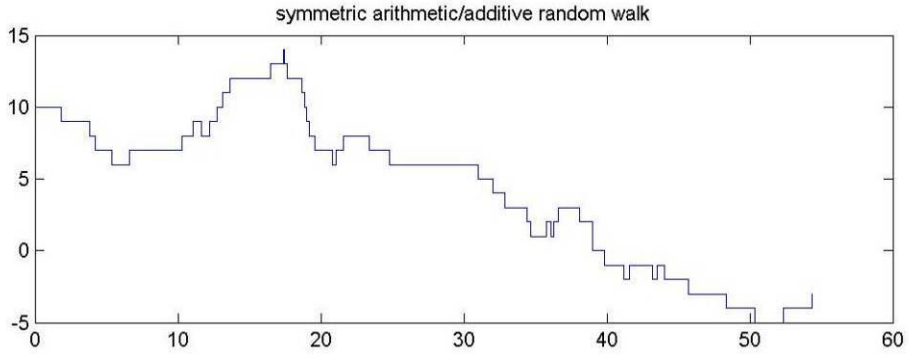


Figure 56: Poisson arithmetic random walk - single simulated path (50 Poisson steps)

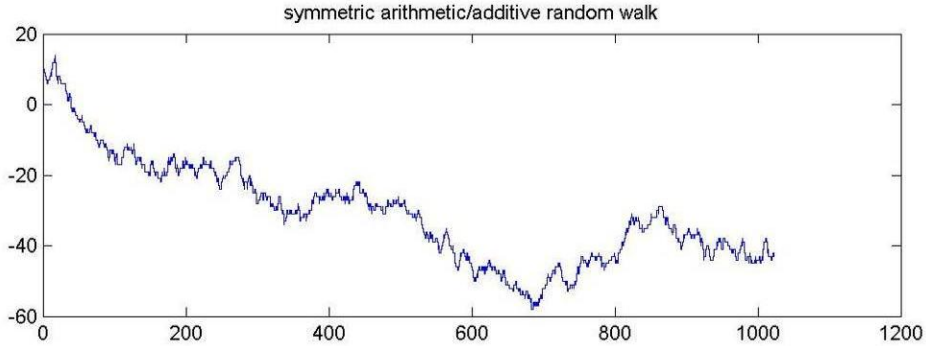


Figure 57: Poisson arithmetic random walk - single simulated path (1,000 Poisson steps)

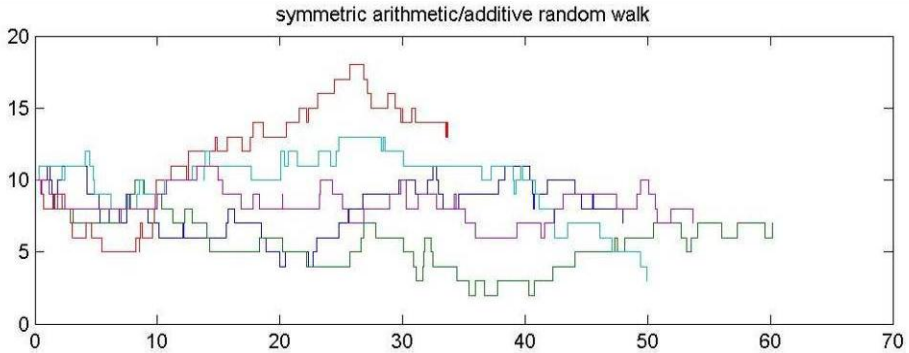


Figure 58: Poisson arithmetic random walk - 5 simulated paths (50 Poisson steps)

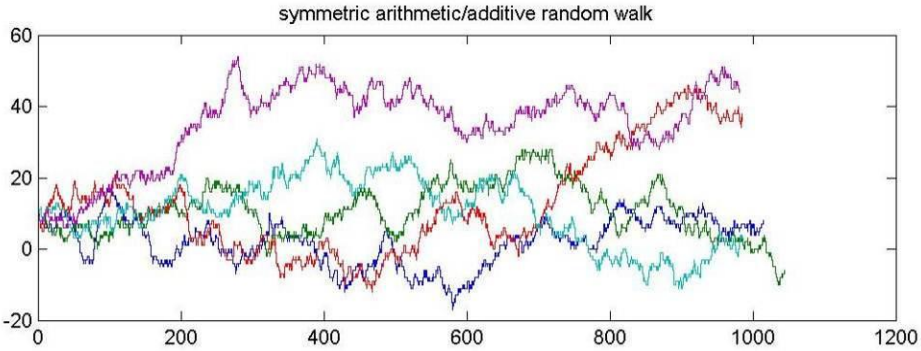


Figure 59: Poisson arithmetic random walk - 5 simulated paths (1,000 Poisson steps)

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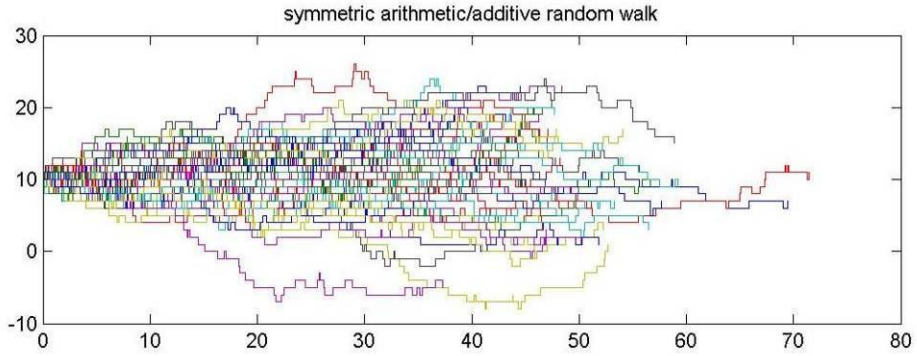


Figure 60: Poisson arithmetic random walk - 50 simulated paths (50 Poisson steps)

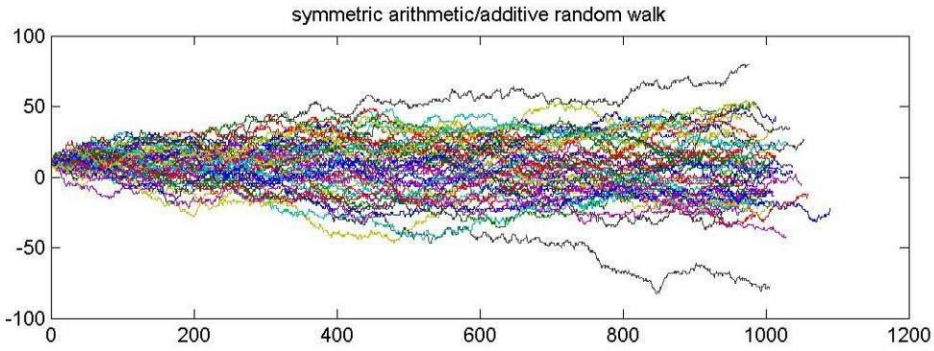


Figure 61: Poisson arithmetic random walk - 50 simulated paths (1,000 Poisson steps)

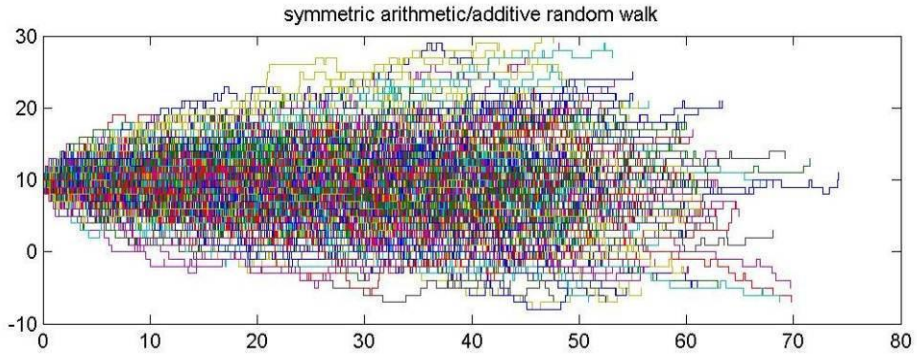


Figure 62: Poisson arithmetic random walk - 500 simulated paths (50 Poisson steps)

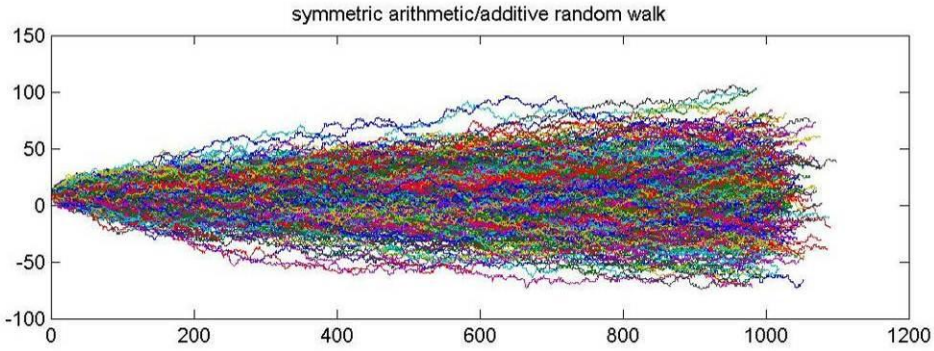


Figure 63: Poisson arithmetic random walk - 500 simulated paths (1,000 Poisson steps)

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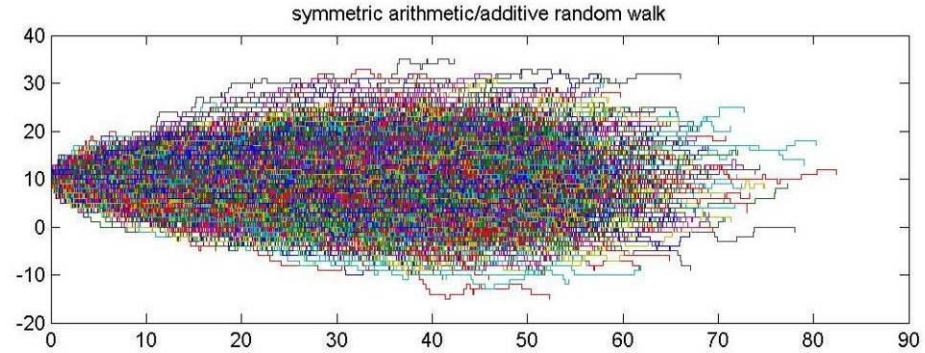


Figure 64: Poisson arithmetic random walk - 5,000 simulated paths (50 Poisson steps)

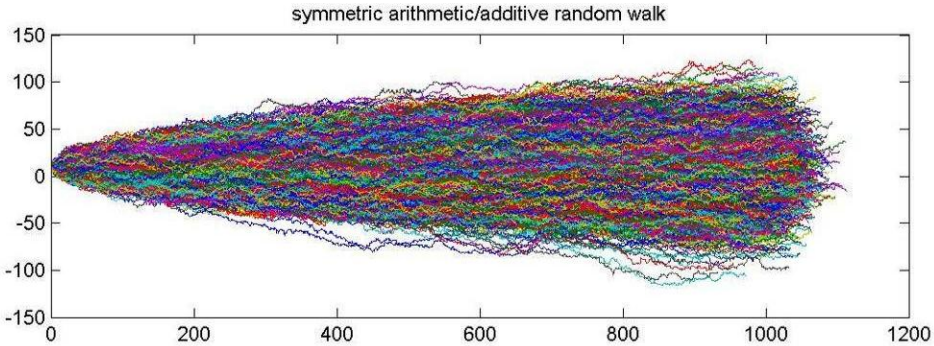


Figure 65: Poisson arithmetic random walk - 5,000 simulated paths (1,000 Poisson steps)

SIMULATED GEOMETRIC POISSON RANDOM WALKS

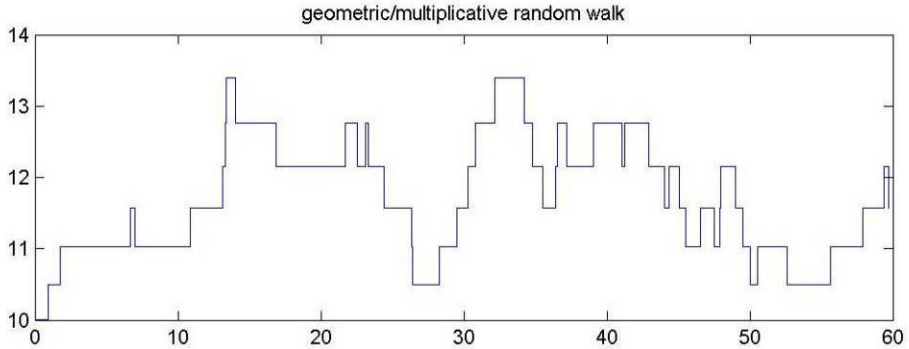


Figure 66: Geometric Poisson random walk - single simulated path (50 Poisson steps)

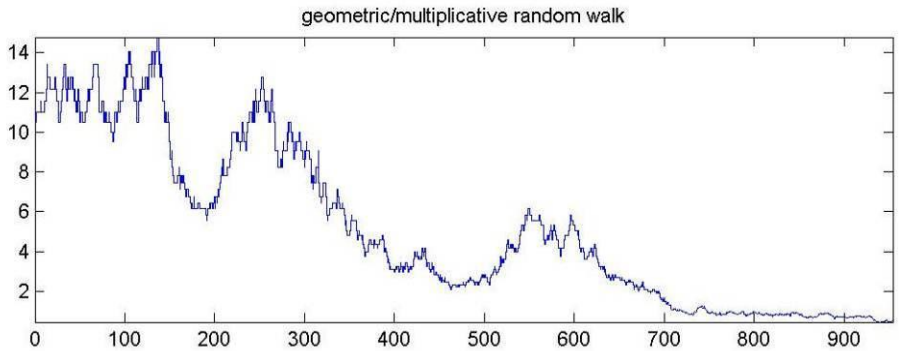


Figure 67: Geometric Poisson random walk - single simulated path (1,000 Poisson steps)

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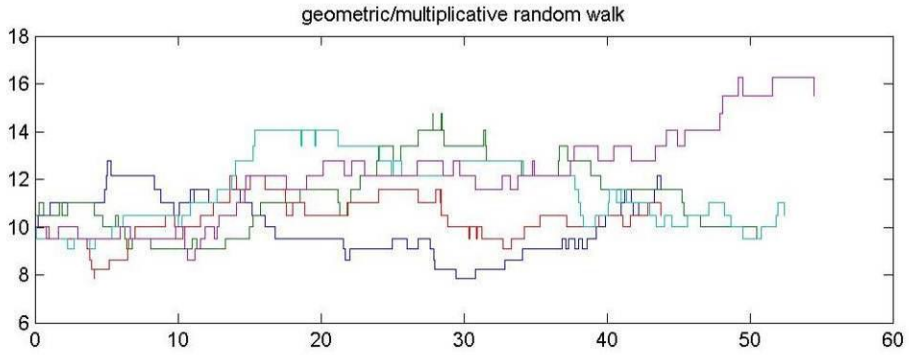


Figure 68: Geometric Poisson random walk - 5 simulated paths (50 Poisson steps)

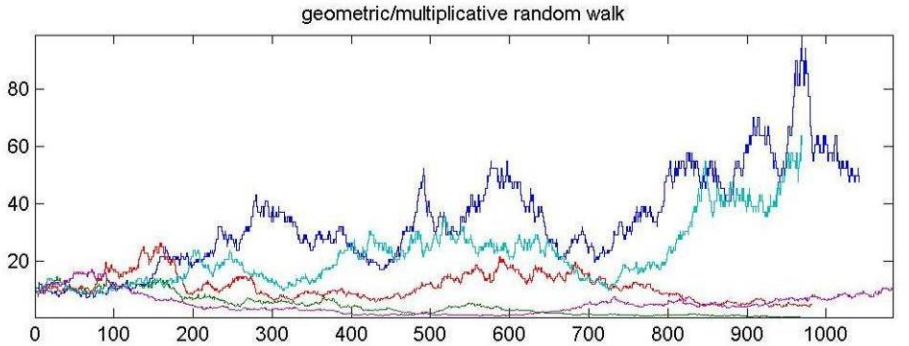


Figure 69: Geometric Poisson random walk - 5 simulated paths (1,000 Poisson steps)

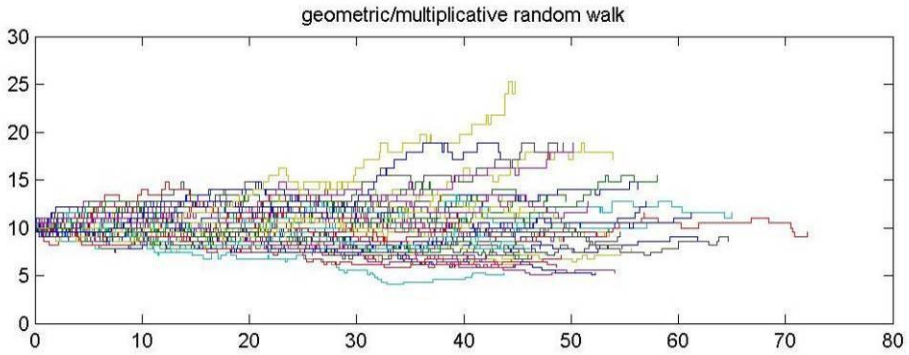


Figure 70: Geometric Poisson random walk - 50 simulated paths (50 Poisson steps)

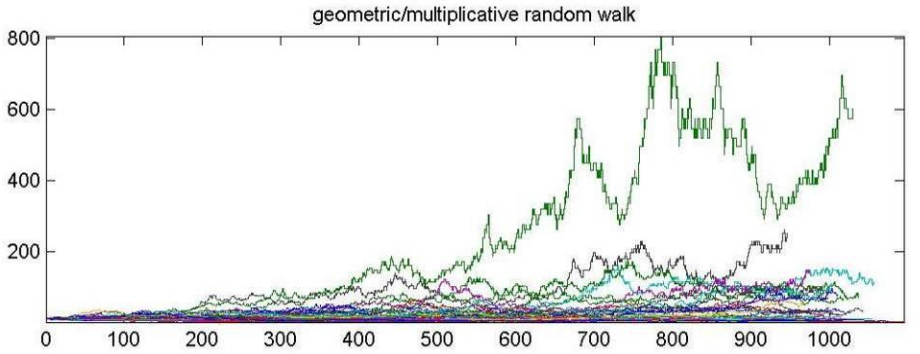


Figure 71: Geometric Poisson random walk - 50 simulated paths (1,000 Poisson steps)

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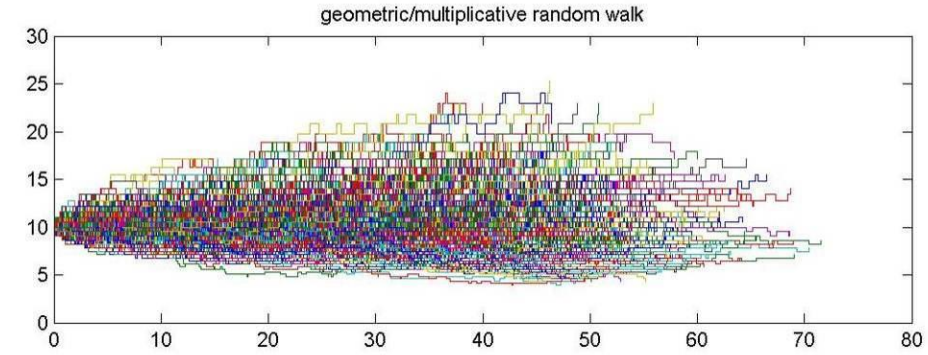


Figure 72: Geometric Poisson random walk - 500 simulated paths (50 Poisson steps)

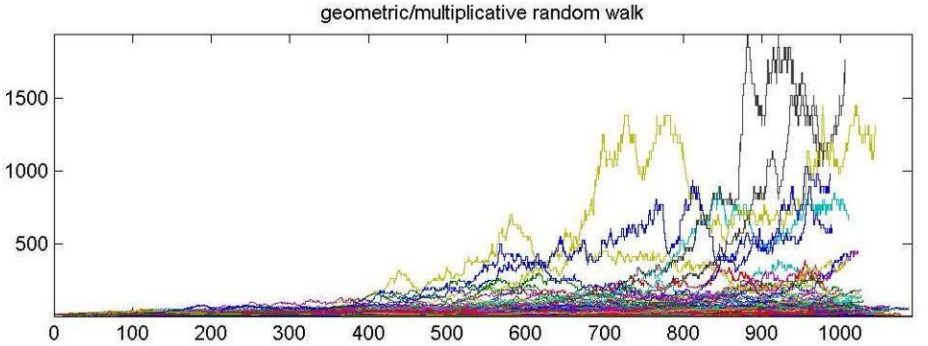


Figure 73: Geometric Poisson random walk - 500 simulated paths (1,000 Poisson steps)

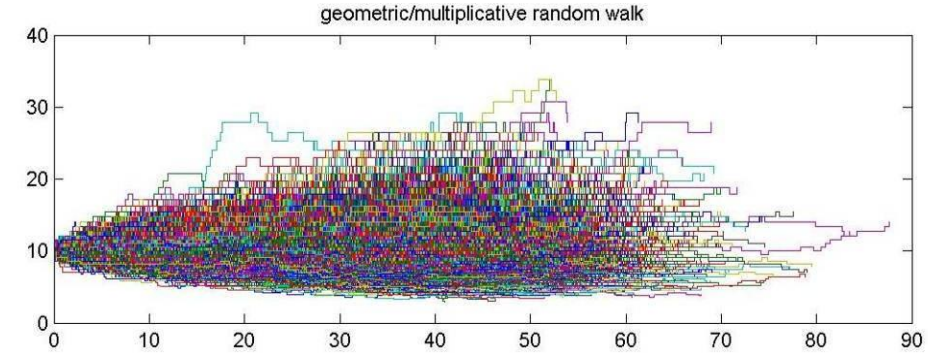


Figure 74: Geometric Poisson random walk - 5,000 simulated paths (50 Poisson steps)

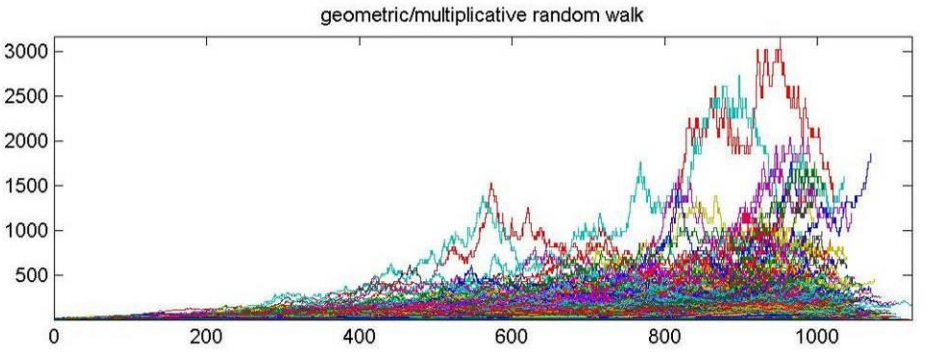


Figure 75: Geometric Poisson random walk - 5,000 simulated paths (1,000 Poisson steps)

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Compendium Extension:

This extension concerns a generalization to the Poisson generator of the Poisson generated random walks. This discussion is related to that for which FOUNDATIONS II discussed when the relations between compounding sums, continuous diffusion, and stochastic volatility processes were considered where variations to discrete Poisson process compounding sums were considered with non-Poisson process compounding sums; namely, first order unit root binomial and negative binomial diffusion, those of the binomial and negative binomial processes, to generate the degree to which the diffusion basis aggregated for a given forward transition.

The extension considered here regards letting the integrated rate intensity vary stochastically as a gamma diffusion (discussed in BOOK 2 as an example of the generalized heat equation), i.e., where the infinitesimal parameter of the Poisson process is stochastic and gamma distributed (gamma-Hilbert space basis function). This results in a transition variable that is generated by a negative binomial process. The mathematical theory is detailed for the transition functions and the transition mgf, and the undiscounted premia are derived for both additive and multiplicative steps.

With the consideration of this extension, there is then, for both additive (summed Σ) and multiplicative (produced Π) steps, three choices as per the discussion of FOUNDATIONS II whereby the Poisson generated walk is the baseline and the alternatives are chosen by reference to the dispersion in the count data for the timing of the convolution of the steps onto the aggregate walk variable. In the context of transaction data, this is level of dispersion in the timing of the transactions; if the timing of the transactions is over-dispersed, then the negative binomial process should be considered and, in the less likely case of under-dispersion, the binomial process.

This forms more rudimentary and summary foundation theory for the more interesting and important variables obtained when the discrete basis of two point discrete steps are replaced by normally distributed basis functions (FOUNDATIONS II of BOOK 1 as earlier) and unit root integrations are modulated and repeated/nested for diffusion of arbitrary order (BOOK 3).

1. Statistical Construction and Forward Transition Distribution

Considering the Poisson generated random walk with transition distribution

$$\begin{aligned} S(t) &\sim S_0 + u \times P(p\lambda t) + d \times \big(P(\lambda t) - P(p\lambda t)\big) \\ &\sim S_0 + u \times P(p\lambda t) + d \times P(q\lambda t) \end{aligned}$$

and transition mgf

$$M_{S(T)|S(t_0)}(\theta) = e^{S_0\theta + \lambda t(pe^{u\theta} + qe^{d\theta} - 1)}$$

which is a Poisson process that returns either a cumulative upstep u with probability p or a cumulative downstep d with probability q whenever a Poisson event occurs. This is a unit root integration of a Poisson basis function with infinitesimal parameter λ that returns a two point distribution for any single step …, a Poisson variant for $\mathbb{Z}$.

Now considering a gamma diffusion, a unit root integration of a gamma basis function. Letting the differential form of the gamma diffusion be the infinitesimal parameter of the Poisson process variant for $\mathbb{Z}$

$$\lambda(t) \sim \Gamma\big(\alpha\, dt, \beta / dt\big)$$

The integration of the Poisson process variant for $\mathbb{Z}$ is then a Poisson process (variant for $\mathbb{Z}$) with the integrated infinitesimal rate

$$\Lambda(t) = \int_0^t \lambda(s) ds \sim \Gamma\big(\alpha t, \beta\big)$$

the integral form of the gamma diffusion. The transition distribution of the gamma diffusion is then

$$f_{\Lambda(t)}(\lambda) = \frac{\lambda^{\alpha t - 1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}$$

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with transition mgf

$$M_{\Lambda(t)}(\theta)=\left(1-\theta\beta\right)^{-\alpha t}$$

The transition mgf of the Poisson process, conditional on the gamma rate intensity

$$M_{S(T)|S(t_0),\Lambda(t)}(\theta)=e^{S_0\theta+\Lambda(t)(pe^{u\theta}+qe^{d\theta}-1)}=e^{S_0\theta+\int_0^t\lambda(s)ds\,(pe^{u\theta}+qe^{d\theta}-1)}$$

which is consistent with the discussion of FOUNDATIONS II regarding stochastic volatility whereby all cumulants (including the variance) sum stochastic subject to the SNR and other cumulant ratios remaining fixed and time invariant. By nested expectation

$$M_{S(T)|S(t_0)}(\theta)=E\left(M_{S(T)|S(t_0),\Lambda(t)}(\theta)\right)=E\left(e^{S_0\theta+\Lambda(t)(pe^{u\theta}+qe^{d\theta}-1)}\right)=e^{S_0\theta}M_{\Lambda(t)}\left(pe^{u\theta}+qe^{d\theta}-1\right)$$

which is then

$$M_{S(T)|S(t_0)}(\theta)=e^{S_0\theta}\left(1+\beta-\beta(pe^{u\theta}+qe^{d\theta})\right)^{-\alpha t}$$

this is the mgf of a negative binomial process (variant for $\mathbb{Z}$). The diffusion variable $S(T)$ is then a unit root integration of a negative binomial (variant for $\mathbb{Z}$) basis function as discussed in FOUNDATIONS II.

The transition mass function is expressible only as an infinite series. When the up and downsteps are equal in magnitude, *u* = -*d*, the probability function is similar to that of the Poisson equivalent. The mass function of the Poisson variant given the gamma diffusion integrated rate intensity

$$p_{Y|\Lambda(t)}(y\,|\,\lambda)=e^{-\Lambda(t)}(\sqrt{p/q})^{y/\Delta t}\sum_{v=0}^{\infty}\frac{(\sqrt{pq})^{v}}{v!}\frac{(\sqrt{pq})^{v+y/\Delta y}}{(v+y/\Delta t)!v!}\lambda^{2v+y/\Delta y}$$

and

$$f_{\Lambda(t)}(\lambda)=\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}$$

so

$$\begin{aligned}
p_Y(y)&=\int_0^{\infty}p_{Y|\Lambda(t)}(y\,|\,\lambda)\,f_{\Lambda(t)}(\lambda)d\lambda\\
&=(\sqrt{p/q})^{y/\Delta t}\sum_{v=0}^{\infty}\frac{(\sqrt{pq})^{v}}{v!}\frac{(\sqrt{pq})^{v+y/\Delta y}}{(v+y/\Delta t)!}\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}e^{-\lambda}\lambda^{2v+y/\Delta y}d\lambda\\
&=(\sqrt{p/q})^{y/\Delta t}\sum_{v=0}^{\infty}\frac{(\sqrt{pq})^{v}}{v!}\frac{(\sqrt{pq})^{v+y/\Delta y}}{(v+y/\Delta t)!}\frac{1}{\beta^{\alpha t}\Gamma(\alpha t)}\int_0^{\infty}e^{-\lambda(1+1/\beta)}\lambda^{\alpha t-1+2v+y/\Delta y}d\lambda\\
&=(\sqrt{p/q})^{y/\Delta t}\sum_{v=0}^{\infty}\frac{(\sqrt{pq})^{v}}{v!}\frac{(\sqrt{pq})^{v+y/\Delta y}}{(v+y/\Delta t)!v!}\frac{1}{\beta^{\alpha t}\Gamma(\alpha t)}\frac{\Gamma(\alpha t+2v+y/\Delta y)}{(1+1/\beta)^{\alpha t+2v+y/\Delta y}}\\
&=(\sqrt{p/q})^{y/\Delta t}\sum_{v=0}^{\infty}\frac{(\sqrt{pq})^{v}}{v!}\frac{(\sqrt{pq})^{v+y/\Delta y}}{(v+y/\Delta t)!}\frac{\Gamma(\alpha t+2v+y/\Delta y)}{\Gamma(\alpha t)}\frac{\beta^{\alpha t+2v+y/\Delta y}}{\beta^{\alpha t}(1+\beta)^{\alpha t+2v+y/\Delta y}}\\
&=(1+\beta)^{-\alpha t}(\sqrt{p/q})^{y/\Delta t}\sum_{v=0}^{\infty}\frac{(\sqrt{pq})^{v}}{v!}\frac{(\sqrt{pq})^{v+y/\Delta y}}{(v+y/\Delta t)!}\frac{\Gamma(\alpha t+2v+y/\Delta y)}{\Gamma(\alpha t)}\frac{\beta^{2v+y/\Delta y}}{(1+\beta)^{2v+y/\Delta y}}\\
&=(1+\beta)^{-\alpha t}(\sqrt{p/q})^{y/\Delta t}\sum_{v=0}^{\infty}\frac{\Gamma(\alpha t+2v+y/\Delta y)}{\Gamma(\alpha t)}\frac{\left(\sqrt{pq}\beta(1+\beta)^{-1}\right)^{v}}{v!}\frac{\left(\sqrt{pq}\beta(1+\beta)^{-1}\right)^{v+y/\Delta y}}{(v+y/\Delta t)!}
\end{aligned}$$

or alternatively

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$$p_Y(y) = (1+\beta)^{-\alpha t}(\sqrt{p/q})^{y/\Delta t}\sum_{v=0}^{\infty}\frac{\Gamma(\alpha t+2v+y/\Delta y)}{(2v+y/\Delta y)!\Gamma(\alpha t)}\frac{(2v+y/\Delta y)!}{v!(v+y/\Delta t)!}\left(\sqrt{pq}\beta(1+\beta)^{-1}\right)^{2v+y/\Delta y}$$

$$= (1+\beta)^{-\alpha t}(\sqrt{p/q})^{y/\Delta t}\left(\sqrt{pq}\beta(1+\beta)^{-1}\right)^{y/\Delta y}\sum_{v=0}^{\infty}\binom{\alpha t-1+2v+y/\Delta y}{2v+y/\Delta y}\binom{2v+y/\Delta y}{v}\left(\sqrt{pq}\beta(1+\beta)^{-1}\right)^{2v+y/\Delta y}$$

$$= (1+\beta)^{-\alpha t}\left(p\beta(1+\beta)^{-1}\right)^{y/\Delta y}\sum_{v=0}^{\infty}\binom{\alpha t-1+2v+y/\Delta y}{2v+y/\Delta y}\binom{2v+y/\Delta y}{v}\left(\sqrt{pq}\beta(1+\beta)^{-1}\right)^{2v}$$

$$= (1+\beta)^{-\alpha t-y/\Delta y}\left(p\beta\right)^{y/\Delta y}\sum_{v=0}^{\infty}\binom{2v+y/\Delta y+\alpha t-1}{2v+y/\Delta y}\binom{2v+y/\Delta y}{v}\left(\sqrt{pq}\beta(1+\beta)^{-1}\right)^{2v}$$

The transition probability functions of the walk variable *S* is then $p_{S(t)|S(0)}(y\,|\,s_0) = p_Y(y-s_0)$ and for a forward transition beginning at time t_0, $p_{S(T)|S(t_0)}(y\,|\,s_0)$ is obtained by replacing t with $T-t_0$ in each occurrence in the functional formulae.

While the probability mass function of the forward transition is not easily evaluated, the limited moments for theoretical undiscounted option premia are. This discussion will conclude with the undiscounted put and call premia for both the additive and multiplicative versions of this form of random walk.

2. Undiscounted Option Premia: Summed walk

The **additive** *t*-period ahead forward walk variable, composed of X^{*N_t} upsteps and $N_t - X^{*N_t}$ downsteps, is

$$S_n = S_0 + uX^{*N_t} + d(N_t - X^{*N_t})$$
$$= S_0 + d\,N_t + (u-d)X^{*N_t}$$

where $N_t \sim P(\Lambda(t))$ and $\Lambda(t) = \int_0^t \lambda(s)ds \sim \Gamma(\alpha t, \beta)$ and, conditional on Λ and N, $N_t = n$, $X^{*n} \sim B(n,p)$

The payoff of an option on the walk variable is then

$\max(S_t - k,\ 0) = \max(S_0 + d\,N_t + (u-d)X^{*N_t} - k,\ 0)$ if it is an option to call the asset to buy for *k*

$\max(k - S_t,\ 0) = \max(k - S_0 - d\,N_t - (u-d)X^{*N_t},\ 0)$ if it is an option to put the asset to sell for *k*

For a call option, with $E(g(X)) = \sum_{x\in X} g(x)\times p_X(x)$, the undiscounted premium is

$$c(t,k) = E\left(\max(S_0 + (u-d)X^{*N_t} + d\,N_t - k), 0)\right)$$
$$= \int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\sum_{n=0}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\sum_{x=0}^{n}{}^nC_x p^x q^{n-x}\times\max\left(S_0 - k + nd + (u-d)x, 0\right)d\lambda$$
$$= \int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\sum_{x=0}^{\infty}\sum_{n=x}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\times{}^nC_x p^x q^{n-x}\times\max\left(S_0 - k + nd + (u-d)x, 0\right)d\lambda$$
$$= \sum_{x=0}^{\infty}\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\max\left(\left(S_0 - k + (u-d)x\right)\sum_{n=x}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}{}^nC_x p^x q^{n-x} + d\sum_{n=x}^{\infty}n e^{-\lambda}\frac{\lambda^n}{n!}{}^nC_x p^x q^{n-x},\ 0\right)d\lambda$$

and from the theory for the Poisson random walk with constant rate intensity

$$c(t,k) = \sum_{x=0}^{\infty}\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\max\left(\left(S_0 - k\right)\frac{e^{-p\lambda}(p\lambda)^x}{x!} + qd\lambda\frac{e^{-p\lambda}(p\lambda)^x}{x!} + (u-d)x\frac{e^{-p\lambda}(p\lambda)^x}{x!},\ 0\right)d\lambda$$
$$= \sum_{x=0}^{\infty}\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\max\left(\left(S_0 - k + (u-d)x\right)\frac{e^{-p\lambda}(p\lambda)^x}{x!} + qd\lambda\frac{e^{-p\lambda}(p\lambda)^x}{x!},\ 0\right)d\lambda$$

noting that

(1) $$\int_0^{\infty}e^{-p\lambda}\frac{(p\lambda)^x}{x!}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}d\lambda = \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

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(2) $\int_0^\infty \lambda e^{-p\lambda}\frac{(p\lambda)^x}{x!}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}d\lambda = \frac{\Gamma(\alpha t+1)}{\Gamma(\alpha t)}\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$

$$p_N(x)=\int_0^\infty e^{-p\lambda}\frac{(p\lambda)^x}{x!}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}d\lambda \qquad p_N(x)=\int_0^\infty \lambda e^{-p\lambda}\frac{(p\lambda)^x}{x!}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}d\lambda$$

$$=\frac{p^x}{x!\beta^{\alpha t}\Gamma(\alpha t)}\int_0^\infty \lambda^{x+\alpha t-1}e^{-\lambda(p+1/\beta)}d\lambda \qquad =\frac{p^x}{x!\beta^{\alpha t}\Gamma(\alpha t)}\int_0^\infty \lambda^{x+\alpha t}e^{-\lambda(p+1/\beta)}d\lambda$$

$$=\frac{p^x}{x!\beta^{\alpha t}\Gamma(\alpha t)}\frac{\Gamma(x+\alpha t)}{(p+1/\beta)^{x+\alpha t}} \qquad =\frac{p^x}{x!\beta^{\alpha t}\Gamma(\alpha t)}\frac{\Gamma(x+\alpha t+1)}{(p+1/\beta)^{x+\alpha t+1}}$$

$$=\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t} \qquad =\frac{\Gamma(\alpha t+1)}{\Gamma(\alpha t)}\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

$$=\binom{x+\alpha t-1}{x}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t} \qquad =\alpha t\binom{x+\alpha t}{x}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

The undiscounted premium can be written as

$$c(t,k)=\sum_{x=0}^\infty \max\left(\left(S_0-k+(u-d)x\right)\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}+\alpha tqd\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1},\ 0\right)$$

The max function is non-zero whenever

$$\left(S_0-k+(u-d)x\right)\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}+\alpha tqd\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}\geq 0$$

$$\left(S_0-k+(u-d)x\right)+qd\frac{\Gamma(x+\alpha t+1)}{\Gamma(x+\alpha t)}\left(\frac{1}{1+\beta p}\right)\geq 0$$

$$(S_0-k)(1+\beta p)+\alpha tqd+\left\{(u-d)(1+\beta p)+qd\right\}x\geq 0$$

$$\left\{(u-d)(1+\beta p)+qd\right\}x\geq (k-S_0)(1+\beta p)-\alpha tqd$$

$$x\geq\frac{(k-S_0)(1+\beta p)-\alpha tqd}{(u-d)(1+\beta p)+qd}$$

so that only terms in the summation for x greater than

$$k^*=\frac{(k-S_0)(1+\beta p)-\alpha tqd}{(u-d)(1+\beta p)+qd}$$

are non-zero, and the formula becomes

$$c(t,k)=\sum_{x=k^*}^\infty\left(\left(S_0-k+(u-d)x\right)\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}+\alpha t\,qd\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}\right)$$

$$=(S_0-k)\sum_{x=k^*}^\infty\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}+(u-d)\sum_{x=k^*}^\infty x\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

$$+\alpha tqd\sum_{x=k^*}^\infty\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

Noting that the weighted sum in x is

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$$\sum_{x=k^*}^{\infty} x\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}=\sum_{x=k^*}^{\infty}\frac{(x+\alpha t-1)!}{(x-1)!(\alpha t-1)!}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

$$=\alpha t\sum_{x=k^*-1}^{\infty}\frac{(x+\alpha t)!}{x!(\alpha t)!}\left(\frac{\beta p}{1+\beta p}\right)^{x+1}\left(\frac{1}{1+\beta p}\right)^{\alpha t}=\alpha t\sum_{x=k^*-1}^{\infty}\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^{x+1}\left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

$$=\alpha t\beta p\sum_{x=k^*-1}^{\infty}\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

so that

$$c(t,k)=(S_0-k)\sum_{x=k^*}^{\infty}\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}+\alpha tqd\sum_{x=k^*}^{\infty}\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

$$+\alpha t\beta p(u-d)\sum_{x=k^*-1}^{\infty}\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

$$=(S_0-k)\overline{F}_{NB(\alpha t,p\beta/(1+p\beta))}(k^*)+\alpha tqd\overline{F}_{NB(\alpha t+1,p\beta/(1+p\beta))}(k^*)+\alpha t\beta p(u-d)\overline{F}_{NB(\alpha t+1,p\beta/(1+p\beta))}(k^*-1)$$

with k^* being the smallest integer equal to or greater than $\left((k-S_0)(1+\beta p)-\alpha qd\right)/\left((u-d)(1+\beta p)+qd\right)$ and where $\overline{F}_{NB(\alpha t,p\beta/(1+p\beta))}(\cdot)$ and $\overline{F}_{NB(\alpha t+1,p\beta/(1+p\beta))}(\cdot)$ are the complementary cumulative distribution functions of negative binomial variables with parameters αt and $p\beta/(1+p\beta)$ and $\alpha t+1$ and $p\beta/(1+p\beta)$ respectively. For a put option

$$p(t,k)=E\left(\max(k-S_0-d\,N_t-(u-d)X^{*N_t}),0\right)$$

$$=\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\sum_{n=0}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\sum_{x=0}^{n}{}^nC_x p^x q^{n-x}\times\max\left(k-S_0-nd-(u-d)x,0\right)d\lambda$$

$$=\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\sum_{x=0}^{\infty}\sum_{n=x}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\times{}^nC_x p^x q^{n-x}\times\max\left(k-S_0-nd-(u-d)x,0\right)d\lambda$$

$$=\sum_{x=0}^{\infty}\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\max\left(\left(k-S_0-(u-d)x\right)\sum_{n=x}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}{}^nC_x p^x q^{n-x}-d\sum_{n=x}^{\infty}ne^{-\lambda}\frac{\lambda^n}{n!}{}^nC_x p^x q^{n-x},\ 0\right)d\lambda$$

with the theory for the Poisson generated random walk put option, summations for the binomial density functions with a Poisson process mixture derived for a call

$$p(t,k)=\sum_{x=0}^{\infty}\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\max\left(\left(k-S_0-(u-d)x\right)\frac{e^{-p\lambda}(p\lambda)^x}{x!}-qd\lambda\frac{e^{-p\lambda}(p\lambda)^x}{x!},\ 0\right)d\lambda$$

$$=\sum_{x=0}^{\infty}\max\left(\left(k-S_0-(u-d)x\right)\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\frac{e^{-p\lambda}(p\lambda)^x}{x!}d\lambda-qd\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\lambda\frac{e^{-p\lambda}(p\lambda)^x}{x!}d\lambda,\ 0\right)d\lambda$$

With

(1) $$\int_0^{\infty}e^{-p\lambda}\frac{(p\lambda)^x}{x!}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}d\lambda=\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

(2) $$\int_0^{\infty}\lambda e^{-p\lambda}\frac{(p\lambda)^x}{x!}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}d\lambda=\frac{\Gamma(\alpha t+1)}{\Gamma(\alpha t)}\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

the formula becomes

$$p(t,k)=\sum_{x=0}^{\infty}\max\left(\left(k-S_0-(u-d)x\right)\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t}-\alpha tqd\frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)}\left(\frac{\beta p}{1+\beta p}\right)^x\left(\frac{1}{1+\beta p}\right)^{\alpha t+1},\ 0\right)$$

For k^* being the largest integer equal to or smaller than

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$$k^* = \frac{(k - S_0)(1+\beta p) - \alpha t q d}{(u-d)(1+\beta p) + qd}$$

$$p(t,k) = \sum_{x=0}^{k^*} \left(\left(k - S_0 - (u-d)x\right) \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)} \left(\frac{\beta p}{1+\beta p}\right)^x \left(\frac{1}{1+\beta p}\right)^{\alpha t} - \alpha t q d \frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)} \left(\frac{\beta p}{1+\beta p}\right)^x \left(\frac{1}{1+\beta p}\right)^{\alpha t+1} \right)$$

and with

(3) $$\sum_{x=0}^{k^*} x \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)} \left(\frac{\beta p}{1+\beta p}\right)^x \left(\frac{1}{1+\beta p}\right)^{\alpha t} = \sum_{x=1}^{k^*} x \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)} \left(\frac{\beta p}{1+\beta p}\right)^x \left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

$$= \alpha t \beta p \sum_{x=0}^{k^*-1} \frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)} \left(\frac{\beta p}{1+\beta p}\right)^x \left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

the formulae can be written in terms of cumulative sums of negative binomial functions, so the undiscounted put option premium becomes

$$p(t,k) = \left(k - S_0\right) \sum_{x=0}^{k^*} \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)} \left(\frac{\beta p}{1+\beta p}\right)^x \left(\frac{1}{1+\beta p}\right)^{\alpha t} - \alpha t q d \sum_{x=0}^{k^*} \frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)} \left(\frac{\beta p}{1+\beta p}\right)^x \left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

$$- \alpha t \beta p (u-d) \sum_{x=0}^{k^*-1} \frac{\Gamma(x+\alpha t+1)}{x!\Gamma(\alpha t+1)} \left(\frac{\beta p}{1+\beta p}\right)^x \left(\frac{1}{1+\beta p}\right)^{\alpha t+1}$$

$$= \left(k - S_0\right) F_{NB(\alpha t, p\beta/(1+p\beta))}\left(k^*\right) - \alpha t q d F_{NB(\alpha t+1, p\beta/(1+p\beta))}\left(k^*\right)$$

$$- \alpha t \beta p (u-d) F_{NB(\alpha t+1, p\beta/(1+p\beta))}\left(k^*-1\right)$$

where k^* is the smallest integer greater than or equal to $\left((k - S_0)(1+\beta p) - \alpha q d\right) / \left((u-d)(1+\beta p) + qd\right)$ and $F_{NB(\alpha t, p\beta/(1+p\beta))}(\cdot)$ and $F_{NB(\alpha t+1, p\beta/(1+p\beta))}(\cdot)$ are the cumulative distribution functions of negative binomial variables with parameters αt and $p\beta / (1 + p\beta)$ and $\alpha t + 1$ and $p\beta / (1 + p\beta)$ respectively.

3. Undiscounted Option Premia: Produced walk

The **multiplicative** *t* period ahead forward walk variable, composed of X^{*N_t} upsteps and $N_t - X^{*N_t}$ downsteps, is

$$S_t = S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}}$$

where $N_t \sim P\left(\Lambda(t)\right)$ and $\Lambda(t) = \int_0^t \lambda(s) ds \sim \Gamma\left(\alpha t, \beta\right)$ and, conditional on Λ and N, $N_t = n$, $X^{*n} \sim B(n, p)$

The payoff of an option on the walk variable is then

$\max(S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}} - k, 0)$ if it is an option to call the asset to buy for *k*

$\max(k - S_0 u^{X^{*N_t}} d^{N_t - X^{*N_t}}, 0)$ if it is an option to put the asset to sell for *k*

By statistical formula, $E\left(g(X)\right) = \sum_{x \in X} g(x) \times p_X(x)$, the undiscounted premium for a call is

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$$c(t,k) = E\left(\max(S_0 u^{X*N_t} d^{N_t - X*N_t} - k), 0)\right)$$

$$= \int_0^\infty \frac{\lambda^{\alpha t-1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)} \sum_{n=0}^\infty e^{-\lambda} \frac{\lambda^n}{n!} \sum_{x=0}^n {}^nC_x p^x q^{n-x} \times \max\left(S_0 u^x d^{n-x} - k, 0\right) d\lambda$$

$$= \int_0^\infty \frac{\lambda^{\alpha t-1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)} \sum_{x=0}^\infty \sum_{n=x}^\infty e^{-\lambda} \frac{\lambda^n}{n!} \times {}^nC_x p^x q^{n-x} \times \max\left(S_0 u^x d^{n-x} - k, 0\right) d\lambda$$

$$= \sum_{x=0}^\infty \int_0^\infty \frac{\lambda^{\alpha t-1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)} \max\left(S_0 \sum_{n=x}^\infty e^{-\lambda} \frac{\lambda^n}{n!} \times^n C_x (pu)^x (qd)^{n-x} - k \sum_{n=x}^\infty e^{-\lambda} \frac{\lambda^n}{n!} \times^n C_x p^x q^{n-x},\ 0 \right) d\lambda$$

from

(1) $$\sum_{n=x}^\infty e^{-\lambda} \frac{\lambda^n}{n!} \times^n C_x p^x q^{n-x} = \frac{e^{-p\lambda}(p\lambda)^x}{x!}$$

(2) $$\sum_{n=x}^\infty e^{-\lambda} \frac{\lambda^n}{n!} \times {}^nC_x (pu)^x (qd)^{n-x} = e^{\lambda((pu+qd)-1)} \frac{e^{-pu\lambda}(pu\lambda)^x}{x!}$$

the call premium becomes

$$c(t,k) = \sum_{x=0}^\infty \int_0^\infty \frac{\lambda^{\alpha t-1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)} \max\left(S_0 e^{\lambda((pu+qd)-1)} \frac{e^{-pu\lambda}(pu\lambda)^x}{x!} - k \frac{e^{-p\lambda}(p\lambda)^x}{x!},\ 0 \right) d\lambda$$

Noting that

(3) $$\int_0^\infty e^{-p\lambda} \frac{(p\lambda)^n}{n!} \frac{\lambda^{\alpha t-1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)} d\lambda = \binom{n+\alpha t-1}{n} \left(\frac{\beta p}{1+\beta p}\right)^n \left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

(4) $$\int_0^\infty e^{\lambda(qd-1)} \frac{(pu\lambda)^x}{x!} \frac{\lambda^{\alpha t-1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)} d\lambda = \left(1+\beta-\beta(qd+pu)\right)^{-\alpha t} \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)} \left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x \left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t}$$

where $$\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)} = 1 - \frac{pu\beta}{1+\beta(1-qd)}$$

$$\int_0^\infty e^{-p\lambda} \frac{(p\lambda)^n}{n!} \frac{\lambda^{\alpha t-1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)} d\lambda$$

$$= \frac{p^n}{n!\beta^{\alpha t}\Gamma(\alpha t)} \int_0^\infty \lambda^{n+\alpha t-1} e^{-\lambda(p+1/\beta)} d\lambda$$

$$= \frac{p^n}{n!\beta^{\alpha t}\Gamma(\alpha t)} \frac{\Gamma(n+\alpha t)}{(p+1/\beta)^{n+\alpha t}}$$

$$= \frac{\Gamma(n+\alpha t)}{n!\Gamma(\alpha t)} \left(\frac{\beta p}{1+\beta p}\right)^n \left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

$$= \binom{n+\alpha t-1}{n} \left(\frac{\beta p}{1+\beta p}\right)^n \left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

$$\int_0^\infty e^{\lambda(qd-1)} \frac{(pu\lambda)^x}{x!} \frac{\lambda^{\alpha t-1} e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)} d\lambda$$

$$= \frac{(pu)^x}{\beta^{\alpha t} x!\Gamma(\alpha t)} \int_0^\infty \lambda^{x+\alpha t-1} e^{-\lambda(1/\beta+(1-qd))} d\lambda$$

$$= \frac{(pu)^x}{\beta^\alpha x!\Gamma(\alpha t)} \frac{\Gamma(x+\alpha t)}{(1/\beta+(1-qd))^{x+\alpha t}}$$

$$= \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)} \frac{(pu\beta)^x}{(1+\beta(1-qd))^{x+\alpha t}}$$

$$= \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)} \left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x \left(\frac{1}{1+\beta(1-qd)}\right)^{\alpha t}$$

$$= \left(1+\beta-\beta(qd+pu)\right)^{-\alpha t} \times \frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)} \left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x \left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t}$$

the premium can be written as a single summation

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$$c(t,k)=\sum_{x=0}^{\infty}\ \max\left(S_0\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}e^{\lambda((pu+qd)-1)}\frac{e^{-pu\lambda}(pu\lambda)^x}{x!}d\lambda-k\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\frac{e^{-p\lambda}(p\lambda)^x}{x!}d\lambda,\ 0\right)$$

$$=\sum_{x=0}^{\infty}\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\max\left(S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha t}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t}-k\left(\frac{p\beta}{1+p\beta}\right)^x\left(\frac{1}{1+p\beta}\right)^{\alpha t},\ 0\right)$$

The max function is non-zero whenever

$$S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha}-k\left(\frac{p\beta}{1+p\beta}\right)^x\left(\frac{1}{1+p\beta}\right)^{\alpha}\geq 0$$

$$S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha}\geq k\left(\frac{p\beta}{1+p\beta}\right)^x\left(\frac{1}{1+p\beta}\right)^{\alpha}$$

$$\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{p\beta}{1+p\beta}\right)^{-x}\geq\frac{k\left(1+\beta-\beta(qd+pu)\right)^{\alpha}}{S_0}\left(\frac{1}{1+p\beta}\right)^{\alpha}\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{-\alpha}$$

$$x\log\left\{\frac{u(1+p\beta)}{1+\beta(1-qd)}\right\}\geq\log\left\{\frac{k}{S_0}\left(\frac{1+\beta(1-qd)}{1+p\beta}\right)^{\alpha}\right\}$$

$$x\geq\log\left\{\frac{k}{S_0}\left(\frac{1+\beta(1-qd)}{1+p\beta}\right)^{\alpha}\right\}\Big/\log\left\{\frac{u(1+p\beta)}{1+\beta(1-qd)}\right\}$$

so that only terms in the summation for x greater than

$$k^*=\log\left\{\frac{k}{S_0}\left(\frac{1+\beta(1-qd)}{1+p\beta}\right)^{\alpha}\right\}\Big/\log\left\{\frac{u(1+p\beta)}{1+\beta(1-qd)}\right\}$$

are non-zero. The undiscounted call option premium is

$$c(t,k)=S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha t}\sum_{x=k^*}^{\infty}\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t}$$
$$-k\sum_{x=k^*}^{\infty}\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{p\beta}{1+p\beta}\right)^x\left(\frac{1}{1+p\beta}\right)^{\alpha t}$$

where k^* is the smallest integer equal to or larger than the standardized exercise price

$$\frac{\log k\left(1+\beta(1-qd)\right)^{\alpha}-\log S_0(1+p\beta)^{\alpha}}{\log u(1+p\beta)-\log\left(1+\beta(1-qd)\right)}=\frac{\log k-\log S_0-\alpha\log(1+p\beta)/\left(1+\beta(1-qd)\right)}{\log u+\log(1+p\beta)/\left(1+\beta(1-qd)\right)}.$$

Written in terms of negative binomial complementary cdfs, the premium is

$$c(t,k)=S_0\left(1+\beta-\beta(qd+pu)\right)^{-\alpha t}\,\overline{F}_{NB(\alpha t,\,pu\beta/[1+p(1-qd)])}\left(k^*\right)-k\,\overline{F}_{NB(\alpha t,\,p\beta/(1+p\beta))}\left(k^*\right)$$

where $\overline{F}_{NB(\alpha t,\,pu\beta/[1+p(1-qd)])}(\cdot)$ and $\overline{F}_{NB(\alpha t,\,p\beta/(1+p\beta))}(\cdot)$ are the complementary cumulative distribution functions of negative binomial variables with parameters αt and $pu\beta/[1+p(1-qd)]$ and αt and $p\beta/(1+p\beta)$ respectively. For a put option

$$p(t,k)=E\left(\max(k-S_0u^{X^*N_t}d^{N_t-X^*N_t}),0)\right)$$

$$=\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\sum_{n=0}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\sum_{x=0}^{n}{}^nC_x p^x q^{n-x}\times\max\left(k-S_0u^xd^{n-x},0\right)d\lambda$$

$$=\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\sum_{x=0}^{\infty}\sum_{n=x}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\times{}^nC_x p^x q^{n-x}\times\max\left(k-S_0u^xd^{n-x},0\right)d\lambda$$

$$=\sum_{x=0}^{\infty}\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\max\left(k\sum_{n=x}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\times{}^nC_x p^x q^{n-x}-S_0\sum_{n=x}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\times{}^nC_x(pu)^x(qd)^{n-x},\ 0\right)d\lambda$$

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with

(1) $$\sum_{n=x}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\times {}^nC_x p^x q^{n-x} = \frac{e^{-p\lambda}(p\lambda)^x}{x!}$$

(2) $$\sum_{n=x}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\times {}^nC_x (pu)^x (qd)^{n-x} = e^{\lambda((pu+qd)-1)}\frac{e^{-pu\lambda}(pu\lambda)^x}{x!}$$

$$p(t,k) = \sum_{x=0}^{\infty}\int_0^{\infty}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}\max\left(k\,\frac{e^{-p\lambda}(p\lambda)^x}{x!} - S_0 e^{\lambda((pu+qd)-1)}\frac{e^{-pu\lambda}(pu\lambda)^x}{x!},\ 0\right)d\lambda$$

and also

(3) $$\int_0^{\infty} e^{-p\lambda}\frac{(p\lambda)^n}{n!}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}d\lambda = \binom{n+\alpha t-1}{n}\left(\frac{\beta p}{1+\beta p}\right)^n\left(\frac{1}{1+\beta p}\right)^{\alpha t}$$

(4) $$\int_0^{\infty} e^{\lambda(qd-1)}\frac{(pu\lambda)^x}{x!}\frac{\lambda^{\alpha t-1}e^{-\lambda/\beta}}{\beta^{\alpha t}\Gamma(\alpha t)}d\lambda = \big(1+\beta-\beta(qd+pu)\big)^{-\alpha t}\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t}$$

$$p(t,k) = \sum_{x=0}^{\infty}\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\max\left(S_0\big(1+\beta-\beta(qd+pu)\big)^{-\alpha t}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t} - k\left(\frac{p\beta}{1+p\beta}\right)^x\left(\frac{1}{1+p\beta}\right)^{\alpha t},0\right)$$

and for k^* being the largest integer equal to or smaller than

$$k^* = \log\left\{\frac{k}{S_0}\left(\frac{1+\beta(1-qd)}{1+p\beta}\right)^{\alpha}\right\}\Big/\log\left\{\frac{u(1+p\beta)}{1+\beta(1-qd)}\right\}$$

the undiscounted put option premium becomes

$$p(t,k) = k\sum_{x=0}^{k^*}\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{p\beta}{1+p\beta}\right)^x\left(\frac{1}{1+p\beta}\right)^{\alpha t}$$
$$-S_0\big(1+\beta-\beta(qd+pu)\big)^{-\alpha t}\sum_{x=0}^{k^*}\frac{\Gamma(x+\alpha t)}{x!\Gamma(\alpha t)}\left(\frac{pu\beta}{1+\beta(1-qd)}\right)^x\left(\frac{1+\beta-\beta(qd+pu)}{1+\beta(1-qd)}\right)^{\alpha t}$$

Written in terms of negative binomial cdfs, both with index αt and the leading term with probability parameter $p\beta/(1+p\beta)$ and the trailing term with $pu\beta/[1+p(1-qd)]$, the undiscounted premium is

$$p(t,k) = k\,F_{NB(\alpha t,\,p\beta/(1+p\beta))}(k^*) - S_0\big(1+\beta-\beta(qd+pu)\big)^{-\alpha t}F_{NB(\alpha t,\,p\beta/[1+p(1-qd)])}(k^*)$$

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